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THE HYPERSPACE OF LOWER SEMICONTINUITY AND THE FIRST POWER OF A TOPOLOGICAL SPACE

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The hyperspace $H_-(E, \tau)$ of lower semicontinuity of a topological space (E, τ) introduced by Z. FROLÍK and M. KATĚTOV in [1] (p. 623) is a subspace of the (first) power $(\mathfrak{P}E, \mathfrak{P}\tau)$ of (E, τ) ("Potenzraum von (E, τ) bezüglich des Limesoperators") introduced by the author in [4] (p. 107) and further discussed in [7] (p. 245). In Section 2, we present this fact (mentioned without proof in Remark 1 of the paper [10] (p. 39)) within a framework of "finitely additive quasitopologies" (see Remark 1 after Proposition 5). Having available the terminology introduced in Section 1, we also make some remarks (in Section 3) on the product of finitely additive quasitopologies (which remarks are useful (see Remark 2) for the generalization of Proposition 8 in [10] (p. 41)). Since the auxiliary considerations for the proof of Proposition 1 will not be needed elsewhere in the present paper, we postpone this proof to Section 4.

1. TERMINOLOGY: FINITELY ADDITIVE QUASITOPOLOGIES, LIMIT OPERATORS, NEIGHBORHOOD OPERATORS, ETC.

Let M be a set. Each filter on a subset of M is called a filter *in M*. A *filtered family in M* is, by definition, an ordered triple $(f, I, \mathfrak{a})$ consisting of a nonempty mapping f into M, its domain I and a filter $\mathfrak{a}$ on I. We denote by $\mathfrak{P}M$, $\Phi_0 M$ and ΦM the class of all subsets of M, the class of all filters in M and the class of all filtered families in M. Filtered families $(f, I, \mathfrak{a})$ are also written in the form $(f(i))_{i\in I,\mathfrak{a}}$. Filtered families in M of the form $(\mathrm{id}_I, I, \mathfrak{a})$ with the identical mapping id_I on I are identified with the filters $\mathfrak{a}$; under this agreement, we have $\Phi_0 M \subseteqq \Phi M$. A set $\mathfrak{a} \subseteqq \mathfrak{P}M$ is called a *quasifilter on M* if and only if $\mathfrak{a}$ is a filter on M or $\mathfrak{a} = \mathfrak{P}M$. Given a set $\mathfrak{a} \subseteqq \mathfrak{P}M$, $\mathscr{H}_M\mathfrak{a}$ denotes (as in [2], p. 321) the set $\{K \mid K \subseteqq M$ and $A \subseteqq K$ for some $A \in \mathfrak{a}\}$ and $\mathscr{G}_M\mathfrak{a}$ denotes (as in [2], p. 322) the set $\{K \mid K \subseteqq M$ and $K \cap A \neq \emptyset$ for all $A \in \mathfrak{a}\}$, while $\mathscr{G}\mathfrak{a}$ stands for $\mathscr{G}_K\mathfrak{a}$ with $K = \bigcup\mathfrak{a}$.

Let E be a set. For each mapping τ on $\mathfrak{P}E$ into $\mathfrak{P}E$ (such a mapping will be called a *quasitopology of E*), we define the statements (τ 1) through (τ 5) by the following lines:

(τ 1) $\tau\emptyset = \emptyset$;

(τ 2) $\tau(A \cup B) = \tau A \cup \tau B$ *for all* $A, B \in \mathfrak{P}E$;

(τ 3) $\tau \bigcup_{A \in \mathfrak{a}} A = \bigcup_{A \in \mathfrak{a}} \tau A$ *for each finite set* $\mathfrak{a} \subseteqq \mathfrak{P}E$;

(τ 4) $A \subseteqq \tau A$ *for all* $A \in \mathfrak{P}E$;

(τ 5) $\tau\tau A \subseteqq \tau A$ *for all* $A \in \mathfrak{P}E$.

For each mapping Lim on $\Phi_0 E$ into $\mathfrak{P}E$, we define the mapping Lim′ by

$$\mathrm{Lim}'\,(f, I, \mathfrak{a}) = \mathrm{Lim}\, f\mathfrak{a}$$

for all $(f, I, \mathfrak{a}) \in \Phi E$ and the statements (Lim 1) through (Lim 3) by the next lines:

(Lim 1) $\mathrm{Lim}\,\mathfrak{a} = \bigcap_{C \in \mathscr{G}\mathfrak{a}} \bigcup_{\mathfrak{b} \in \Phi_0 C} \mathrm{Lim}\,\mathfrak{b}$ *for all* $\mathfrak{a} \in \Phi_0 E$;

(Lim 2) $x \in \mathrm{Lim}\,\{\{x\}\}$ *for all* $x \in E$;

(Lim 3) *If* $(f, I, \mathfrak{a}) \in \Phi E$ *and* $i \to (g_i, K_i, \mathfrak{b}_i)$ $(i \in I)$ *is a mapping on* I *into* ΦE *such that*

$$f(i) \in \mathrm{Lim}'\,(g_i, K_i, \mathfrak{b}_i) \quad \textit{for all} \quad i \in I\,,$$

then

$$\mathrm{Lim}'\,(f, I, \mathfrak{a}) \subseteqq \mathrm{Lim}'\,(\mathop{\mathrm{S}}_{i \in I} g_i,\ \mathop{\mathrm{S}}_{i \in I} K_i,\ {}^{\mathfrak{a}}\!\mathop{\mathrm{S}}_{i \in I} \mathfrak{b}_i)$$

