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ON DENJOY PROPERTY AND ON APPROXIMATE PARTIAL DERIVATIVES

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1. An interesting property of a derivative function f in the class of real functions of a single real variable is that if f is nonnegative almost everywhere in an interval then f is nonnegative everywhere in that interval from which the following property of f can easily be deduced:

(*) given any closed interval I, each of the sets $I \cap \{x : f(x) > \alpha\}$ and $I \cap \{x : f(x) < \alpha\}$ is either void or of positive measure for arbitrary values of α.

This property of the derivative function f together with its Darboux property and belonging to Baire class 1 implies the Denjoy property of f [5, 3] viz.

(**) given any closed interval I, the set $I \cap \{x : \alpha < f(x) < \beta\}$ is either void or of positive measure for arbitrary α and β, $\alpha < \beta$.

In [11] it is shown that the properties (*) and (**) are equivalent for functions of single variable in Baire class 1. In the present paper we extend this result to functions of several variables. It is interesting to note that MIŠÍK [8] proved that if a function f belongs to the class $\mathscr{M}_2$ (for definition see also [17]) then it satisfies the property (**). Since a function f belongs to the class $\mathscr{M}_2$ iff f is in Baire class 1 satisfying the condition (*), this result also follows from the result of Mišík. Mišík proved his result with the help of Darboux property which is established in [9] (see also [10]). Since the properties (*) and (**) are very close to each other, and because of the intrinsic value of these properties, we give here a direct proof. Finally, in the light of these results some properties of the approximate partial derivatives are studied.

2. Throughout the paper X is the euclidean n-space R^n, where R is the set of all real numbers. We shall follow the standard definitions of closed and open intervals, spheres, cubes, etc. given in [13; p. 57]. By an interval (resp. sphere) in X we mean a set $\mathscr{G} \subset X$ such that $\overline{\mathscr{G}}$ (the closure of $\mathscr{G}$) is a closed nondegenerate interval (resp. sphere) and $\mathscr{G}^0$ (the interior of $\mathscr{G}$) is an open interval (resp. sphere) with the same principal vertices (resp. with the same centre and radius), i.e. $(\overline{\mathcal{G}})^0 \subset \mathcal{G} \subset \overline{\mathcal{G}}$. $\mu(A)$ will denote the Lebesgue measure of the measurable set $A \subset X$. $\mathcal{B}$ will denote the base for the topology of X consisting of all open intervals (or, spheres) in X. Following [2] we shall say that a function f on X has Darboux property relative to $\mathcal{B}$ if $f(\overline{U})$ is connected for every $U \in \mathcal{B}$. We need the following

Definition. A measurable set $E \subset X$ is said to satisfy *the property $\mathcal{C}$ relative to $\mathcal{B}$* iff $\overline{U} \cap E$ is either void or of positive measure for every $U \in \mathcal{B}$.

A measurable function f on X is said to satisfy *the property $\mathcal{C}_*$ relative to $\mathcal{B}$* iff for arbitrary $\alpha \in R$ each of the sets

$$\{x : f(x) > \alpha\} \quad \text{and} \quad \{x : f(x) < \alpha\}$$

satisfies the property $\mathcal{C}$ relative to $\mathcal{B}$, ($\mathcal{M}'_2$ property of Zahorski - Mišík).

The function f is said to satisfy *the property $\mathcal{C}_{**}$ relative to $\mathcal{B}$* iff for arbitrary α, $\beta \in R$, $\alpha < \beta$, the set

$$\{x : \alpha < f(x) < \beta\}$$

satisfies the property $\mathcal{C}$ relative to $\mathcal{B}$, (Denjoy property).

We shall consider extended real valued functions on X. $\mathcal{F}$, $\mathcal{B}_1$, $\mathcal{M}'_2(\mathcal{B})$, $\mathcal{D}'(\mathcal{B})$ and $\mathcal{D}(\mathcal{B})$ will denote respectively the class of finite functions, the class of functions of Baire type 1, the class of functions satisfying the property $\mathcal{C}_*$ relative to $\mathcal{B}$, the class of functions satisfying the property $\mathcal{C}_{**}$ relative to $\mathcal{B}$, and the class of functions satisfying Darboux property relative to $\mathcal{B}$.

