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ON A GENERALIZED DIRECTED-DIVERGENCE FUNCTION

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1. INTRODUCTION AND DEFINITIONS

Consider three finite discrete probability distributions $P = (p_1, \ldots, p_n)$, $p_i \geqq 0$, $\sum_{i=1}^{n} p_i = 1$, $Q = (q_1, \ldots, q_n)$, $q_i \geqq 0$, $\sum_{i=1}^{n} q_i = 1$ and $R = (r_1, \ldots, r_n)$, $r_i \geqq 0$, $\sum_{r=1}^{n} r_i = 1$. For these probability distributions, we defined in an earlier paper [7] the following generalized directed-divergence of type β ($\beta \neq 1$):

$$(1) \qquad I_n^\beta \begin{pmatrix} p_1, \ldots, p_n \\ q_1, \ldots, q_n \\ r_1, \ldots, r_n \end{pmatrix} = (2^{\beta-1} - 1) \left[\sum_{i=1}^{n} p_i q_i^{\beta-1} r_i^{1-\beta} - 1 \right], \quad \beta \neq 1 .$$

For $\beta \to 1$, (1) gives

$$(2) \qquad I_n^1 \begin{pmatrix} p_1, \ldots, p_n \\ q_1, \ldots, q_n \\ r_1, \ldots, r_n \end{pmatrix} = \sum_{i=1}^{n} p_i \log (q_i/r_i) .$$

For $P \equiv Q$, (2) reduces to the well-known directed-divergence [3]

$$(3) \qquad I_n^1 \begin{pmatrix} p_1, \ldots, p_n \\ p_1, \ldots, p_n \\ r_1, \ldots, r_n \end{pmatrix} = \sum_{i=1}^{n} p_i \log (p_i/r_i) .$$

The object of this paper is to generalize further the quantities defined by (1) and (2) by introducing the concept of generalized directed-divergence and to characterize this by means of a functional equation. The generalizations are given in the following definitions:

Definition 1. A real valued function f on $I \times I \times I$ where $I = [0, 1]$, is called a generalized directed-divergence function of order α $(\neq 1)$ and type β $(\beta \neq 1)$ if f is a solution of the functional equation

$$(4)\quad f(x, y, z) + (1 - x)^{\alpha} (1 - y)^{\beta - 1} (1 - z)^{1-\beta} f\left(\frac{u}{1 - x}, \frac{v}{1 - y}, \frac{w}{1 - z}\right) =$$

$$= f(u, v, w) + (1 - u)^{\alpha} (1 - v)^{\beta - 1} (1 - w)^{1-\beta} f\left(\frac{x}{1 - u}, \frac{y}{1 - v}, \frac{z}{1 - w}\right),$$

for $x, y, z, u, v, w \in [0, 1)$ with $x + u, y + v, z + w \in I$, satisfying further

$$(5)\qquad f(0, 0, 0) = f(1, 1, 1),$$

and

$$(6)\qquad f(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = (2^{1-\alpha} - 1)(2^{\beta-1} - 1)^{-1}.$$

Definition 2. If f is a generalized directed-divergence function of order α $(\neq 1)$ and type β $(\neq 1)$ as defined above, then the generalized directed-divergence of order α $(\neq 1)$ and type β $(\neq 1)$ is defined by the expression:

$$(7)\qquad I^{\beta}_{\alpha,n}\begin{pmatrix} p_1, \ldots, p_n \\ q_1, \ldots, q_n \\ r_1, \ldots, r_n \end{pmatrix} = \sum_{i=2}^{n} P_i^{\alpha} Q_i^{\beta-1} R_i^{1-\beta} f\left(\frac{p_i}{P_i}, \frac{q_i}{Q_i}, \frac{r_i}{R_i}\right)$$

where $P_i = p_1 + \ldots + p_i$, $Q_i = q_1 + \ldots + q_i$, $R_i = r_1 + \ldots + r_i$, for $i = 1, 2, \ldots$ $\ldots, n$ with $P_n = Q_n = R_n = 1$.

In § 2 the solution of (4) and an expression for (7) are given. In § 3, some interesting special cases are pointed out.