(for the terminology, see [3], p. 396, and [2], pp. 325, 330). For each mapping $\mathfrak{V}$ on E into $\mathfrak{P}\mathfrak{P}E$, we define the mapping $\mathrm{Int}_{\mathfrak{V}}$ by

$$\mathrm{Int}_{\mathfrak{V}}\, X = \{y \mid y \in E \textit{ and } X \in \mathfrak{V}y\}$$

for all $X \in \mathfrak{P}E$ and the statements ($\mathfrak{V}$ 1) through ($\mathfrak{V}$ 3) by the following lines:

($\mathfrak{V}$ 1) $\mathfrak{V}x$ *is a quasifilter on* E *for all* $x \in E$;

($\mathfrak{V}$ 2) "*if* $V \in \mathfrak{V}x$, *then* $x \in V$" *for all* $x \in E$;

($\mathfrak{V}$ 3) $\mathrm{Int}_{\mathfrak{V}}\, X \subseteqq \mathrm{Int}_{\mathfrak{V}}\, \mathrm{Int}_{\mathfrak{V}}\, X$ *for all* $X \in \mathfrak{P}E$.

Clearly, "(τ 1) and (τ 2)" *holds if and only if* (τ 3).

We define $\mathscr{T}E$ to be the class of all mappings τ on $\mathfrak{P}E$ into $\mathfrak{P}E$ such that (τ 3) holds, $\mathscr{L}E$ to be the class of all mappings Lim on $\Phi_0 E$ into $\mathfrak{P}E$ such that (Lim 1) holds, $\mathscr{V}E$ to be the class of all mappings $\mathfrak{V}$ on E into $\mathfrak{P}\mathfrak{P}E$ such that ($\mathfrak{V}$ 1) holds. The elements of $\mathscr{T}E$, $\mathscr{L}E$, $\mathscr{V}E$ are called *finitely additive quasitopologies, limit operators, neighborhood operators* (respectively) *of E*. Given a $\tau \in \mathscr{T}E$, τ is called a *pretopology* (a *topology*) of E if and only if (τ 4) ((τ 4) and (τ 5)) holds (hold).

We define the mappings $\tau \to \mathrm{Lim}_\tau$ $(\tau \in \mathscr{T}E)$, $\mathrm{Lim} \to \tau_{\mathrm{Lim}}$ $(\mathrm{Lim} \in \mathscr{L}E)$, $\mathrm{Lim} \to \mathfrak{V}_{\mathrm{Lim}}$ $(\mathrm{Lim} \in \mathscr{L}E)$, $\mathfrak{V} \to \mathrm{Lim}_{\mathfrak{V}}$ $(\mathfrak{V} \in \mathscr{V}E)$, $\mathfrak{V} \to \tau_{\mathfrak{V}}(\mathfrak{V} \in \mathscr{V}E)$, $\tau \to \mathfrak{V}_\tau$ $(\tau \in \mathscr{T}E)$ by the following six lines (for each set M, $\bigcap_M$ denotes the intersection symbol related to the base set M, e.g. $\bigcap_M \emptyset = M$):

$$\mathrm{Lim}_\tau\, \mathfrak{a} = \bigcap_{C \in \mathscr{G}\mathfrak{a}} \tau C \quad \textit{for all} \quad \mathfrak{a} \in \Phi_0 E;$$

$$\tau_{\mathrm{Lim}} X = \bigcup_{\mathfrak{a} \in \Phi_0 X} \mathrm{Lim}\, \mathfrak{a} \quad \textit{for all} \quad X \in \mathfrak{P}E;$$

$$\mathfrak{V}_{\mathrm{Lim}} x = \bigcap\nolimits_{\mathfrak{P}E} \{\mathscr{H}_E \mathfrak{a} \mid \mathfrak{a} \in \Phi_0 E \textit{ and } x \in \mathrm{Lim}\, \mathfrak{a}\} \quad \textit{for all} \quad x \in E;$$

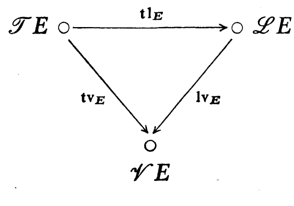
$$\text{“}x \in \mathrm{Lim}_{\mathfrak{V}}\, \mathfrak{a} \textit{ if and only if } \mathfrak{V}x \subseteqq \mathscr{H}_E \mathfrak{a}\text{”} \textit{ for all } x \in E \textit{ and all } \mathfrak{a} \in \Phi_0 E;$$

$$\text{“}x \in \tau_{\mathscr{V}} Y \textit{ if and only if } Y \in \mathscr{G}_E \mathfrak{V}x\text{”} \textit{ for all } x \in E \textit{ and all } Y \in \mathfrak{P}E;$$

$$\mathfrak{V}_\tau x = \mathscr{G}_E \{Y \mid x \in \tau Y \textit{ and } Y \in \mathfrak{P}E\} \textit{ for all } x \in E.$$

We denote these six mappings by the symbols tl_E, lt_E, lv_E, vl_E, vt_E, tv_E. Let $\mathscr{S}$ be the class of all sets. The symbols $\mathscr{T}$, $\mathscr{L}$, $\mathscr{V}$ denote the mappings $M \to \mathscr{T}M$, $M \to \mathscr{L}M$, $M \to \mathscr{V}M$ $(M \in \mathscr{S})$, the symbols tl, lt, lv, vl, vt, tv the mappings $M \to \mathrm{tl}_M, \ldots, M \to \mathrm{tv}_M$ $(M \in \mathscr{S})$. For each $M \in \mathscr{S}$ and each $\mathrm{Lim} \in \mathscr{L}M$, we abbreviate $\mathrm{Lim}'\,(f, I, \mathfrak{a})$, for each $(f, I, \mathfrak{a}) \in \Phi M$, by $\mathrm{Lim}\,(f, I, \mathfrak{a})$, but we still distinguish between Lim′ and Lim as mappings.