We shall write

$$E(f; *, \alpha) = \{x : x \in X; f(x) < \alpha\},$$
$$E(f; \beta, *) = \{x : x \in X; f(x) > \beta\},$$
$$E(f; \alpha, \beta) = \{x : x \in X; \alpha < f(x) < \beta\}.$$

When there is no confusion, we shall simply write $E(*, \alpha)$ etc. instead of $E(f; *, \alpha)$ etc. For the definition of metric density, approximate continuity, and approximate partial derivatives, which we shall consider in the paper, we refer to the book of Saks [13]. We shall prove our results when $\mathcal{B}$ consists of all open intervals. The case when $\mathcal{B}$ consists of all open spheres is similar.

3. Theorem 1. $\mathcal{M}'_2(\mathcal{B}) \cap B_1 \subset \mathcal{D}'(\mathcal{B}) \cap B_1$.

Proof. We shall prove by contradiction. Suppose $f \in M'_2(\mathcal{B}) \cap \mathcal{B}_1$ and suppose that there are $A \in \mathcal{B}$, α, $\beta \in R$, $\alpha < \beta$, such that

$$\bar{A} \cap E(\alpha, \beta) \neq 0, \quad \mu[\bar{A} \cap E(\alpha, \beta)] = 0.$$

Let

$$F_\alpha = \{p : p \in \bar{A}; f(p) \leqq \alpha\}, \quad F_\beta = \{p : p \in \bar{A}; f(p) \geqq \beta\}.$$

Let $\mathscr{G}$ be any interval contained in F_α. Then since $\bar{\mathscr{G}} \cap E(\alpha, *) \subset \bar{\mathscr{G}} - \mathscr{G}^0$ and $\mu[\bar{\mathscr{G}} - \mathscr{G}^0] = 0$, we conclude $\mu[\bar{\mathscr{G}} \cap E(\alpha, *)] = 0$. Since the set $E(\alpha, *)$ has the property $\mathscr{C}$, $\bar{\mathscr{G}} \cap E(\alpha, *) = 0$. Hence $\bar{\mathscr{G}} \subset F_\alpha$. Thus if $\mathscr{G}$ is any interval such that $\mathscr{G} \subset F_\alpha$ then $\bar{\mathscr{G}} \subset F_\alpha$. Similarly if $\mathscr{G}$ is any interval contained in F_β then $\bar{\mathscr{G}} \subset F_\beta$.

Let $\{\mathscr{G}\}$ be the family of all intervals $\mathscr{G}$ such that either $\mathscr{G} \subset F_\alpha$ or $\mathscr{G} \subset F_\beta$. Let $P = \bar{A} - \cup \mathscr{G}^0$, where the union is taken over all $\mathscr{G} \in \{\mathscr{G}\}$, and where the interior $\mathscr{G}^0$ is taken relative to $\bar{A}$. Then P is closed.

We can conclude $P \subset \bar{F}_\alpha \cap \bar{F}_\beta$. For, if possible, let $p_0 \in P$ but $p_0 \notin \bar{F}_\alpha$. Then there is an open interval S, containing p_0 such that $S \cap F_\alpha = 0$. Since $\mu[\bar{A} \cap E(\alpha, \beta)] = 0$, $\mu[\bar{A} \cap \bar{S} \cap E(*, \beta)] = 0$ and hence $\bar{A} \cap \bar{S} \cap E(*, \beta) = 0$. So, $\bar{A} \cap \bar{S} \subset F_\beta$. Hence $\bar{A} \cap S \subset \mathscr{G}$ for some $\mathscr{G} \in \{\mathscr{G}\}$ and therefore $p_0 \in \mathscr{G}^0$ which is a contradiction, since $p_0 \in P$. Thus $P \subset \bar{F}_\alpha$. Similarly $P \subset \bar{F}_\beta$. Hence $P \subset \bar{F}_\alpha \cap \bar{F}_\beta$.