The results (4) and (7) unify and generalize all the existing results in this direction. For example (7) reduces to (i) Shannon's entropy when $y = z$ and $\alpha = \beta \to 1$ in (4); (ii) informatic function of typed introduced in [2] for $x = y = z$ and $\beta = 1$ in (4); (iii) directed divergence [1], [5].

2. THIS SECTION DEALS WITH TWO THEOREMS WHICH GIVE f AND $I^{\beta}_{\alpha,n}$ RESPECTIVELY

Theorem 1. *If f is a solution of the functional equation* (4) *satisfying the additional conditions* (5) *and* (6), *then f is given by*

$$(8)\quad f(x, y, z) = [x^{\alpha} y^{\beta-1} z^{1-\beta} + (1 - x)^{\alpha} (1 - y)^{\beta-1} (1 - z)^{1-\beta} - 1](2^{\beta-1} - 1)^{-1},$$

for all $x, y, z \in I$ and conversely.

Remark 1. The solution f of (4) has the form (8), provided we use the notation $0^\gamma = 0$ (for $\gamma \neq 0$). Only to put f in the form given by (8), this notation is used, but nowhere in the proof of the theorem, $0^\gamma = 0$ is used.

Proof. Putting $x = 0$, $y = 0$ and $z = 0$ in (5), we get

$$[1 - (1-u)^\alpha (1-v)^{\beta-1} (1-w)^{1-\beta}] f(0,0,0) = 0\,, \quad \text{for} \quad u, v, w \in [0, 1)\,,$$

which in turn gives $f(0, 0, 0) = 0$. Hence from (4), we have

$$f(1,1,1) = f(0,0,0) = 0\,. \tag{9}$$

Replacing u, v and w in (4) by $1 - x$, $1 - y$ and $1 - z$ respectively and using (9), we obtain

$$f(x,y,z) = f(1-x,\ 1-y,\ 1-z)\,, \quad \text{for all} \quad x, y, z \in (0,1)\,. \tag{10}$$

Setting

$$p = \frac{u}{1-x}\,, \quad q = \frac{v}{1-y}\,, \quad r = \frac{w}{1-z}\,, \quad \xi = 1 - x\,, \quad \eta = 1 - y \quad \text{and} \quad \zeta = 1 - z$$

in (4), (4) can be rewritten as

$$\begin{aligned} f(1-\xi,\ 1-\eta,\ 1-\zeta) + \xi^\alpha \eta^{\beta-1} \zeta^{1-\beta} f(p,q,r) = f(p\xi, q\eta, r\zeta) + \\ + (1-p\xi)^\alpha (1-q\eta)^{\beta-1} (1-r\zeta)^{1-\beta} f\left(\frac{1-\xi}{1-p\xi}, \frac{1-\eta}{1-q\eta}, \frac{1-\zeta}{1-r\zeta}\right), \end{aligned} \tag{11}$$

for all $p, q, r \in I$, $\xi, \eta, \zeta \in (0, 1]$, such that $p\xi \neq 1$, $q\eta \neq 1$ and $r\zeta \neq 1$.

From (10) and (11), we get

$$\begin{aligned} f(\xi,\eta,\zeta) + \xi^\alpha \eta^{\beta-1} \zeta^{1-\beta} f(p,q,r) = f\,(p\xi, q\eta, r\zeta) + \\ + (1-p\xi)^\alpha (1-q\eta)^{\beta-1} (1-r\zeta)^{1-\beta} f\left(\frac{1-\xi}{1-p\xi}, \frac{1-\eta}{1-q\eta}, \frac{1-\zeta}{1-r\zeta}\right), \end{aligned} \tag{12}$$

holding for $p, q, r \in I$ and $\xi, \eta, \zeta \in (0, 1)$.

Define

$$\begin{aligned} F(p,q,r,\xi,\eta,\zeta) = \\ = f(\xi,\eta,\zeta) + [\xi^\alpha \eta^{\beta-1} \zeta^{1-\beta} + (1-\xi)^\alpha (1-\eta)^{\beta-1} (1-\zeta)^{1-\beta}] f(p,q,r)\,, \end{aligned} \tag{13}$$

where $p, q, r, \xi, \eta, \zeta \in (0, 1)$.

