Ján Jakubík
Normal prime filters of a lattice ordered group


Persistent URL: [http://dml.cz/dmlcz/101219](http://dml.cz/dmlcz/101219)

Terms of use:

© Institute of Mathematics AS CR, 1974

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library [http://dml.cz](http://dml.cz)
NORMAL PRIME FILTERS OF A LATTICE ORDERED GROUP

JÁN JAKUBÍK, Košice

(Received January 9, 1973)

The representation of lattice ordered groups as subdirect products of linearly ordered groups was investigated in several papers ([1], [4]—[7], [9]—[13]). Let \( P \) be the positive cone of a lattice ordered group \( G \) and let \( W \) be the union of all normal prime filters of \( P \), \( K_0 = \{ x \in G : |x| \notin W \} \). Then \( K_0 \) is an \( l \)-ideal of \( G \) and \( G/K_0 \) is the largest quotient group of \( G \) that can be represented as a subdirect product of linearly ordered groups (cf. [1]).

Banaschewski [1] remarks that it might be of interest to have a characterization of \( W \) and \( K_0 \) internally in terms of the elements of \( G \) and that it remains as an open question whether \( W \) is the set of all elements \( 0 < a \in G \) such that

\[
(x_1 + a - x_1) \land \ldots \land (x_k + a - x_k) > 0 \quad \text{for any} \quad x_i \in G .
\]

Let \( W_1 \) be the set of all strictly positive elements \( a \) of \( G \) satisfying (1). In this paper it will be shown that there exists an \( l \)-group \( G \) with two generators such that \( W \neq W_1 \), thus answering in the negative the above mentioned question (§2). An internal characterization of the sets \( W \) and \( K_0 \) for any \( l \)-group \( G \) will be given in §3.

Let \( G \) be a subdirect product of linearly ordered groups. Then for each \( 0 < a \in G \) and any finite set \( x_1, \ldots, x_n \in G \) the relation (1) is valid. In §4 we show that for an infinite set \( \{ x_i \} (i \in I) \) such that \( \bigwedge_{i \in I} (x_i + a - x_i) \) exists, the relation

\[
\bigwedge_{i \in I} (x_i + a - x_i) > 0
\]

need not hold.

The standard terminology for lattice ordered groups will be used (cf. Birkhoff [2], Fuchs [3]). The lattice ordered groups will be written additively though they are not assumed to be abelian.

1. PRELIMINARIES

Let us recall some definitions and results that we shall use.

Let \( G \) be a lattice ordered group with the positive cone \( P \). A proper subset \( \emptyset \neq Q \) of \( P \) will be called a prime filter in \( P \) if
(i) $x, y \in Q$ implies $x \land y \in Q$;
(ii) $x \in Q$ and $z \in P$, $z \geq x$ implies $z \in Q$;
(iii) $x, y \in P$ and $x + y \in Q$ implies $x \in Q$ or $y \in Q$.

The relation between prime filters of $P$ and homomorphisms of $G$ into totally ordered groups is described by the following proposition:

(*) ([1], Proposition 1.) For any homomorphism $f$ with $f(G) \neq \{0\}$ of $G$ into a linearly ordered group $T$, $Q(f) = \{x \in P : f(x) > 0\}$ is normal prime filter and $\text{Ker}(f) = \{x \in G : |x| \notin Q(f)\}$. Conversely, for any normal prime filter $Q$ in $P$ there exists an epimorphism $f$ from $G$ into a linearly ordered group such that $Q = Q(f)$, namely the natural homomorphism $G \to G/K$ where $K$ is the $l$-ideal $\{x \in G : |x| \in Q\}$.

For any $A \subseteq G$ we denote

$$A^a = \{x \in G : |x| \land |a| = 0 \text{ for each } a \in A\}.$$  

The set $A^a$ will be called a polar of $G$. Each polar is a convex $l$-subgroup of $G$. The following theorem was proven in [13]:

(**) A lattice ordered group $G$ is a subdirect union of linearly ordered groups if and only if each polar of $G$ is normal.