Let $p_0 \in P$ and let I be a closed interval containing p_0 in its interior. We shall show that $I \cap P \cap F_\alpha$ and $I \cap P \cap F_\beta$ are non void. Since $p_0 \in P$, $p_0 \in \bar{F}_\alpha$. Hence there is a point q of F_α in I. If $q \in P$, then $q \in I \cap P \cap F_\alpha$ and the assertion follows. If $p_0 \in F_\alpha$, then also $p_0 \in I \cap P \cap F_\alpha$. So, we suppose that $p_0 \notin F_\alpha$, $q \notin P$. Since $q \notin P$, there is $\mathscr{G} \in \{\mathscr{G}\}$ such that $q \in \mathscr{G}^0$, $\mathscr{G} \subset F_\alpha$ and $p_0 \notin \mathscr{G}$. Let $S_r(q)$ denote the open cube with center q and each side of length $2r$. Consider the collection of all open cubes $\{S_r(q)\}$ for different values of $r > 0$. There is one and only one such cube, say $S_{r_0}(q)$, which will contain p_0 on its faces. Since $q \in \mathscr{G}^0$, there is r_1, $0 < r_1 < r_0$, such that $\bar{A} \cap \cap S_{r_1}(q) \subset \mathscr{G}^0$. So, $\bar{A} \cap I \cap S_r(q) \subset F_\alpha$ for all r, $0 < r \leqq r_1$. Also $\bar{A} \cap I \cap S_{r_0}(q) \not\subset \not\subset F_\alpha$; for, if $\bar{A} \cap I \cap S_{r_0}(q) \subset F_\alpha$ then p_0 being a point on the boundary of $S_{r_0}(q)$ would belong to F_α which is against our supposition. Thus there is a real number say σ, $0 < \sigma \leqq r_0$, such that if $0 < r < \sigma$, then $\bar{A} \cap I \cap S_r(q) \subset F_\alpha$ and if $r > \sigma$ then $\bar{A} \cap I \cap S_r(q) \not\subset F_\alpha$. Since $S_\sigma(q) = \bigcup_{0<r<\sigma} S_r(q)$, $\bar{A} \cap I \cap S_\sigma(q) \subset F_\alpha$ and hence $\bar{A} \cap I \cap \bar{S}_\sigma(q) \subset F_\alpha$. We conclude that the set $I \cap \bar{S}_\sigma(q) \cap P$ is nonvoid. For, if $I \cap \bar{S}_\sigma(q) \cap P = 0$, then every $p \in \bar{A} \cap I \cap [\bar{S}_\sigma(q) - S_\sigma(q)]$ is contained in an open interval $\mathscr{G}^0 \subset \mathscr{G} \in \{\mathscr{G}\}$, $\mathscr{G} \subset F_\alpha$, and since $\bar{A} \cap I \cap [\bar{S}_\sigma(q) - S_\sigma(q)]$ is closed, there is a finite set $\mathscr{G}_1^0, \mathscr{G}_2^0, \ldots, \mathscr{G}_n^0$, such that

$$\bar{A} \cap I \cap [\bar{S}_\sigma(q) - S_\sigma(q)] \subset \bigcup_{i=1}^{n} \mathscr{G}_i^0, \quad \mathscr{G}_i \subset F_\alpha .$$

and hence it is possible to obtain $r_2 > \sigma$ such that $\bar{A} \cap I \cap S_{r_2}(q) \subset F_\alpha$ which is a contradiction. Thus $I \cap \bar{S}_\sigma(q) \cap P \neq 0$. Since $\bar{A} \cap I \cap \bar{S}_\sigma(q) \subset F_\alpha$, we have $I \cap \bar{S}_\sigma(q) \cap P \subset I \cap P \cap F_\alpha$ and hence $I \cap P \cap F_\alpha \neq 0$. Similar argument can be applied to show that $I \cap P \cap F_\beta \neq 0$. From this we get

$$\inf_{p \in I \cap P} f(p) \leqq \alpha, \quad \sup_{p \in I \cap P} f(p) \geqq \beta .$$

So, p_0 is not a point of continuity of $f \mid P$. Since f is of Baire class 1, this is a contradiction.