We shall prove that

$$F(p,q,r,\xi,\eta,\zeta) = F(\xi,\eta,\zeta,p,q,r)\,. \tag{14}$$

Now (12) and (13) yield

$$(15)\qquad F(p, q, r, \xi, \eta, \zeta) = f(p\xi, q\eta, r\zeta) + (1 - p\xi)^{\alpha}(1 - q\eta)^{\beta-1}(1 - r\zeta)^{1-\beta}\,.$$
$$\cdot\left\{f\left(\frac{1-\xi}{1-p\xi}, \frac{1-\eta}{1-q\eta}, \frac{1-\zeta}{1-r\zeta}\right) + \left(\frac{1-\xi}{1-p\xi}\right)^{\alpha}\left(\frac{1-\eta}{1-q\eta}\right)^{\beta-1}\left(\frac{1-\zeta}{1-r\zeta}\right)^{1-\beta} f(p, q, r)\right\},$$

for all $p, q, r, \xi, \eta, \zeta \in (0,1)$. Allowing

$$A = \frac{1-\xi}{1-p\xi}, \quad B = \frac{1-\eta}{1-q\eta} \quad \text{and} \quad C = \frac{1-\zeta}{1-r\zeta}$$

and making use of (10) and (12), we have

(16)

$$f\left(\frac{1-\xi}{1-p\xi}, \frac{1-\eta}{1-q\eta}, \frac{1-\zeta}{1-r\zeta}\right) + \left(\frac{1-\xi}{1-p\xi}\right)^{\alpha}\left(\frac{1-\eta}{1-p\eta}\right)^{\beta-1}\left(\frac{1-\zeta}{1-r\zeta}\right)^{1-\beta} f(p, q, r) =$$
$$= f(A, B, C) + A^{\alpha}B^{\beta-1}C^{1-\beta}f(p, q, r) = f(pA, qB, rC) +$$
$$+ (1 - pA)^{\alpha}(1 - qB)^{\beta-1}(1 - rC)^{1-\beta} f\left(\frac{1-A}{1-pA}, \frac{1-B}{1-qB}, \frac{1-C}{1-rC}\right) =$$
$$= f(1 - pA, 1 - qB, 1 - rC) + (1 - pA)^{\alpha}(1 - qB)^{\beta-1}(1 - rC)^{1-\beta}\,.$$
$$\cdot f\left(\frac{1-A}{1-pA}, \frac{1-B}{1-qB}, \frac{1-C}{1-rC}\right) = f\left(\frac{1-p}{1-p\xi}, \frac{1-q}{1-q\eta}, \frac{1-r}{1-r\zeta}\right) +$$
$$+ \left(\frac{1-p}{1-p\xi}\right)^{\alpha}\left(\frac{1-q}{1-q\eta}\right)^{\beta-1}\left(\frac{1-r}{1-rC}\right)^{1-\beta} f(\xi, \eta, \zeta)\,.$$

The equation (15) and (16) indeed go to show that (14) is true for all $p, q, r, \xi, \eta, \zeta \in (0, 1)$. Thus from (13) and (14) result

$$(17)\quad f(p, q, r) + [p^{\alpha}q^{\beta-1}r^{1-\beta} + (1 - p)^{\alpha}(1 - q)^{\beta-1}(1 - r)^{1-\beta}] f(\xi, \eta, \zeta) =$$
$$= f(\xi, \eta, \zeta) + [\xi^{\alpha}\eta^{\beta-1}\zeta^{1-\beta} + (1 - \xi)^{\alpha}(1 - \eta)^{\beta-1}(1 - \zeta)^{1-\beta}] f(p, q, r),$$

where $\xi, \eta, \zeta, p, q, r \in (0, 1)$.

Now (17) with $\xi = \frac{1}{2}$, $\eta = \frac{1}{2}$ and $\zeta = \frac{1}{2}$ and (6) give

$$(18)\qquad f(p, q, r) =$$
$$= [p^{\alpha}q^{\beta-1}r^{1-\beta} + (1 - p)^{\alpha}(1 - q)^{\beta-1}(1 - r)^{1-\beta} - 1](2^{\beta-1} - 1)^{-1},$$

for all $p, q, r \in (0, 1)$.

Next, we will show that (18) remains true for all $p, q, r \in I$, that is, (8) is valid.