Proposition 1. *The mappings* tl_E, lt_E, lv_E, vl_E, vt_E, tv_E *are one-to-one onto* $\mathscr{L}E$, $\mathscr{T}E$, $\mathscr{V}E$, $\mathscr{L}E$, $\mathscr{T}E$, $\mathscr{V}E$ *(respectively), one has* $\mathrm{lt}_E = \mathrm{tl}_E^{-1}$, $\mathrm{vl}_E = \mathrm{lv}_E^{-1}$, $\mathrm{tv}_E = \mathrm{vt}_E^{-1}$, *and the diagram*



is commutative.

Proof in Section 4.

Proposition 2. *Under the mappings* tl_E *and* tv_E, *the class of all pretopologies of E corresponds to the class* $\{\mathrm{Lim} \mid \mathrm{Lim} \in \mathscr{L}E$ *and* (Lim 2)$\}$ *and to the class* $\{\mathfrak{V} \mid \mathfrak{V} \in \mathscr{V}E$ *and* ($\mathfrak{V}$ 2)$\}$. *Under the mappings* tl_E *and* tv_E, *the class of all topologies of E corresponds to the class* $\{\mathrm{Lim} \mid \mathrm{Lim} \in \mathscr{L}E$ *and* (Lim 2) *and* (Lim 3)$\}$ *and to the class* $\{\mathfrak{V} \mid \mathfrak{V} \in \mathscr{V}E$ *and* ($\mathfrak{V}$ 2) *and* ($\mathfrak{V}$ 3)$\}$.

Proof. [4], §§ 3–4; [8], p. 159, “Satz 4”; [3], § 1. □

The classical method of generating neighborhood operators by means of local subbases of them (see, e.g., ČECH [1], p. 242) reflects, within the present general framework, in the following construction: Let $\mathfrak{S}$ be a mapping on the set E into $\mathfrak{P}\mathfrak{P}E$. Define the mapping $\mathfrak{W}$ by

$$\mathfrak{W}x = \mathscr{H}_E\{\bigcap_E \mathfrak{r} \mid \mathfrak{r} \text{ is a finite subset of } \mathfrak{S}x\}$$

for all $x \in E$. Then, $\mathfrak{W} \in \mathscr{V}E$, and we call $\mathfrak{W}$ the *neighborhood operator of E generated by* $\mathfrak{S}$.

Assume, for the remainder of this paper except for Section 4, that $\tau \in \mathscr{T}E$ be given and the limit operator Lim and the neighborhood operator $\mathfrak{V}$ be defined by Lim $=$ $= \mathrm{tl}_E\tau$ and $\mathfrak{V} = \mathrm{tv}_E\tau$ (use of Proposition 1). We remark that, *for each* $(f, I, \mathfrak{a}) \in \Phi E$, $\mathrm{Lim}\,(f, I, \mathfrak{a}) = \bigcap_{C \in \mathscr{G}\mathfrak{a}} \tau f(C)$ *and, for each* $x \in E$, $x \in \mathrm{Lim}\,(f, I, \mathfrak{a})$ *holds if and only if, for each* $V \in \mathfrak{V}x$, $f(i) \in V$ *holds for* $\mathfrak{a}$*-almost all* $i \in I$ (i.e. for all elements i of some set $A \in \mathfrak{a}$).

Let $M \subseteqq E$. Then, the *trace* τ_M *of* τ *in* M is defined by $\tau_M(X) = M \cap (\tau X)$ for all $X \in \mathfrak{P}M$, and one has $\tau_M \in \mathscr{T}M$. Define the limit operator Lim_M by $\mathrm{Lim}_M = \mathrm{tl}_M\,\tau_M$ (use of Proposition 1); then $\mathrm{Lim}_M\,(f, I, \mathfrak{a}) = M \cap \mathrm{Lim}\,(f, I, \mathfrak{a})$ holds for all $(f, I, \mathfrak{a}) \in$ $\in \Phi M$. (Proof. $\mathrm{Lim}_M\,(f, I, \mathfrak{a}) = \bigcap_{C \in \mathscr{G}\mathfrak{a}} \tau_M f(C) = \bigcap_{C \in \mathscr{G}\mathfrak{a}} M \cap (\tau f(C)) = M \cap \bigcap_{C \in \mathscr{G}\mathfrak{a}} \tau f(C) =$ $= M \cap \mathrm{Lim}\,(f, I, \mathfrak{a})$. □ (Cf. also [6], p. 316, "Satz 5", and [8], p. 159, "Satz 4".))

One defines the mapping $\liminf_\tau$, being called the *limit inferior induced by* τ, by $\liminf_\tau\,(f, I, \mathfrak{a}) = \bigcap_{C \in \mathscr{G}\mathfrak{a}} \tau \bigcup_{i \in C} f(i)$ for all $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}E)$. Then, for each $x \in E$, $x \in \liminf_\tau\,(f, I, \mathfrak{a})$ *holds if and only if, for each* $V \in \mathfrak{V}x$, $V \cap f(i) \neq \emptyset$ *holds for* $\mathfrak{a}$*-almost all* $i \in I$. It turns out (see [5], p. 98, "Satz 1", and [8], p. 159, "Satz 4") that the composition

$$L := \mathfrak{P} \circ (\liminf_\tau)_{\Phi_0(\mathfrak{P}E)}$$

of the restriction of $\liminf_\tau$ to the class $\Phi_0(\mathfrak{P}E)$ with the mapping $\mathfrak{P}$ assigning to each set its power set is a member of $\mathscr{L}(\mathfrak{P}E)$. We define the finitely additive quasitopology $\mathfrak{P}\tau$ by $\mathfrak{P}\tau = \mathrm{lt}_{\mathfrak{P}E}\,L$ (use of Proposition 1) and call $\mathfrak{P}\tau$ to be the (*first*) *power of the finitely additive quasitopology* τ. (One has to be careful not to mix up $\mathfrak{P}\tau$ with the dower set of the set τ!)