The converse of Theorem 1 is not true. For, consider $n = 1$ and the function g on R defined as

$$g(x) = 0\,, \quad \text{if} \quad x \neq 0\,,$$
$$\quad = \infty\,, \quad \text{if} \quad x = 0\,.$$

Then $g \in B_1$; for $g = \lim_{i\to\infty} f_i$, where

$$f_i(x) = i(1 - ix)\,, \quad \text{if} \quad 0 \leqq x < \frac{1}{i}\,,$$
$$= i(1 + ix)\,, \quad \text{if} \quad -\frac{1}{i} < x < 0\,,$$
$$= 0\,, \quad \text{elsewhere}\,.$$

Also $g \in \mathscr{D}'(\mathscr{B})$. But $g \notin \mathscr{M}'_2(\mathscr{B})$. (For $n > 1$, functions on X can easily be constructed with the help of the function g so as to satisfy these requirements.) The converse, however, is true for finite functions or Darboux functions. For finite functions this follows from Denjoy property. If $f \in \mathscr{D}(\mathscr{B})$, $U \in \mathscr{B}$, $\alpha \in R$ and if $\bar{U} \cap E(*, \alpha) \neq 0$, then, either $f(x) < \alpha$ for all $x \in \bar{U}$, in which case the result follows, or there is one $x \in \bar{U}$ such that $f(x) \geqq \alpha$ and hence by the Darboux property there is $\xi \in \bar{U}$ such that $-\infty < f(\xi) < \alpha$ and the result follows from the Denjoy property. Thus we get

Theorem 2.

$$\mathscr{D}'(\mathscr{B}) \cap [\mathscr{F} \cup \mathscr{D}(\mathscr{B})] \subset \mathscr{M}'_2(\mathscr{B})$$

Again we remark that the inclusion cannot be reversed. For example we refer to [8, 14, 7].

The following theorem is also known [17, 9]; but we give a different proof.

Theorem 3.

$$\mathscr{M}'_2(\mathscr{B}) \cap \mathscr{B}_1 \subset \mathscr{D}(\mathscr{B}) \cap \mathscr{B}_1$$

Proof. Suppose that $f \in \mathscr{M}'_2(\mathscr{B}) \cap \mathscr{B}_1$ but $f \notin \mathscr{D}(\mathscr{B})$. Then there are $A \in \mathscr{B}$ and $\alpha \in R$ such that the sets

$$\bar{A} \cap E(*, \alpha) \neq 0 \neq \bar{A} \cap E(\alpha, *)\,, \quad \bar{A} \subset E(*, \alpha) \cup E(\alpha, *)\,.$$

Since $f \in \mathscr{M}'_2(\mathscr{B})$, we have

$$\mu[\bar{A} \cap E(*, \alpha)] > 0\,, \quad \mu[\bar{A} \cap E(\alpha, *)] > 0\,.$$

Let $\{\mathscr{G}\}$ be the collection of all intervals $\mathscr{G}$ such that $\mathscr{G} \subset \bar{A} \cap E(*, \alpha)$ or $\mathscr{G} \subset \bar{A} \cap E(\alpha, *)$. Then the set $P = \bar{A} - \cup \mathscr{G}^0$, where the union is taken over all $\mathscr{G} \in \{\mathscr{G}\}$, and where the interior $\mathscr{G}^0$ is taken relative to $\bar{A}$, is closed. Since f is in Baire class 1, there is $p_0 \in P$ such that $f(x)$ tends to $f(p_0)$ as x tends to p_0 through P. Since $\bar{A} = [\bar{A} \cap E(*, \alpha)] \cup [\bar{A} \cap E(\alpha, *)]$, $p_0 \in \bar{A} \cap E(*, \alpha)$ or $p_0 \in \bar{A} \cap E(\alpha, *)$. Suppose $p_0 \in \bar{A} \cap E(*, \alpha)$. Then $f(p_0) < \alpha$. By the above property of f there is $\delta > 0$ such that $I_\delta \cap P \subset E(*, \alpha)$, where I_δ is the closed cube with p_0 as centre and each side of length 2δ.