2. AN EXAMPLE

Lemma 1. Let $\varphi$ be a homomorphism of a lattice ordered group $G$ into a linearly ordered group $H$. Let $a, x_1, x_2 \in G$ such that

(2) $$(x_1 + a - x_1) \land (x_2 + a - x_2) = 0.$$  

Then $\varphi(a) = 0$.

Proof. From (2) it follows

$$\varphi(x_1 + a - x_1) \land \varphi(x_2 + a - x_2) = 0$$

and since $H$ is linearly ordered we have either $\varphi(x_1 + a - x_1) = 0$ or $\varphi(x_2 + a - x_2) = 0$; because $\varphi(a)$ is conjugate of $\varphi(x_1 + a - x_1)$ we infer that $\varphi(a) = 0$.

Let $Z$ be the additive group of all integers with the natural linear order. Let

$$F = \prod_{i \in I} Z_i \ (i \in I)$$

be the complete direct product of $l$-groups $Z_i = Z$ for each $i \in I$ where $I$ is the set of all integers. The elements $f \in F$ are written in the form $f = (\ldots, f(i), \ldots) \ (i \in I)$. For any integer $n$ put

$$p_n f = (\ldots, b(i), \ldots) \ (i \in I)$$
with \( b(i) = f(i - n) \). Let \( m \) be a fixed positive integer, \( m > 1 \). We denote by \( F_m \) the set of all \( f \in F \) such that for any \( i, j, k \in I \) satisfying \( i - j = km \) we have \( f(i) = f(j) \).

Let \( G \) be the set of all pairs \( (n, f) \), \( n \in \mathbb{Z}, f \in F_m \). We put \( (n_1, f_1) < (n_2, f_2) \) if either \( n_1 < n_2 \), or \( n_1 = n_2 \) and \( f_1 < f_2 \). We define the operation \( + \) in \( G \) by the rule

\[
(n_1, f_1) + (n_2, f_2) = (n_1 + n_2, p_n f_1 + f_2).
\]

Then \( G \) is a lattice ordered group that is generated by two elements (cf. [8]). Let \( j \in I, 0 \leq j < m \). Let \( \overline{0} \) and \( \overline{1} \) be neutral element of \( F_m \) and of \( G \), respectively. Further let \( f_j \in F_m \) such that \( f_j(i) = 1 \) when \( i - j = km \) for some \( k \in I \) and \( f_j(i) = 0 \) otherwise. Put \( a_j = (0, f_j) \), \( x_1 = \overline{0}, x_2 = (1, \overline{0}) \). Then we have

\[
(x_1 + a_j - x_1) \land (x_2 + a_j - x_2) = 0.
\]

According to Lemma 1 and (*) we obtain \( a_j \in K_0 \). Since \( K_0 \) is an \( I \)-ideal of \( G \) (cf. [1]) we infer that the element \( a = \sum a_j \) \((j = 0, 1, \ldots, m - 1)\) belongs to \( K_0 \), hence \( a \notin W \).

For each \( x \in G \),

\[
x + a - x = a,
\]

thus \( a \) fulfils (1) and so \( a \in W_1 \). Therefore \( W_1 \neq W \).

\section{3. CONSTRUCTION OF \( K_0 \)}

Let \( G \) be a lattice ordered group. We define by induction subsets \( K_n \) and \( \overline{K}_n \subset G \) \((n = 1, 2, \ldots)\) as follows. We put \( K_1 = \overline{K}_1 = \{0\} \). If \( K_{n-1} \) and \( \overline{K}_{n-1} \) are already defined we define \( K_n \) to be the set of all elements \( 0 \leq a \in G \) such that

\[
(x_1 + a - x_1) \land (x_2 + a - x_2) \in \overline{K}_{n-1}
\]

for some \( x_1, x_2 \in G \). Further let \( \overline{K}_n \) be the subsemigroup of \( G \) generated by \( K_n \); i.e., \( \overline{K}_n \) is the set of all elements of \( G \) that can be written in the form \( b = a_1 + \ldots + a_m \) for some \( a_1, \ldots, a_m \in K_n \) and some positive integer \( m \).

