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TRANSLATION STRUCTURES AND GROUP PARTITIONS

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Following the papers of V. HAVEL [2], [3] we shall present in this article an example of a non-planar quasifield which describes a special translation structure in the sense of ANDRÉ [1]. The author wishes to express his gratitude to V. Havel for suggesting this research and for his valuable advice.

Definition 1. A set of non-trivial (i.e. of order > 1) proper subgroups (called components) $\boldsymbol{G}_\alpha = (G_\alpha, +)$, $\alpha \in I$ of a group $\boldsymbol{G} = (G, +)$ is said to be a *partition* if

(1) $\boldsymbol{G} = \bigcup_{\alpha \in I} G_\alpha$,

(2) $G_\alpha \cap G_\beta$ consists only of the neutral element 0 whenever α, β are distinct indices from I.

A partition $(\boldsymbol{G}_\alpha)_{\alpha \in I}$ of G is called a *τ-partition* if

(3) there are pairwise distinct indices $\alpha, \beta, \gamma \in I$ such that $\boldsymbol{G}_\alpha + \boldsymbol{G}_\nu = \boldsymbol{G}$, $\forall \nu \in I \setminus \{\alpha\}$ and $\boldsymbol{G}_\beta + \boldsymbol{G}_\gamma = \boldsymbol{G}$.

A partition $(\boldsymbol{G}_\alpha)_{\alpha \in I}$ of $\boldsymbol{G}$ is called a *π-partition* if

(4) there exist distinct indices α, $\beta \in I$ such that $\boldsymbol{G}_\alpha + \boldsymbol{G}_\nu = \boldsymbol{G}$, $\forall \nu \in I \setminus \{\alpha\}$, $\boldsymbol{G}_\beta + \boldsymbol{G}_\nu = \boldsymbol{G}$, $\forall \nu \in I \setminus \{\beta\}$.

A partition $(\boldsymbol{G}_\alpha)_{\alpha \in I}$ of $\boldsymbol{G}$ is called a *congruence* if

(5) $\boldsymbol{G}_\alpha + \boldsymbol{G}_\beta = \boldsymbol{G}$ whenever α, β are distinct indices from I.

Remark. Evidently every congruence is a π-partition, every π-partition is a τ-partition and every τ-partition is a partition.

Definition 2. By an *algebraic τ-system* we mean an algebraic system $(Q, +, \cdot)$ such that

(i) $(Q, +)$ is an Abelian group (with neutral element 0);

(ii) $\forall a, b \in Q, a \neq 0, \exists! x \in Q, x . a = b,$
$\forall a, b \in Q, a \neq 0, \exists\ y \in Q, a . y = b;$

(iii) there exists a neutral element $1 \in Q \setminus \{0\}$ for the groupoid $(Q, \cdot)$;

(iv) 0 is a multiplicating zero, i.e., $0 . a = a . 0 = 0, \forall a \in Q$;

(v) the right distributivity holds, i.e., $a(b + c) = a . b + a . c, \forall a, b, c \in Q$.

An algebraic τ-system $(Q, +, \cdot)$ is called an *algebraic π-system* (a – not necessarily planar – *quasifield*) if

(vi) $Q \setminus \{0\}$ is a loop.

An algebraic π-system $(Q, +, \cdot)$ is called a *planar quasifield* if the following *planarity condition* holds true

(vii) $\forall a, b, c \in Q, a \neq b, \exists! x \in Q, -a . x + b . x = c.$

A π-system in which (vii) is violated is called also a non-planar quasifield.

Theorem 1. *Let $\boldsymbol{Q} = (Q, +, \cdot)$ be an algebraic τ-system. Then there is a τ-partition of the group $(Q, +) \times (Q, +)$* (this partition will be denoted by $\mathscr{P}(\boldsymbol{Q})$) *such that*

(i) *$\mathscr{P}(\boldsymbol{Q})$ is a π-partition if and only if $\boldsymbol{Q}$ is a π-system,*

(ii) *$\mathscr{P}(\boldsymbol{Q})$ is a congruence if and only if $\boldsymbol{Q}$ is a planar quasifield.*

The proof is a routine; $\mathscr{P}(Q)$ is composed of $G_m = \{(x, m . x) \mid x \in Q\}$, $\forall m \in Q$ and of $G_\infty = \{(0, x), x \in Q\}$.

Theorem 2. *Let $\mathscr{P} = (G_\iota)_{\iota \in I}$ be a τ-partition of a group $\boldsymbol{G}$* (following the notation of Definition 1). *Let us choose an* (arbitrary) *element $e \in G_\alpha \setminus \{0\}$. Then there exists an algebraic τ-system $\boldsymbol{Q}_e(\mathscr{P}) = (G_\alpha, +, \cdot)$ such that*

(i) *$\boldsymbol{Q}_e(\mathscr{P})$ is an algebraic π-system if and only if $\mathscr{P}$ is a π-partition;*

(ii) *$\boldsymbol{Q}_e(\mathscr{P})$ is a quasifield if and only if $\mathscr{P}$ is a congruence.*

Cf. [3] for the proof.