The equation (11) with $\xi = 1, \eta = 1$ and $\zeta = \frac{1}{2}$ gives,

$$(19) \qquad f(0, 0, \tfrac{1}{2}) + 2^{\beta-1} f(p, q, r) = \\ = f\left(p, q, \frac{r}{2}\right) + (1-p)^{\alpha}(1-q)^{\beta-1}\left(1-\frac{r}{2}\right)^{1-\beta} f\left(0, 0, \frac{1}{2-r}\right),$$

for all $p, q \in [0, 1)$ and $r \in I$.

For $r = \frac{1}{2}$, (19) becomes,

$$(20) \qquad f(0, 0, \tfrac{1}{2}) + 2^{\beta-1} f(p, q, \tfrac{1}{2}) = \\ = f(p, q, \tfrac{1}{4}) + (1-p)^{\alpha}(1-q)^{\beta-1}(\tfrac{3}{4})^{1-\beta} f(0, 0, \tfrac{2}{3})$$

where $p, q \in [0, 1)$.

Taking $p = \frac{1}{2}$ and $q = \frac{1}{2}$ in (20) and using (6), we have

$$(21) \qquad f(0, 0, \tfrac{1}{2}) + 2^{\beta-1}(2^{1-\alpha} - 1)(2^{\beta-1} - 1)^{-1} = \\ = f(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}) + 2^{1-\alpha-\beta}(\tfrac{3}{4})^{1-\beta} f(0, 0, \tfrac{2}{3}).$$

From (18), (20) for $p, q \in (0,1)$ and (21), result

$$(22) \qquad f(0, 0, \tfrac{1}{2}) = 1.$$

Now (22) in (19) with $r = 0$ gives,

$$(23) \quad f(p, q, 0) = [(1-p)^{\alpha}(1-q)^{\beta-1} - 1](2^{\beta-1} - 1)^{-1}, \quad \text{for all} \quad p, q \in [0, 1).$$

Taking $\xi = 1$, $q = 0$ and $r = 0$ in (11), (11) becomes,

$$(24) \qquad f(0, 1-\eta, 1-\zeta) + \eta^{\beta-1}\zeta^{1-\beta} f(p, 0, 0) = \\ = f(p, 0, 0) + (1-p)^{\alpha} f(0, 1-\eta, 1-\zeta),$$

where $p \in [0, 1)$, and $\eta, \zeta \in (0, 1]$.

In view of (23), (24) gives

$$(25) \qquad f(0, 1-\eta, 1-\zeta) = (\eta^{\beta-1}\zeta^{1-\beta} - 1)(2^{\beta-1} - 1)^{-1}, \quad \text{for} \quad \eta, \zeta \in (0, 1].$$

Letting $p = 1, \eta = 1$ and $\zeta = 1$ in (11), we get

$$(26) \qquad f(1-\xi, 0, 0) + \xi^{\alpha} f(1, q, r) = \\ = f(\xi, q, r) + (1-\xi)^{\alpha}(1-q)^{\beta-1}(1-r)^{1-\beta} f(1, 0, 0),$$

for $\xi \in (0, 1)$, $q, r \in [0, 1)$.

By (23) and (26) with $q = 0$ and $r = 0$, we get

$$f(1, 0, 0) = (1 - 2^{\beta - 1})^{-1} . \tag{27}$$

Again from (26), using (18), (23) and (27), we obtain

$$f(1, q, r) = (q^{\beta - 1} r^{1 - \beta} - 1)(2^{\beta - 1} - 1)^{-1} , \quad \text{for} \quad q, r \in (0, 1) . \tag{28}$$

$q = 0$, $\xi = 1$ and $\zeta = 1$ in (11) give,

$$\begin{aligned} f(0, 1 - \eta, 0) + \eta^{\beta - 1} f(p, 0, r) = \\ = f(p, 0, r) + (1 - p)^{\alpha} (1 - r)^{1 - \beta} f(0, 1 - \eta, 0) , \end{aligned} \tag{29}$$

where $\eta \in (0, 1]$ and $p, r \in [0, 1)$.