We assert that $\bar{A} \cap I_\delta \cap \mathscr{G}^0 = 0$ for all $\mathscr{G} \in \{\mathscr{G}\}$, $\mathscr{G} \subset E(\alpha, *)$. For, suppose that $\bar{A} \cap I_\delta \cap \mathscr{G}^0 \neq 0$ for some $\mathscr{G} \in \{\mathscr{G}\}$, $\mathscr{G} \subset E(\alpha, *)$. Then there is $q \in A^0 \cap I_\delta^0 \cap \mathscr{G}^0$. Also $p_0 \notin \mathscr{G}^0$. Let $\{S_r(q)\}$ be the collection of all open cubes with center q and each side of length $2r$, for different $r > 0$. Then as in the proof of Theorem 1 there is a real number σ, such that if $0 < r < \sigma$, then $\bar{A} \cap I_\delta \cap S_r(q) \subset E(\alpha, *)$ and if $r > \sigma$, then $\bar{A} \cap I_\delta \cap S_r(q) \not\subset E(\alpha, *)$ and the set $I_\delta \cap \bar{S}_\sigma(q) \cap P$ is nonvoid. Let $s \in I_\delta \cap \cap \bar{S}_\sigma(q) \cap P$. Since $\bar{A} \cap I_\delta \cap S_r(q) \subset E(\alpha, *)$ for $0 < r < \sigma$, $\bar{A} \cap I_\delta \cap S_\sigma(q) \subset \subset E(\alpha, *)$ and since $f \in \mathscr{M}'_2(\mathscr{B})$, $\bar{A} \cap I_\delta \cap \bar{S}_\sigma(q) \subset E(\alpha, *)$. So, $s \in E(\alpha, *)$. But this contradicts $I_\delta \cap P \subset E(*, \alpha)$. Thus

$$\bar{A} \cap I_\delta \cap \mathscr{G}^0 = 0 \quad \text{for all} \quad \mathscr{G} \in \{\mathscr{G}\}, \quad \mathscr{G} \subset E(\alpha, *).$$

Since $\bar{A} \cap [E(\alpha, *) - \cup \mathscr{G}^0] \subset P$, where the union is taken over all $\mathscr{G} \in \{\mathscr{G}\}$, $\mathscr{G} \subset \subset E(\alpha, *)$, and since $I_\delta \cap P \subset E(*, \alpha)$, we infer that $\bar{A} \cap I_\delta \cap E(\alpha, *) = 0$ and hence $\bar{A} \cap I_\delta \subset E(*, \alpha)$. Since $p_0 \in \bar{A} \cap I_\delta^0$, $p_0 \in \mathscr{G}^0$ for some $\mathscr{G} \in \{\mathscr{G}\}$, $\mathscr{G} \subset E(*, \alpha)$. This contradicts the fact that $p_0 \in P$. This completes the proof.

It is known that the inclusion in Theorem 3 cannot be reversed [17]. From Theorems 1 and 3, it is natural to ask if there is any relation between the classes $\mathscr{D}'(\mathscr{B}) \cap \mathscr{B}_1$ and $\mathscr{D}(\mathscr{B}) \cap \mathscr{B}_1$. For finite functions clearly $f \in \mathscr{D}'(\mathscr{B}) \cap \mathscr{B}_1$ implies $f \in \mathscr{D}(\mathscr{B}) \cap \mathscr{B}_1$ and hence

$$\mathscr{D}'(\mathscr{B}) \cap \mathscr{B}_1 \cap \mathscr{F} \subset \mathscr{D}(\mathscr{B}) \cap \mathscr{B}_1 \cap \mathscr{F}.$$