Proposition 3. *Let* $M \subseteqq E$. *If* τ *is a pretopology (topology), then* τ_M *is a pretopology (topology).* τ *is a pretopology (topology) if and only if* $\mathfrak{P}\tau$ *is a pretopology (topology).*

Proof. The assertion about τ_M is obvious, while that about $\mathfrak{P}\tau$ is contained in "Satz 18" in [4], p. 108 (see also [4], p. 82). □

2. MORE ON THE FIRST POWER OF A FINITELY ADDITIVE QUASITOPOLOGY

It is of general interest but (except for Remark 2) not being used in this paper that the notion of the trace and that of the first power of a finitely additive quasitopology are compatible with each other in the sense of Proposition 4, where indices indicate traces and $\mathfrak{P}(\tau_M)$ denotes the first power of τ_M.

Proposition 4. *If $M \subseteqq E$, then $\mathfrak{P}(\tau_M) = (\mathfrak{P}\tau)_{\mathfrak{P}M}$.*

Proof. For all $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}M)$, we have the equations

$$\mathfrak{P} \bigcap_{C \in \mathscr{G}\mathfrak{a}} (M \cap \tau \bigcup_{i \in C} f(i)) = \mathfrak{P}(M \cap \bigcap_{C \in \mathscr{G}\mathfrak{a}} \tau \bigcup_{i \in C} f(i)) = (\mathfrak{P}M) \cap (\mathfrak{P} \bigcap_{C \in \mathscr{G}\mathfrak{a}} \tau \bigcup_{i \in C} f(i)) . \square$$

Next, we present a construction of $\mathfrak{P}\tau$ by means of the given neighborhood operator $\mathfrak{B}$.

Let the mapping $\mathfrak{S}$ be defined by

$$\mathfrak{S}X = \{\mathscr{G}_E\{Y\} \mid Y \in \bigcup_{x \in X} \mathfrak{B}x\} \quad \text{for all} \quad X \in \mathfrak{P}E .$$

Let $\mathfrak{W}$ denote the neighborhood operator of $\mathfrak{P}E$ generated by $\mathfrak{S}$. Designate by $\mathfrak{P}^0E$ the set $(\mathfrak{P}E) \setminus \{\emptyset\}$ and by $\mathfrak{S}^0$ the restriction of the mapping $\mathfrak{S}$ to the set $\mathfrak{P}^0E$ (as domain), furthermore by $\mathfrak{W}^0$ the neighborhood operator of $\mathfrak{P}^0E$ generated by $\mathfrak{S}^0$. Then, we obtain Proposition 5, where $(\mathfrak{P}\tau)_{\mathfrak{P}^0E}$ denotes the trace of $\mathfrak{P}\tau$ in $\mathfrak{P}^0E$.

Proposition 5. $\mathfrak{W} = \mathrm{tv}_{(\mathfrak{P}E)}(\mathfrak{P}\tau)$ *and* $\mathfrak{W}^0 = \mathrm{tv}_{(\mathfrak{P}^0E)}(\mathfrak{P}\tau)_{(\mathfrak{P}^0E)}$.

Proof. Define $\mathrm{Lim}_{\mathfrak{W}}$ and $\mathrm{Lim}_{\mathfrak{W}^0}$ by $\mathrm{Lim}_{\mathfrak{W}} = \mathrm{vl}_{(\mathfrak{P}E)}\mathfrak{W}$ and $\mathrm{Lim}_{\mathfrak{W}^0} = \mathrm{vl}_{(\mathfrak{P}^0E)}\mathfrak{W}^0$. By Proposition 1, it suffices to show that, for all $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}E)$,

$$\text{(1)} \qquad \mathrm{Lim}_{\mathfrak{W}}(f, I, \mathfrak{a}) = \mathfrak{P} \liminf_\tau (f, I, \mathfrak{a})$$

and, if $f(i) \neq \emptyset$ for all $i \in I$,

$$\text{(2)} \qquad \mathrm{Lim}_{\mathfrak{W}^0}(f, I, \mathfrak{a}) = (\mathfrak{P}^0E) \cap (\mathfrak{P} \liminf_\tau (f, I, \mathfrak{a})) .$$