\textbf{Lemma 2.} \( \bigcup_{n=1}^{\infty} K_n \subset K_0 \).

\textbf{Proof.} Obviously \( \overline{K}_1 \subset K_0 \); assume that \( \overline{K}_{n-1} \subset K_0 \). Let \( a \in K_n \). Then

\[
(x_1 + a - x_1) \land (x_2 + a - x_2) = a_1 + \ldots + a_m
\]

for some elements \( x_1, x_2 \in G, a_1, \ldots, a_m \in \overline{K}_{n-1} \). If \( a \in F \) for some normal prime filter \( F \) of \( G \), then some \( a_i \) belongs to \( F \) and hence \( a_i \in W \), which is a contradiction. Hence \( a \in K_0 \). Since \( K_0 \) is a subgroup of \( G \) we infer that \( \overline{K}_n \subset K_0 \).
Lemma 3. For each positive integer $n$, $K_n$ and $\mathcal{K}_n$ are convex subsets of $G$ containing 0.

Proof. The assertion is obviously valid for $n = 1$; assume that the assertion holds for $n - 1$. Let $a, b, x_1, x_2 \in G$, $0 \leq b \leq a$ and let (4) hold. Then we have

$$0 \leq (x_1 + b - x_1) \land (x_2 + b - x_2) \leq (x_1 + a - x_1) \land (x_2 + a - x_2).$$

Because $\mathcal{K}_{n-1}$ is convex by the assumption, we obtain

$$(x_1 + b - x_1) \land (x_2 + b - x_2) \in \mathcal{K}_{n-1}$$

and therefore $b \in K_n$. Thus $K_n$ is a convex subset of $G$. Let $y \in \mathcal{K}_n$, $z \in G$, $0 \leq z \leq y$. Then there are elements $y_1, \ldots, y_n \in K_n$ with $y_1 + \ldots + y_n = y$. Further there are elements $z_1, \ldots, z_n$ with $0 \leq z_1 \leq y$, $z = z_1 + \ldots + z_n$ (cf. [3]). Thus $z_1 \in K_n$ and hence $z \in \mathcal{K}_n$. Therefore $\mathcal{K}_n$ is convex in $G$.

Lemma 4. For each positive integer $n$, $K_n$ and $\mathcal{K}_n$ are normal subsets of $G$.

Proof. Obviously $K_1 = \mathcal{K}_1 = \{0\}$ is normal. Assume that $K_{n-1}$ and $\mathcal{K}_{n-1}$ are normal for some $n > 1$. Let $a \in K_n$, $x_1, x_2 \in G$ such that (4) holds and let $b \in G$. By putting $x_3 = b + x_1 - b$, $x_4 = b + x_2 - b$, $a' = b + a - b$ we obtain from (4)

$$(x_3 + a' - x_3) \land (x_4 + a' - x_4) \in \mathcal{K}_{n-1},$$

thus $b + a - b \in K_n$. Hence $K_n$ is a normal subset of $G$. From this it follows immediately that $\mathcal{K}_n$ is a normal subset of $G$ as well.

From Lemma 2 and Lemma 3 we obtain that $\mathcal{K} = \bigcup K_n (n = 1, 2, 3, \ldots)$ is a convex normal subset of $G$ containing 0; since each $\mathcal{K}_n$ is a subsemigroup of $G$, the set $\mathcal{K}$ is a subsemigroup of $G$.

It is easy to verify that if $A$ is a convex subsemigroup of $G$ containing 0 then the set

$$B = \{x \in G : -a \leq x \leq a \text{ for some } a \in A\} = \{x \in G : |x| \in A\}$$

is a convex $l$-subgroup of $G$; if, moreover, $A$ is normal in $G$, then $B$ is an $l$-ideal of $G$. Thus we have the assertion:

Lemma 5. The set $K = \{x \in G : |x| \in \bigcup K_n\}$ is an $l$-ideal of $G$.