With the help of (25) and (29), we have

$$f(p, 0, r) = [(1 - p)^{\alpha} (1 - r)^{1 - \beta} - 1](2^{\beta - 1} - 1)^{-1} , \quad \text{for} \quad p, r \in [0, 1) . \tag{30}$$

For $q = 1$, $\xi = 1$ and $\zeta = 1$, (11) gives

$$\begin{aligned} f(0, 1 - \eta, 0) + \eta^{\beta - 1} f(p, 1, r) = \\ = f(p, \eta, r) + (1 - p)^{\alpha} (1 - \eta)^{\beta - 1} (1 - r)^{1 - \beta} f(0, 1, 0) , \end{aligned} \tag{31}$$

for $\eta \in (0, 1)$ and $p, r \in [0, 1)$.

From (25) and (31) with $p = 0$, $r = 0$, we get

$$f(0, 1, 0) = (1 - 2^{\beta - 1})^{-1} . \tag{32}$$

Now (31), (32), (25) and (18) yield

$$f(p, 1, r) = (p^{\alpha} r^{1 - \beta} - 1)(2^{\beta - 1} - 1)^{-1} , \quad \text{for} \quad p, r \in (0, 1) . \tag{33}$$

With $p = 1$, (11) becomes

$$\begin{aligned} f(1 - \xi, 1 - \eta, 1 - \zeta) + \xi^{\alpha} \eta^{\beta - 1} \zeta^{1 - \beta} f(1, q, r) = \\ = f(\xi, q\eta, r\zeta) + (1 - \xi)^{\alpha} (1 - q\eta)^{\beta - 1} (1 - r\zeta)^{1 - \beta} f\left(1, \frac{1 - \eta}{1 - q\eta}, \frac{1 - \zeta}{1 - r\zeta}\right) , \end{aligned} \tag{34}$$

where $q, r \in I$, $\xi \in (0, 1)$, $\eta, \zeta \in (0, 1]$ such that $q\eta \neq 1$ and $r\zeta \neq 1$.

Now $r = 1$ and $\eta = 1$ in (34) imply

$$\begin{aligned} f(1 - \xi, 0, 1 - \zeta) + \xi^{\alpha} \zeta^{1 - \beta} f(1, q, 1) = \\ = f(\xi, q, \zeta) + (1 - \xi)^{\alpha} (1 - q)^{\beta - 1} (1 - \xi)^{1 - \beta} f(1, 0, 1) , \end{aligned} \tag{35}$$

for $q \in [0, 1)$, $\xi, \zeta \in (0, 1)$.

Consequently (30) and (35) with $q = 0$, give

$$f(1, 0, 1) = (1 - 2^{\beta-1})^{-1}. \tag{36}$$

Employing (30), (18), (35) and (36), we obtain

$$f(1, q, 1) = (q^{\beta-1} - 1)(2^{\beta-1} - 1)^{-1}, \quad \text{for} \quad q \in [0, 1). \tag{37}$$

Taking $\eta = 1$ and $r = 0$, (34) gives

$$\begin{aligned} f(1 - \xi, 0, 1 - \zeta) + \xi^{\alpha}\zeta^{1-\beta} f(1, q, 0) = \\ = f(\xi, q, 0) + (1 - \xi)^{\alpha}(1 - q)^{\beta-1} f(1, 0, 1 - \zeta), \end{aligned} \tag{38}$$

for $\xi \in (0, 1)$, $\zeta \in (0, 1]$ and $q \in [0, 1)$.

In view of (23), (27) and (38) with $\zeta = 1$, we get

$$f(1, q, 0) = (1 - 2^{\beta-1})^{-1}, \quad \text{for} \quad q \in [0, 1). \tag{39}$$

Again from (23), (30), (38) and (39), we obtain

$$f(1, 0, 1 - \zeta) = (1 - 2^{\beta-1})^{-1}, \quad \text{for} \quad \zeta \in (0, 1]. \tag{40}$$

Putting $q \approx 1$, (34) gives

$$\begin{aligned} f(1 - \xi,\ 1 - \eta,\ 1 - \zeta) + \xi^{\alpha}\eta^{\beta-1}\zeta^{1-\beta} f(1, 1, r) = \\ = f(\xi, \eta, r\zeta) + (1 - \xi)^{\alpha}(1 - \eta)^{\beta-1}(1 - r\zeta)^{1-\beta} f\left(1, 1, \frac{1 - \zeta}{1 - r\zeta}\right), \end{aligned} \tag{41}$$

where $\xi, \eta \in (0, 1)$, $\zeta \in (0, 1]$, $r \in I$ with $\zeta r \neq 1$.