1. First, we prove (1). a) Assume $X \in \mathrm{Lim}_{\mathfrak{W}}(f, I, \mathfrak{a})$, $x \in X$ and $U \in \mathfrak{B}x$. Then $\mathscr{G}_E\{U\} \in \mathfrak{W}X$, consequently $f(i) \in \mathscr{G}_E\{U\}$, thus $U \cap f(i) \neq \emptyset$ for $\mathfrak{a}$-almost all $i \in I$, therefore, by the choice of U, $x \in \liminf_\tau (f, I, \mathfrak{a})$, thus, by the choice of x, $X \in \mathfrak{P} \liminf_\tau (f, I, \mathfrak{a})$. b) Assume $X \in \mathfrak{P} \liminf_\tau (f, I, \mathfrak{a})$ and $W \in \mathfrak{W}X$. Then, there is a finite set $\mathfrak{r} \subseteqq \mathfrak{S}X$ such that $\bigcap_{(\mathfrak{P}E)}\mathfrak{r} \subseteqq W$. Let $Q \in \mathfrak{r}$; then, for some $x \in X$, there is a $Y \in \mathfrak{B}x$ such that $Q = \mathscr{G}_E\{Y\}$. By the choice of X, x and Y, there exists an $A_Q \in \mathfrak{a}$ with $f(i) \in Q$ for all $i \in A_Q$. In such a way, we choose a family $(A_Q)_{Q \in \mathfrak{r}}$ (admitting the empty family in the case $\mathfrak{r} = \emptyset$). Then, for all $j \in \bigcap_{I, Q \in \mathfrak{r}} A_Q$, $f(j) \in \bigcap_{(\mathfrak{P}E)} \mathfrak{r} \subseteqq W$, therefore $f(i) \in W$ for $\mathfrak{a}$-almost all $i \in I$, thus, by the choice of W, $X \in \mathrm{Lim}_{\mathfrak{W}}(f, I, \mathfrak{a})$.

2. Let $X \neq \emptyset$ and $f(i) \neq \emptyset$ for all $i \in I$. Then, one obtains a proof of (2) from the preceding part 1 if one replaces there everywhere $\mathfrak{W}$ and $\mathfrak{P}E$ by $\mathfrak{W}^0$ und $\mathfrak{P}^0E$; consider that $\mathfrak{S}^0X = \mathfrak{S}X$. □

Given a mapping σ on $\mathfrak{P}E$ into $\mathfrak{P}E$, we call (E, σ) a *quasitopological space*; if and only if σ is a finitely additive quasitopology, a pretopology, a topology, we call (E, σ) a *finitely additive quasitopological space*, a *pretopological space* (ČECH [1], p. 237: "closure space"), a *topological space.*

Remark 1. If τ is a pretopology (topology), then $(\mathfrak{P}\tau)_{(\mathfrak{P}^0E)}$ is a pretopology (topology) of $\mathfrak{P}^0E$ by Proposition 3, and, by Proposition 5 and the construction of $\mathfrak{W}^0$, the pretopological (topological) space $(\mathfrak{P}^0E, (\mathfrak{P}\tau)_{(\mathfrak{P}^0E)})$ coincides with the *hyperspace of lower semicontinuity*, $H_-(E, \tau)$, *of* (E, τ) defined in ČECH [1], p. 623, Definition 34 A.1.

3. ON THE PRODUCT OF FINITELY ADDITIVE QUASITOPOLOGIES

In [5], the author has introduced (in a slightly different language) the product of finitely additive quasitopologies (even of arbitrary quasitopologies) without discussing its construction by means of neighborhood operators. Here, we give such a construction (usual for pretopologies, see ČECH [1], p. 289, Definition 17 C.1) in full generality.

Let $(E_d, \tau_d)_{d \in D}$ be a family (D a set) of finitely additive quasitopological spaces (E_d, τ_d), and define Lim_d und $\mathfrak{B}_d$ by $\mathrm{Lim}_d = \mathrm{tl}_{E_d} \tau_d$ and $\mathfrak{B}_d = \mathrm{tv}_{E_d} \tau_d$ for all $d \in D$ (cf. Proposition 1). For abbreviation, we set $\mathrm{P}_{d \in D} E_d = P$ (= cartesian product of the family $(E_d)_{d \in D}$ of sets E_d). The mapping L defined, now, by

$$L(\mathfrak{a}) = \mathrm{P}_{d \in D} \mathrm{Lim}_d (\mathrm{pr}_d\, \mathfrak{a}) \quad \text{for all} \quad \mathfrak{a} \in \Phi_0 P$$

(where pr_d denotes the d-th projection mapping on P) turns out (see [5], p. 364, "Satz 1", and [8], p. 159, "Satz 4") to be a member of $\mathscr{L}P$. We define the finitely additive quasitopology $\mathrm{P}_{d \in D} \tau_d$ of P by $\mathrm{P}_{d \in D} \tau_d = \mathrm{tl}_P L$ and call it the *product of the family* $(\tau_d)_{d \in D}$ *of finitely additive quasitopologies.* (One has to be careful not to mix up $\mathrm{P}_{d \in D} \tau_d$ with the cartesian product of the sets τ_d!)

The notion of the product of finitely additive quasitopologies is compatible with that of a trace of a finitely additive quasitopology in the sense of Proposition 6, where $\mathrm{P}_{d \in D} (\tau_d)_{M_d}$ designates the product of the traces $(\tau_d)_{M_d}$ of the τ_d in M_d.

Proposition 6. *Let* $(M_d)_{d \in D}$ *be a family of sets* $M_d \subseteq E_d$. *Then,* $(\mathrm{P}_{d \in D} \tau_d)_{\mathrm{P}_{d \in D} M_d} = \mathrm{P}_{d \in D} (\tau_d)_{M_d}$.

Proof (see also [6], p. 317, "Satz 6"). The equation

$$(\mathop{\mathrm{P}}_{d\in D} M_d) \cap (\mathop{\mathrm{P}}_{d\in D} \mathrm{Lim}_d (f_i(d))_{i\in I,\mathfrak{a}}) = \mathop{\mathrm{P}}_{d\in D} (M_d \cap \mathrm{Lim}_d (f_i(d))_{i\in I,\mathfrak{a}})$$

holds for all $(f_i)_{i\in I,\mathfrak{a}} \in \Phi(\mathop{\mathrm{P}}_{d\in D} M_d)$. □

Next, we reconstruct the product $\mathop{\mathrm{P}}_{d\in D} \tau_d$ by means of the neighborhood operators $\mathfrak{B}_d$.