But this relation is not true in general. In fact, neither of the relations

$$\mathscr{D}'(\mathscr{B}) \cap \mathscr{B}_1 \subset \mathscr{D}(\mathscr{B}) \quad \text{and} \quad \mathscr{D}(\mathscr{B}) \cap \mathscr{B}_1 \subset \mathscr{D}'(\mathscr{B})$$

is valid. For, as is mentioned, the function g on R, where $g(x) = 0$, $x \neq 0$ and $g(0) = \infty$, is such that $g \in \mathscr{D}'(\mathscr{B}) \cap \mathscr{B}_1$ but $g \notin \mathscr{D}(\mathscr{B})$; and the function f on R constructed in [4] is such that $f \in \mathscr{D}(\mathscr{B}) \cap \mathscr{B}_1$ and f vanishes almost everywhere (but not identically vanishing) which shows that $f \notin \mathscr{D}'(\mathscr{B})$. For $n > 1$, functions can easily be constructed with the help of these functions to satisfy the requirements.

It is known that if $f : X \to R$ is an approximately continuous function then f is of Baire class 1 [6]. Also for each $\alpha \in R$ the sets $\{x : f(x) < \alpha\}$ and $\{x : f(x) > \alpha\}$ satisfy the property $\mathscr{C}$. For, if $p \in \bar{I} \cap \{x : f(x) > \alpha\}$ where $I \in \mathscr{B}$, then by the definition of approximate continuity, the set $\{x : f(x) > \alpha\}$ has metric density 1 at p and hence $\bar{I} \cap \{x : f(x) > \alpha\}$ is of positive measure. So, from Theorem 3 we get

Corollary. *If $f : X \to R$ is approximately continuous then f has the Darboux property.*

This is proved in [12] by considering interval function.

4. We shall further require some additional notations. If $x = (x_1, x_2, \dots, x_n)$ is a point in X then f_{x_1} will denote, for fixed x_1, the function ϕ on R^{n-1} defined by

$$\phi(x_2, x_3, \dots, x_n) = f_{x_1}(x_2, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) .$$

Also if $\bar{I}$ is a closed interval in R^n given by $\bar{I} = [a_1 b_1; a_2, b_2; \dots; a_n, b_n]$ then $\bar{I}_1$ and $\bar{I}^1$ will denote the closed intervals in R and R^{n-1} given by $\bar{I}_1 = [a_1, b_1]$ and $\bar{I}^1 = [a_2, b_2; \dots; a_n, b_n]$, respectively.

Theorem 4. *If $f : X \to R$ is such that the approximate partial derivative w.r. to x_1, $(\partial f/\partial x_1)_{\text{ap}}$ exists (possibly infinite), except perhaps on a denumerable subset and f is Darboux function of Baire class 1 relative to x_1 (i.e. when $x_2, x_3, \dots, x_n$ remain fixed) and f_{x_1} is approximately continuous in R^{n-1} for fixed x_1 then $(\partial f/\partial x_1)_{\text{ap}} \in \mathscr{M}'_2(\mathscr{B})$.*

Proof. We shall prove that the set $E((\partial f/\partial x_1)_{\text{ap}}; *, \alpha)$ satisfies the property $\mathscr{C}$ relative to $\mathscr{B}$ for arbitrary $\alpha \in R$. The proof for $E((\partial f/\partial x_1)_{\text{ap}}; \alpha, *)$ is similar.

We suppose $\alpha = 0$; for $\alpha \neq 0$, we are to apply this argument to the function $h : X \to R$ given by

$$h(x_1, x_2, \dots, x_n) = f(x_1, \dots, x_n) - \alpha x_1 .$$

(This is permissible, since the sum of a Darboux - Baire-1 function and a continuous function is a Darboux - Baire-1 function).