Combining (23) and (41) with $\zeta = 1$ and $r = 0$, we have

$$f(1, 1, 0) = (1 - 2^{\beta-1})^{-1}. \tag{42}$$

Again from (23), (42), (18) and (41) with $\zeta = 1$, we get

$$f(1, 1, r) = (2^{\beta-1} - 1)^{-1}(r^{1-\beta} - 1), \quad \text{for} \quad r \in (0, 1). \tag{43}$$

Taking $r \approx 1$, (11) becomes

$$\begin{aligned} f(1 - \xi,\ 1 - \eta,\ 1 - \zeta) + \xi^{\alpha}\eta^{\beta-1}\zeta^{1-\beta} f(p, q, 1) = \\ = f(p\xi, q\eta, \zeta) + (1 - p\xi)^{\alpha}(1 - q\eta)^{\beta-1}(1 - \zeta)^{1-\beta} f\left(\frac{1 - \xi}{1 - p\xi}, \frac{1 - \eta}{1 - q\eta}, 1\right), \end{aligned} \tag{44}$$

for $\xi, \eta \in (0, 1]$, $\zeta \in (0, 1)$, $p, q \in I$ such that $p\xi \neq 1$ and $q\eta \neq 1$.

Letting $\eta = 1$ and $\xi = 1$ in (44), we have

$$f(0, 0, 1 - \zeta) + \zeta^{1-\beta} f(p, q, 1) = \\ = f(p, q, \zeta) + (1 - p)^{\alpha} (1 - q)^{\beta-1} (1 - \zeta)^{1-\beta} f(0, 0, 1), \tag{45}$$

where $p, q \in [0, 1)$ and $\zeta \in (0, 1)$.

From (25) with $\eta = 1$ and (45) with $p = q = 0$, we have

$$f(0, 0, 1) = (1 - 2^{\beta-1})^{-1}. \tag{46}$$

With the help of (22), (46) and (18), (19) with $r = 1$ and $p, q \in (0, 1)$ gives

$$f(p, q, 1) = (p^{\alpha} q^{\beta-1} - 1)(2^{\beta-1} - 1)^{-1}, \quad \text{where} \quad p, q \in (0, 1). \tag{47}$$

Again from (45) with $q = 0$, (30) and (46), we get

$$f(p, 0, 1) = (1 - 2^{\beta-1})^{-1}, \quad \text{for} \quad p \in (0, 1). \tag{48}$$

Setting $p = 0$ and $\eta = 1$ in (44), we have

$$f(1 - \xi, 0, 1 - \zeta) + \xi^{\alpha} \zeta^{1-\beta} f(0, q, 1) = \\ = f(0, q, \zeta) + (1 - q)^{\beta-1} (1 - \zeta)^{1-\beta} f(1 - \xi, 0, 1)$$

for $\xi \in (0, 1]$, $\zeta \in (0, 1)$, $q \in [0, 1)$, which with the help of (30), (25) and (48) gives

$$f(0, q, 1) = (1 - 2^{\beta-1})^{-1}, \quad \text{for} \quad q \in (0, 1). \tag{49}$$

For $q = 1$ (44) becomes

$$f(1 - \xi, 1 - \eta, 1 - \zeta) + \xi^{\alpha} \eta^{\beta-1} \zeta^{1-\beta} f(p, 1, 1) = \\ = f(p\xi, \eta, \zeta) + (1 - p\xi)^{\alpha} (1 - \eta)^{\beta-1} (1 - \zeta)^{1-\beta} f\left(\frac{1 - \xi}{1 - p\xi}, 1, 1\right), \tag{50}$$

where $\xi \in (0, 1]$, $\eta, \zeta \in (0, 1)$ and $p \in I$ such that $p\xi \neq 1$.