Let the mapping $\mathfrak{S}$ be defined by

$$\mathfrak{S}f = \{\mathrm{pr}_d^{-1} V \mid d \in D \text{ and } V \in \mathfrak{B}_d f(d)\} \quad \text{for all} \quad f \in P\,.$$

Let $\mathfrak{W}$, now, denote the neighborhood operator of P generated by $\mathfrak{S}$. We remark that for each $f \in P$

$$\begin{aligned} (3) \qquad & \{\textstyle\bigcap_E \mathfrak{r} \mid \mathfrak{r} \text{ is a finite subset of } \mathfrak{S}f\} = \\ & = \{\mathop{\mathrm{P}}_{d\in D} \varphi(d) \mid \varphi \in \mathop{\mathrm{P}}_{d\in D} \mathfrak{B}_d f(d) \text{ and, for some finite set } G \subseteqq D, \\ & \qquad \varphi(d) = E_d \text{ for all } d \in D \setminus G\}. \end{aligned}$$

Proposition 7. $\mathfrak{W} = \mathrm{tv}_P(\mathop{\mathrm{P}}_{d\in D} \tau_d)$.

Proof. We define the limit operator $\mathrm{Lim}_{\mathfrak{W}}$ of P by $\mathrm{Lim}_{\mathfrak{W}} = \mathrm{vl}_P\, \mathfrak{W}$ (use of Proposition 1). By Proposition 1, it suffices to show that

$$\mathrm{Lim}_{\mathfrak{W}} (f_i)_{i\in I,\mathfrak{a}} = \mathop{\mathrm{P}}_{d\in D} \mathrm{Lim}_d (f_i(d))_{i\in I,\mathfrak{a}}$$

holds for all $(f_i)_{i\in I,\mathfrak{a}} \in \Phi P$. Let $(f_i)_{i\in I,\mathfrak{a}} \in \Phi P$ and $f \in P$.

1. Assume $f \in \mathrm{Lim}_{\mathfrak{W}} (f_i)_{i\in I,\mathfrak{a}}$. Let $c \in D$ und $V \in \mathfrak{B}_c f(c)$. Define a mapping φ by letting $\varphi(d) = E_d$ for all $d \in D \setminus \{c\}$ and $\varphi(c) = V$. Then, by (3), $\mathop{\mathrm{P}}_{d\in D} \varphi(d) \in \mathfrak{W}f$. Thus, there is an $A \in \mathfrak{a}$ such that $f_i \in \mathop{\mathrm{P}}_{d\in D} \varphi(d)$, especially $f_i(c) \in V$, for all $i \in A$. Therefore, by the choice of V, $f(c) \in \mathrm{Lim}_c (f_i(c))_{i\in I,\mathfrak{a}}$, thus, by the choice of c, $f \in \mathop{\mathrm{P}}_{d\in D} \mathrm{Lim}_d (f_i(d))_{i\in I,\mathfrak{a}}$.

2. Assume $f \in \mathop{\mathrm{P}}_{d\in D} \mathrm{Lim}_d (f_i(d))_{i\in I,\mathfrak{a}}$, and let $W \in \mathfrak{W}f$. Then, there exists (by (3)) a mapping $\varphi \in \mathop{\mathrm{P}}_{d\in D} \mathfrak{B}_d f(d)$ and a finite set $G \subseteqq D$ such that $\varphi(d) = E_d$ for all $d \in D \setminus G$ and $\mathop{\mathrm{P}}_{d\in D} \varphi(d) \subseteqq W$. By the choice of f and φ, there is, for each $d \in D$, a set $A_d \in \mathfrak{a}$ with

$$(4) \qquad f_i(d) \in \varphi(d) \quad \text{for all} \quad i \in A_d\,.$$

Define A by $A = \bigcap_{I,\, d\in G} A_d$ (admitting the trivial case $G = \emptyset$); then $A \in \mathfrak{a}$ (since G is finite and $\mathfrak{a}$ is a filter on I), and one obtains $f_i(d) \in \varphi(d)$ for all $i \in A$ and all $d \in D$, using (4) in the case $G \neq \emptyset$. Therefore, $f_i \in \mathop{\mathrm{P}}_{d\in D} \varphi(d) \subseteqq W$ holds for $\mathfrak{a}$-almost all $i \in I$, thus, by the choice of W, $f \in \mathrm{Lim}_{\mathfrak{W}} (f_i)_{i\in I,\mathfrak{a}}$. □

Remark 2. In the paper [10], the author has written everything in a way that the proofs there remain valid — up to slight modifications — also in a setting of finitely additive quasitopologies instead of topologies (see Remark 4 in [10], p. 43). Of course, given finitely additive quasitopological spaces (E, τ) and (F, σ), a mapping φ on E into F is called (τ, σ)-*continuous* if and only if $\varphi(\tau X) \subseteq \sigma(\varphi X)$ holds for all $X \in \mathfrak{P}E$. In terms of limit operators, this definition reflects in the assertion that

$$\varphi \textit{ is } (\tau, \sigma)\textit{-continuous if and only if } \varphi(\mathrm{Lim}_\tau(f, I, \mathfrak{a})) \subseteq \mathrm{Lim}_\sigma(\varphi \circ f, I, \mathfrak{a})$$