If $\mathscr{P} = (G_\alpha)_{\alpha \in I}$ is a partition of a group $\boldsymbol{G}$ (the notation here as wellas in the sequel follows Definition 1) then the *associated* translation structure $\mathfrak{S}(\mathscr{P})$ is constructed as follows:

points of $\mathfrak{S}(\mathscr{P})$ are precisely the elements of G;

lines of $\mathfrak{S}(\mathscr{P})$ are precisely of the form $a + G_\nu$, $\forall a \in G$, $\nu \in I$;

parallelity $\|$ of lines is introduced by

$$a + G_\gamma \| b + G_\delta \Leftrightarrow \gamma = \delta .$$

Theorem 3. *Let $\mathscr{P}$ be a partition of a group* **G**. *Then $\mathfrak{S}(\mathscr{P})$ is a translation plane exactly if $\mathscr{P}$ is a congruence.*

The proof is well-known. Cf. e.g. [2].

Definition 3. A *τ-translation structure* is a translation structure such that

(τ) there are classes $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ of parallel lines with the following property: each line of $\mathscr{L}_1$ intersects each line which is not from $\mathscr{L}_1$ and each line of $\mathscr{L}_2$ intersects each line from $\mathscr{L}_3$.

A *π-translation structure* is a translation structure satisfying the following axiom:

(π) there exist two classes $\mathscr{L}_1 \neq \mathscr{L}_2$ of parallel lines such that each line of $\mathscr{L}_i$ intersects each line which is not from $\mathscr{L}_i$ $(i = 1, 2)$.

Theorem 4. *Let $\mathscr{P}$ be a partition of a group* **G**. *Then $\mathfrak{S}(\mathscr{P})$ is a τ-translation structure or a π-translation structure if and only if $\mathscr{P}$ is a τ-partition or a π-partition, respectively.*

The proof is a routine.

In the paper [4], E. H. Davis described π-translation structures which are coordinatized by non-planar quasifields with an associative multiplication (i.e. by non-planar nearfields) and found collineations of them. We can state

Theorem 5. *Let $\mathbf{Q} = (Q, +, \cdot)$ be a quasifield with an associative multiplication* (i.e. a nearfield). *For $\mathscr{P}(\mathbf{Q})$ let there exist pairwise distinct components $G^{(1)}$, $G^{(2)}$, $G^{(3)}$ such that for $\forall i = 1, 2, 3$, $G^{(i)} + G' = G$ for all components $G' \neq G^{(i)}$. Then $\mathbf{Q}$ is planar.*

Proof. The proof follows from that of Theorem 3.2, [4]. The condition $G^{(i)} + G' = G$ implies that each line of the form $a + G^{(i)}$ intersects each line $b + G'$ if $G' \neq G^{(i)}$, $(i = 1, 2, 3)$.

One of the lines $0 + G^{(i)}$ has the equation of the form $y = n . x + k$ with $n \neq 0$. If the point P does not lie on the line $0 + G^{(i)}$ then every line through P intersects the line $0 + G^{(i)}$ which is a contradiction with Theorem 3,2 of [4].

It seems that Theorem 5 does not hold without the assumption of associative multiplication.

Example of a non-planar non-associative and non-distributive (i.e., without the left distributivity) quasifield:

Let $K^*(x)$ be the field from [5] where K has the form $M(\theta)$ with M a subfield of K

and θ transcendental over M, η a non-trivial automorphism of K fixing θ. To each $\xi \in K^*(x) \setminus \{0\}$ there exists a positive integer n and an integer h such that ξ can be expressed in the form

$$\xi = a_h(\theta)\, x^{h/n} + a_{h+1}(\theta)\, x^{(h+1)/n} + \ldots$$

with $a_i(\theta) \in K$ for all $i \in \{h, h+1, \ldots\}$, $a_h(\theta) \neq 0$. Define $O_1(\xi)$ as $\tilde{h}$, where $\tilde{h}/\tilde{n}$ is a irreducible form of h/n (as in [5]) and $\delta(\xi)$ as the degree of the numerator of $a_h(\theta)$ Further put

$$\begin{aligned} \eta(\xi) &= \eta(a_h(\theta))\, x^{h/n} + \eta(a_{h+1}(\theta))\, x^{(h+1)/n} + \ldots, \\ \eta(0) &= 0, \\ T(\xi) &= a_h(\theta + 1)\, x^{h/n} + a_{h+1}(\theta + 1)\, x^{(h+1)/n} + \ldots, \\ T(0) &= 0. \end{aligned}$$

It holds

$$O_1(\eta(\xi)) = O_1(\xi), \quad O_1(T(\xi)) = O_1(\xi), \quad \delta(T(\xi)) = \delta(\eta(\xi)) = \delta(\xi).$$

If we define the new multiplication $\odot$ on $K^*(x)$ by

$$\begin{aligned} \xi \odot \zeta &= \xi \,.\, \eta^{O_1(\xi)}(T^{\delta(\xi)}(\zeta)) \quad &\text{whenever} \quad \zeta \neq 0 \\ 0 \odot \zeta &= 0 &\text{for all} \quad \zeta \end{aligned}$$

then $(K^*(x), +, \odot)$ is the quasifield which has all desired properties which may be proved similarly as in [4], [5].

The non-planarity is guaranteed by nonsolvability of

$$y = \theta \odot y + \theta.$$

The preceding construction originated from the connection of constructions of a planar quasifield from [5] and a non-planar nearfield from [4].

It is easy to prove that $(K^*(x), +, \odot)$ is a non-planar π-system.

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