Let $I \in \mathscr{B}$ and let

$$\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \bar{I} \cap E\left(\left(\frac{\partial f}{\partial x_1}\right)_{\text{ap}}; *, 0\right).$$

Set $g(x_1) = f(x_1, \xi_2, \dots, \xi_n)$. Then g is a Darboux function of Baire class 1 from R to R. Also the approximate derivative g'_{ap} of g exists (possibly infinite) except on a denumerable subset of R. Finally, if $g'_{\text{ap}} \geqq 0$ almost everywhere in a closed interval $[a, b] \subset R$ then by a known result [1, 15], g is non decreasing in $[a, b]$, and hence $g'_{\text{ap}} \geqq 0$ everywhere in $[a, b]$. From this fact and from the fact that $\xi_1 \in \bar{I}_1$, $g'_{\text{ap}}(\xi_1) < 0$ we conclude that the set $\bar{I}_1 \cap \{x_1 : g'_{\text{ap}}(x_1) < 0\}$ is of positive linear measure. Hence there are two points η_1 and ζ_1 in I_1 such that $\eta_1 < \zeta_1$ and $g(\eta_1) > g(\zeta_1)$, that is, $f(\eta_1, \xi_2, \dots, \xi_n) > f(\zeta_1, \xi_2, \dots, \xi_n)$. Let $\varepsilon = \{f(\eta_1, \xi_2, \dots, \xi_n) - f(\zeta_1, \xi_2, \dots, \xi_n)\}/2$. Since f_{η_1} and f_{ζ_1} are approximately continuous from R^{n-1} to R, each of the sets

$$S = \{(x_2, \dots, x_n) : f_{\eta_1}(x_2, \dots, x_n) > f_{\eta_1}(\xi_2, \dots, \xi_n) - \varepsilon\}$$

and

$$T = \{(x_2, \ldots, x_n) : f_{\zeta_1}(x_2, \ldots, x_n) < f_{\zeta_1}(\xi_2, \ldots, \xi_n) + \varepsilon\}$$

has metric density 1 at the point $(\xi_2\xi_3 \ldots \xi_n) \in R^{n-1}$. Now $\bar{I}^1$ is an interval on R^{n-1} containing the point $(\xi_2\xi_3 \ldots \xi_n)$. Also the set $S \cap T$ has metric density 1 at $(\xi_2\xi_3 \ldots \xi_n)$ in R^{n-1}. Hence $\bar{I}^1 \cap (S \cap T)$ is of positive measure in R^{n-1}. Let $(k_2, k_3, \ldots, k_n) \in \in \bar{I}^1 \cap (S \cap T)$. Then we have

$$(1) \qquad f(\eta_1, k_2, \ldots, k_n) > \frac{f_{\eta_1}(\xi_2, \ldots, \xi_n) + f_{\zeta_1}(\xi_2, \ldots, \xi_n)}{2} > f(\zeta_1, k_2, \ldots, k_n).$$

We conclude from this relation that the set

$$(2) \qquad \left\{x_1 : \eta_1 \leqq x_1 \leqq \zeta_1; \left(\frac{\partial f}{\partial x_1}\right)_{\text{ap}} (x_1, k_2, \ldots, k_n) < 0\right\}$$

is of positive linear measure. For, if this set is of linear measure zero, then the function $\phi(x_1) = f(x_1, k_2, \ldots, k_n)$ is such that $\phi'_{\text{ap}} \geqq 0$ almost everywhere in the interval $[\eta_1, \zeta_1]$ and hence, as above, ϕ is nondecreasing in $[\eta_1, \zeta_1]$ which contradicts (1). So, the set defined in (2) is of positive linear measure for every point $(k_2, k_3 \ldots k_n)$ in $\bar{I}^1 \cap (S \cap T)$. Since $\bar{I}^1 \cap (S \cap T)$ is of positive measure in R^{n-1}, we conclude that the set

$$(3) \qquad \left\{x = (x_1 \ldots x_n); x_1 \in \bar{I}_1; (x_2 \ldots x_n) \in \bar{I}^1 \cap (S \cap T); \left(\frac{\partial f}{\partial x_1}\right)_{\text{ap}} (x) < 0\right\}$$

is of positive measure in R^n. Since the set in (3) is a subset of the set

$$(4) \qquad \bar{I} \cap E\left(\left(\frac{\partial f}{\partial x_1}\right)_{\text{ap}}; *, 0\right),$$

the set in (4) is of positive measure. This completes the proof.