for each $(f, I, \mathfrak{a}) \in \Phi E$, where $\mathrm{Lim}_\tau = \mathrm{tl}_E\, \tau$ and $\mathrm{Lim}_\sigma = \mathrm{tl}_F\, \sigma$ (see Proposition 1); and it is clear, how the definition looks like in terms of neighborhood operators. Based on this definition of continuity for mappings, everything else (more precisely: the definition of (τ, σ)-continuity of a mapping *from* E into F, Definitions 1 und 2, Propositions 1 through 8, Theorems 1 through 3) in the paper [10] can be carried over to finitely additive quasitopological spaces, including the proofs, word by word except for the following change (necessary by the fact that $\mathfrak{V}x$ with $x \in E$ can degenerate to $\mathfrak{P}E$ under the present situation): In part 1 of the proof of Proposition 5 in [10], p. 40, one has to replace the words "We have" by the words "If $\mathfrak{a} = \mathfrak{P}M$, then, for some $V \in \mathfrak{a}$ (choose $V = \emptyset$), $U \cap h(z) \neq \emptyset$ holds for all $z \in V$. Let $\mathfrak{a} \neq \mathfrak{P}M$. Then, we have". Proposition 7 of the present paper serves as a lemma within the proof of the generalized Proposition 8 in [10], p. 41, and the present Proposition 4 replaces the statement (0) in [10], p. 36 (which serves as a lemma for the Propositions 4 and 7 in [10], p. 39 and 41).

4. PROOF OF PROPOSITION 1

Let E be a set fixed for the whole section and, down to the Lemma 2, $\mathfrak{a} \subseteq \mathfrak{P}E$. $\mathfrak{a}$ is called a *grill on* E if and only if

(5) $\mathfrak{a} \neq \emptyset,\ \emptyset \notin \mathfrak{a},\ \mathscr{H}_E\mathfrak{a} = \mathfrak{a}$ *and, for all sets* A, B, *if* $A \cup B \in \mathfrak{a}$, *then* $A \in \mathfrak{a}$ *or* $B \in \mathfrak{a}$,

a *quasigrill on* E if and only if $\mathfrak{a}$ is a grill on E or $\mathfrak{a} = \emptyset$ (i.e., if and only if (5) without the requirement $\mathfrak{a} \neq \emptyset$ holds).

Lemma 1. *Let* $\mathfrak{a} \neq \emptyset$, $\emptyset \notin \mathfrak{a}$ *and* $\mathscr{H}_E\mathfrak{a} = \mathfrak{a}$. *Then* $\mathfrak{a}$ *is a filter on* E (*grill on* E) *if and only if* $\mathscr{G}_E\mathfrak{a}$ *is a grill on* E (*filter on* E).

Proof. [2], p. 323, "Satz 3". □

Lemma 2. *Let* $\mathscr{H}_E\mathfrak{a} = \mathfrak{a}$. *Then* $\mathfrak{a}$ *is a quasifilter on* E (*quasigrill on* E) *if and only if* $\mathscr{G}_E\mathfrak{a}$ *is a quasigrill on* E (*quasifilter on* E).

Proof. Use Lemma 1 and $\mathscr{G}_E\mathfrak{P}E = \emptyset$, $\mathscr{G}_E\emptyset = \mathfrak{P}E$. □

For each mapping τ on $\mathfrak{P}E$ into $\mathfrak{P}E$, define the mapping $(\in \tau)[\,]$ by $(\in \tau)[x] = \{M \mid M \in \mathfrak{P}E$ and $x \in \tau M\}$ for all $x \in E$. (In other terms, $(\in \tau)[\,]$ is the mapping on E into $\mathfrak{P}\mathfrak{P}E$ induced, in the indicated sense, by the relation $\in \tau$ which be defined by $x(\in \tau) M$ if and only if $x \in \tau M$ (for all $x \in E$ and all $M \in \mathfrak{P}E$).)

By the definitions, it is clear that

Lemma 3. *$\tau \in \mathscr{T}E$ if and only if $(\in \tau)[x]$ is a quasigrill on E for each $x \in E$.*

Lemma 4. tv_E *is a one-to-one mapping on $\mathscr{T}E$ onto $\mathscr{V}E$ and we have* $\mathrm{vt}_E = (\mathrm{tv}_E)^{-1}$.

Proof. 1. By the Lemmas 2 and 3, it is clear that tv_E maps $\mathscr{T}E$ into $\mathscr{V}E$ and vt_E maps $\mathscr{V}E$ into $\mathscr{T}E$.

2. Let $\tau \in \mathscr{T}E$ and $M \in \mathfrak{P}E$. Then, one has, for all $x \in E$, the logical chain

$$\begin{aligned} x \in (\mathrm{vt}_E((\mathrm{tv}_E)(\tau)))\, M &\Leftrightarrow M \in \mathscr{G}_E \mathscr{G}_E((\in\tau)[x]) \\ &\Leftrightarrow M \in (\in\tau)[x] \\ &\Leftrightarrow x \in \tau M\,, \end{aligned}$$

since for the mappings $\mathscr{G}_E$ and $\mathscr{H}_E$ on $\mathfrak{P}\mathfrak{P}E$ into $\mathfrak{P}\mathfrak{P}E$ the equation $\mathscr{G}_E \circ \mathscr{G}_E = \mathscr{H}_E$ holds (see [2], p. 323, "Korollar 1"). Thus, $\mathrm{vt}_E \circ \mathrm{tv}_E = \mathrm{id}_{(\mathscr{T}E)}$ (= identical mapping on $\mathscr{T}E$).