$\xi = 1$, $p = 0$ in (50) together with (25) gives

$$f(0, 1, 1) = (1 - 2^{\beta-1})^{-1}. \tag{51}$$

Utilizing (46) with $\xi = 1$, (47), (14) and (25) we have

$$f(p, 1, 1) = (p^{\alpha} - 1)(2^{\beta-1} - 1)^{-1}, \quad \text{for} \quad p \in (0, 1). \tag{52}$$

For $q = 1$ (11) gives

$$(53) \qquad f(1 - \xi, 1 - \eta, 1 - \zeta) + \xi^{\alpha}\eta^{\beta-1}\zeta^{1-\beta} f(p, 1, r) =$$

$$= f(p\xi, \eta, r\zeta) + (1 - p\xi)^{\alpha} (1 - \eta)^{\beta-1} (1 - r\zeta)^{1-\beta} f\left(\frac{1 - \xi}{1 - p\xi}, 1, \frac{1 - \zeta}{1 - r\zeta}\right),$$

for $\xi, \zeta \in (0, 1]$, $\eta \in (0, 1)$, $p, r \in I$ such that $p\xi \neq 1$ and $r\zeta \neq 1$.

Using (53) with $\xi = 1$, $\zeta = 1$ and $r = 0$, (32), (25) and (23), we get

$$(54) \qquad f(p, 1, 0) = (1 - 2^{\beta-1})^{-1}, \quad \text{for} \quad p \in (0, 1).$$

Finally from (53) with $p = 0$, $\xi = 1$ and $\zeta = 1$, (23), (25) and (32) we have

$$(55) \qquad f(0, 1, r) = (1 - 2^{\beta-1})^{-1}, \quad \text{for} \quad r \in (0, 1).$$

Combining (5), (23), (25), (27), (28), (30), (32), (33), (36), (37), (39), (40), (42), (43), (46), (47), (48), (49), (51), (52), (54) and (55), we conclude that f has the form given by (8) (with the notation $0^{\alpha} = 0$, $\alpha \neq 0$) and that (10) is true for all $x, y, z \in I$.

The converse can be easily verified by straight forward calculation.

Theorem 2. *The generalized directed-divergence of order* α $(\neq 1)$ *and type* β $(\neq 1)$ *is given by*

$$(56) \qquad I^{\beta}_{\alpha,n}\begin{pmatrix} p_1, \dots, p_n \\ q_1, \dots, q_n \\ r_1, \dots, r_n \end{pmatrix} = (2^{\beta-1} - 1)^{-1} \left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta-1} r_i^{1-\beta} - 1\right].$$

Proof. Equations (7) and (8) give,

$$I^{\beta}_{\alpha,n}\begin{pmatrix} p_1, \dots, p_n \\ q_1, \dots, q_n \\ r_1, \dots, r_n \end{pmatrix} =$$

$$= (2^{\beta-1} - 1)^{-1} \sum_{i=2}^{n} \left[p_i^{\alpha} q_i^{\beta-1} r_i^{1-\beta} + P_{i-1}^{\alpha} Q_{i-1}^{\beta-1} R_{i-1}^{1-\beta} - P_i^{\alpha} Q_i^{\beta-1} R_i^{1-\beta}\right] =$$

$$= (2^{\beta-1} - 1)^{-1} \left[\sum_{i=2}^{n} p_i^{\alpha} q_i^{\beta-1} r_i^{1-\beta} + P_1^{\alpha} Q_1^{\beta-1} R_1^{1-\beta} - P_n^{\alpha} Q_n^{\beta-1} R_n^{1-\beta}\right] =$$

$$= (2^{\beta-1} - 1)^{-1} \left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta-1} r_i^{1-\beta} - 1\right].$$

This proves theorem 2.

3. SPECIAL CASES

An interesting special case of (56) is obtained for $\alpha = \beta$ and $Q \equiv R$. In this case, (56) reduces to the entropy of type β,

$$I(P) = (2^{\beta-1} - 1)^{-1} \left(\sum_{i=1}^{n} p_i^\beta - 1\right), \quad \beta \neq 1\,. \tag{58}$$

which was discussed in [2], [3] and [8] respectively.

Another interesting special case of (56) is obtained by replacing β by $2 - \beta$ and then taking $\alpha = \beta$, $P \equiv Q$; and is given by

$$I_{\beta,n}^{2-\beta}\begin{pmatrix} p_1, \ldots, p_n \\ p_1, \ldots, p_n \\ r_1, \ldots, r_n \end{pmatrix} = (2^{1-\beta} - 1)^{-1} \left(\sum_{i=1}^{n} p_i r_i^{\beta-1} - 1\right), \quad \beta \neq 1\,. \tag{59}$$

The quantity given in (59) is the inaccuracy of type β discussed earlier by us in [6].

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