It is not known whether the approximate partial derivatives of functions satisfying the hypothesis of Theorem 4 belong to Baire class 1. If this is true then it follows from Theorems 4, 1 and 3 that such approximate partial derivatives will enjoy Denjoy property and Darboux property. However, if $f : X \to R$ is such that the approximate partial derivative w.r. to x_1, $(\partial f/\partial x_1)_{\text{ap}}$ exists finitely and f_{x_1} is continuous from R^{n-1} to R for each fixed x_1 then $(\partial f/\partial x_1)_{\text{ap}}$ is of Baire class 1 [16]. Hence we get from Theorems 4, 1 and 3

Corollary. *If $f : X \to R$ is such that the approximate partial derivative w.r. to x_1, $(\partial f/\partial x_1)_{\text{ap}}$ exists and for each fixed x_1, f_{x_1} is continuous from R^{n-1} to R, then*

$$\left(\frac{\partial f}{\partial x_1}\right)_{\text{ap}} \in \mathscr{D}'(\mathscr{B}) \cap \mathscr{D}(\mathscr{B}).$$

This result is an improvement of Theorem 2 in [11].

Reference

[1] *A. M. Bruckner:* An affirmative answer to a problem of Zahorski and some consequences, Mich. Math. J. *13* (1966), 15—26.

[2] *A. M. Bruckner* and *J. B. Bruckner:* Darboux transformations, Trans. Amer. Math Soc. *128* (1967), 103—111.

[3] *J. A. Clarkson:* A property of derivatives, Bull. Amer. Math. Soc. *53* (1947), 124—125.

[4] *H. T. Croft:* A note on Darboux continuous functions, J. London Math. Soc. *38* (1963), 9—10.

[5] *A. Denjoy:* Sur une propriete des fonctions derivees, Enseignement Math. *18* (1916), 320—328.

[6] *C. Goffman* and *D. Waterman:* Approximately continuous transformations Proc. Amer. Math. Soc. *12* (1961), 116—121.

[7] *J. S. Lipiński:* Sur la classe $\mathcal{M}'_2$, Časopis Pěst. Mat. *93* (1968), 222—226.

[8] *L. Mišík:* Über die Klasse $\mathcal{M}_2$, Časopis Pěst. Mat. *91* (1966), 389—393.

[9] *L. Mišík:* Über die Funktionen der ersten Baireschen Klasse mit der Eigenschaft von Darboux, Mat. Fyz. Časopis SAV *14* (1964), 44—48.

[10] *L. Mišík:* Über die Eigenschaft von Darboux und einigen Klasse von Funktionen, Rev. Roum. Math. Pures et Appl. *11* (1966), 411—430.

[11] *S. N. Mukhopadhyay:* On a certain property of the derivative, Fund. Math. *67* (1970), 279—284.

[12] *C. J. Neugebauer:* Darboux property for functions of several variables, Trans. Amer. Math. Soc. *107* (1963), 30—37.

[13] *S. Saks:* Theory of the Integral, 1937.

[14] *T. Šalát:* Remarks on Denjoy property and $\mathcal{M}'_2$ property of real functions, Časopis Pěst. Mat. *96* (1971), 391—397.

[15] *T. Swiatkowski:* On the conditions of monotonicity of functions, Fund. Math. *59* (1966), 189—201.

[16] *L. E. Snyder:* The Baire classification of ordinary and approximate partial derivatives, Proc. Amer. Math. Soc. *17* (1966), 115—123.

[17] *Z. Zahorski:* Sur la premiere derivee, Trans. Amer. Math. Soc. *69* (1950), 1—54.

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