Lemma 6. Each polar of the factor $l$-group $G/K$ is normal.

Proof. Let

$$C \subset G/K, \quad D = \{Y \in G/K : X \land Y = K \text{ for each } X \in C\},$$

$Z \in G/K$. Assume that $Y_1 = Z + Y - Z \notin D$ for some $Y \in D$. Then there is $X \in C$ such that $Y_1 \land X = U_1 > K$. Let $x \in X$, $y \in Y$, $z \in Z$. Thus $y_1 = z + y - z \in Y_1$
and \( u_1 = y_1 \wedge x \in U_1, u_1 \notin K \). Denote \( u = -z + u_1 + z \). From

\[
0 < u_1 \leq y_1 = z + y - z
\]

we obtain \( 0 < u \leq y \) and hence

\[
u_1 + K \leq x + K, \quad u + K \leq y + K.
\]

Therefore

\[
K \leq (u_1 \wedge u) + K = (u_1 + K) \wedge (u + K) \leq (x + K) \wedge (y + K) = K
\]

and so \( u_1 \wedge u \in K \). Since \( u_1 \wedge u \geq 0 \), we have \( u_1 \wedge u \in \mathbb{K}_n \) for some positive integer \( n \) and hence, according to the definition of \( K_{n+1} \), we obtain \( u_1 \in K_{n+1} \subseteq K \). Thus \( U_1 = K \), a contradiction.

**Lemma 7.** \( K_0 \subseteq K \).

**Proof.** From Lemma 7 and (***) it follows that \( G/K \) is a subdirect product of linearly ordered groups. Since \( G/K_0 \) is the largest quotient group of \( G \) that can be represented as a subdirect product of linearly ordered groups (cf. [1]) we obtain \( K_0 \subseteq K \).

According to Lemma 2 and Lemma 8, \( K_0 = K \). Hence we have the following internal characterization of the sets \( W \) and \( K_0 \):

**Theorem.** \( K_0 = \{ x \in G : |x| \in \bigcup \mathbb{K}_n \ (n = 1, 2, 3, \ldots) \} \), \( W = P - K_0 \).

### 4. MEETS OF CONJUGATE ELEMENTS

Let \( G \) be a lattice ordered group that is a subdirect product of linearly ordered groups. From the results of Banaschewski [1] (cf. also the Introduction) it follows that for each \( 0 < a \in W \) and any finite set \( X = \{ x_1, \ldots, x_n \} \subseteq G \),

\[
\bigwedge_{x \in X} (x_i + a - x_i) > 0.
\]

We show by an example, that for an infinite set \( X \subseteq G \) we can have

\[
\bigwedge_{x \in X} (x_i + a - x_i) = 0.
\]

Let \( Z, F, \) and \( p_n \) be as in §2. We denote by \( F_0 \) the set of all \( f \in F \) such that the set

\[
s(f) = \{ i \in I : f(i) \neq 0 \}
\]

is finite. Let \( G \) be the set of all pairs \( (n, f) \) with \( n \in Z, f \in F_0 \). We define the operation \(+\) in \( G \) by (3), §2. Then \( G \) is a group. Further we put \( (n, f) > 0 \) if either (i) \( n > 0 \), or (ii) \( n = 0, s(f) \neq \emptyset \) and \( f(i_0) > 0 \) where \( i_0 \) is the least element of \( s(f) \). Then \( G \)
turns out to be a linearly ordered group. Let $n \in \mathbb{Z}, f, f^n \in F_0$ such that $f_0(i) = 0$ for each $i \in I, f''(i) = 0$ for each $i \in I, i \neq n$ and $f''(n) = 1$. Put $a = (0, f^0), x_n = (n, f_0)$. We have $-x_n = (-n, f_0)$ and

$$x_n + a - x_n = (0, f^{-n}),$$

hence

$$\land(x_i + a - x_i) = 0,$$

where $i$ runs over the set of all positive integers.

References


Author's address: 040 01 Košice, Švermova 5, ČSSR (VŠT v Košiciach).