3. Let $\mathfrak{V} \in \mathscr{V}E$ and $x \in E$. Then, one has, for all $V \subseteq E$, the logical chain

$$\begin{aligned} V \in (\mathrm{tv}_E((\mathrm{vt}_E)(\mathfrak{V})))\, x &\Leftrightarrow V \in \mathscr{G}_E(\in (\mathrm{vt}_E(\mathfrak{V})))[x] \\ &\Leftrightarrow V \in \mathscr{G}_E \mathscr{G}_E \mathfrak{V}x \\ &\Leftrightarrow V \in \mathfrak{V}x\,, \end{aligned}$$

since $\mathscr{G}_E \circ \mathscr{G}_E = \mathscr{H}_E$. Thus, $\mathrm{tv}_E \circ \mathrm{vt}_E = \mathrm{id}_{(\mathscr{V}E)}$ (= identical mapping on $\mathscr{V}E$).

1, 2 and 3 assure the assertion. □

Lemma 5. tl_E *is a one-to-one mapping on $\mathscr{T}E$ onto $\mathscr{L}E$, and one has* $\mathrm{lt}_E = (\mathrm{tl}_E)^{-1}$.

Proof. §§ 3 and 4 in [4], pp. 104–107, and [8], p. 159, "Satz 4"; or [9], p. 370, "Korollar zu Satz 6". □

Lemma 6. $x \in \mathrm{Lim}_{\mathfrak{V}}(\mathfrak{V}x)$ *holds for all $(x, \mathfrak{V}) \in E \times (\mathscr{V}E)$ with the property $\mathfrak{V}x \neq \mathfrak{P}E$.*

Proof. If $\mathfrak{V}x \neq \mathfrak{P}E$, then $\mathfrak{V}x \in \Phi_0 E$ and $\mathfrak{V}x \subseteq \mathfrak{V}x$; furthermore, use the definition of $\mathrm{Lim}_{\mathfrak{V}}$ in Section 1. □

Lemma 7. $\mathfrak{V}x = \bigcap_{\mathfrak{P}E}\{\mathscr{H}_E\mathfrak{a} \mid x \in \operatorname{Lim}_{\mathfrak{V}} \mathfrak{a}$ *and* $\mathfrak{a} \in \Phi_0 E\}$ *for all* $x \in E$ *and all* $\mathfrak{V} \in \mathscr{V}E$.

Proof. 1. Let $V \in \mathfrak{V}x$. If $\mathfrak{a} \in \Phi_0 E$ and $x \in \operatorname{Lim}_{\mathfrak{V}} \mathfrak{a}$, then $\mathfrak{V}x \subseteq \mathscr{H}_E\mathfrak{a}$, therefore $V \in \mathscr{H}_E\mathfrak{a}$.

2. If $V \in \bigcap_{\mathfrak{P}E}\{\mathscr{H}_E\mathfrak{a} \mid x \in \operatorname{Lim}_{\mathfrak{V}} \mathfrak{a}$ and $\mathfrak{a} \in \Phi_0 E\}$, then $V \in \mathfrak{V}x$ holds by Lemma 6 if $\mathfrak{V}x \neq \mathfrak{P}E$, while $V \in \mathfrak{V}x$ holds trivially if $\mathfrak{V}x = \mathfrak{P}E$. □

Lemma 8. $\operatorname{tl}_E \circ \operatorname{vt}_E = \operatorname{vl}_E$ *and* $\operatorname{lv}_E = (\operatorname{vl}_E)^{-1}$.

Proof. 1. Let $\mathfrak{V} \in \mathscr{V}E$, $\mathfrak{a} \in \Phi_0 E$, $y \in E$ and put, for abbreviation, $\bigcup\mathfrak{a} = I$. Then,

$$\begin{aligned} y \in \operatorname{Lim}_{\mathfrak{V}} \mathfrak{a} &\Leftrightarrow \mathfrak{V}y \subseteq \mathscr{H}_E\mathfrak{a} \Leftrightarrow \mathscr{G}_E\mathfrak{a} \subseteq \mathscr{G}_E\mathfrak{V}y \\ &\Leftrightarrow (C \in \mathscr{G}_I\mathfrak{a} \Rightarrow C \in \mathscr{G}_E\mathfrak{V}y) \\ &\Leftrightarrow y \in \bigcap_{C \in \mathscr{G}_I\mathfrak{a}} \{x \mid x \in E \text{ and } C \in \mathscr{G}_E\mathfrak{V}x\} \\ &\Leftrightarrow y \in ((\operatorname{tl}_E \circ \operatorname{vt}_E)(\mathfrak{V}))\,\mathfrak{a}\,. \end{aligned}$$

(For the proof of the second $\Leftrightarrow$ in this logical chain one uses (1.12) in [2], p. 322, and that $\mathscr{G}_E \circ \mathscr{H}_E = \mathscr{G}_E$, $\mathscr{G}_E \circ \mathscr{G}_E = \mathscr{H}_E$ and $\mathscr{H}_E \circ \mathscr{H}_E = \mathscr{H}_E$ hold for the mappings $\mathscr{G}_E$ and $\mathscr{H}_E$ on $\mathfrak{P}\mathfrak{P}E$ into $\mathfrak{P}\mathfrak{P}E$.)

Thus, $\operatorname{vl}_E = \operatorname{tl}_E \circ \operatorname{vt}_E$.

2. By Lemma 7, we have $\operatorname{lv}_E \circ \operatorname{vl}_E = \operatorname{id}_{\mathscr{V}E}$ (= identical mapping on $\mathscr{V}E$). Since, by Lemma 4, Lemma 5 and part 1 of this proof, the range of vl_E is equal to the domain of lv_E, we have shown that $\operatorname{lv}_E = (\operatorname{vl}_E)^{-1}$. □

Proposition 1 is now proven by Lemma 4, Lemma 5 and Lemma 8.

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