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ON MAPPINGS OF A MANIFOLD INTO A LIE GROUP

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In what follows I am concerned with the following problem: Let G be a Lie group, g its Lie algebra, M a manifold and φ a g-valued 1-form over M; under what conditions is there a mapping $\Phi : M \to G$ such that $\varphi = \Phi_* \omega$, ω being the Maurer-Cartan form of G? I study just the formal aspects of this question using the cohomology language; see, p. ex., V. GUILLEMIN and S. STERNBERG, *Deformation Theory of Pseudogroup Structures* (Memoirs of the AMS, No 64, 1966).

The paper has been written during my stay at the State University and the Pedagogical Institute at Vilnius, USSR.

Let g be a Lie algebra over $\mathscr{R}$ and M a differentiable manifold of class C^∞. Denote by a^p $(p = 0, 1, \ldots)$ the sheaf of g-valued p-forms on M, let $A^p = \Gamma(a^p, M)$ be the $\mathscr{R}$-module of the sections of a^p over M. Further, be given $\varphi \in A^1$ satisfying

$$\text{(1)} \qquad d\varphi(X, Y) = -[\varphi(X), \varphi(Y)]$$

for arbitrary vector fields X, Y on M. We are going to use the following definition of the exterior differential: for $\omega \in a^p$, $d\omega \in a^{p+1}$ is given by

$$\text{(2)} \qquad d\omega(X_1, \ldots, X_{p+1}) = \sum(-1)^{i+1} X_i\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) + \\ + \sum_{i<j}(-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).$$

Definition 1. The operator

$$\text{(3)} \qquad \delta_\varphi^p \equiv \delta : a^p \to a^{p+1}$$

be defined by

$$\text{(4)} \qquad \delta\omega(X_1, \ldots, X_{p+1}) = \\ = d\omega(X_1, \ldots, X_{p+1}) + \sum(-1)^{i+1} [\varphi(X_i), \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})].$$

Proposition 1. *We have*

$$\delta^2 = 0 . \tag{5}$$

Proof. Let $\omega \in a^p$, the form $\Omega \in a^{p+1}$ be defined by

$$\Omega(X_1, \ldots, X_{p+1}) = \sum(-1)^{i+1} [\varphi(X_i), \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})] . \tag{6}$$

Then

$$\begin{aligned} \mathrm{d}\Omega(X_1, \ldots, X_{p+2}) &= \sum(-1)^i [\varphi(X_i), \mathrm{d}\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+2})] + \\ &+ \sum_{i<j}(-1)^{i+j+1} [\mathrm{d}\varphi(X_i, X_j), \omega(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+2})] \end{aligned} \tag{7}$$

and $\delta\omega = \mathrm{d}\omega + \Omega$, i.e.,

$$\begin{aligned} \delta^2\omega(X_1, \ldots, X_{p+2}) &= \mathrm{d}\Omega(X_1, \ldots, X_{p+2}) + \\ &+ \sum (-1)^{i+1} [\varphi(X_i), \mathrm{d}\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+2}) + \Omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+2})] = \\ &= \sum_{i<j} (-1)^{i+j+1} [\mathrm{d}\varphi(X_i. X_j), \omega(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+2})] + \\ &+ \sum (-1)^{i+1} [\varphi(X_i), \Omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+2})] = \\ &= \sum_{i<j} (-1)^{i+j} [[\varphi(X_i), \varphi(X_j)], \omega(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+2})] + \\ &+ \sum_{i<j}(-1)^{i+j+1} [\varphi(X_i), [\varphi(X_j), \omega(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+2})]] + \\ &+ \sum_{i<j}(-1)^{i+j} [\varphi(X_j), [\varphi(X_i), \omega(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+2})]] = 0 . \end{aligned}$$

The details of the proof are omitted.

Proposition 2. (Poincaré lemma.) *Let $\omega \in a^p$ $(p \geqq 1)$ be defined in a neighborhood $U \subset M$ of the point $m \in M$, and let $\delta\omega = 0$. Then there is a neighborhood $U_1 \subset U$ of m and $\tau \in a^{p-1}$ defined on U_1 such that $\delta\tau = \omega$ on U_1.*

Proof. Write again $\delta\omega = \mathrm{d}\omega + \Omega$, Ω being defined by (6). The proposition follows from the Poincaré lemma for d if $\mathrm{d}\Omega = 0$ is a consequence of $\mathrm{d}\omega + \Omega = 0$. But this follows from (7).

Thus we get

Theorem 1. *Let $\mathscr{S}_\varphi \subset a^0$ be the sheaf of the solutions of the equation*

$$\delta s \equiv \mathrm{d}s + [\varphi, s] = 0 ; \tag{8}$$

then

$$0 \to \mathscr{S}_\varphi \to a^0 \xrightarrow{\delta} a^1 \xrightarrow{\delta} \ldots \tag{9}$$

is the resolution of $\mathscr{S}_\varphi$.

Definition 2. Denote by $B^p_\varphi(M, g)$ $(p = 1, 2, \ldots)$ the vector space of the forms of the type $\delta\omega$ with $\omega \in A^{p-1}$; let $Z^p_\varphi(M, g)$ $(p = 0, 1, \ldots)$ be the vector space of the forms $\omega' \in A^p$ satisfying $\delta\omega' = 0$. The *cohomological groups* be defined by

$$(10) \qquad \mathscr{H}^p_\varphi(M, g) = Z^p_\varphi(M, g)/B^p_\varphi(M, g) \quad \text{for} \quad p = 1, 2, \ldots;$$
$$\mathscr{H}^0_\varphi(M, g) = Z^0_\varphi(M, g)\,.$$

Definition 3. The form $\omega_1 \in A^1$ is called an *infinitesimal deformation* of φ if $\omega_1 \in Z^1_\varphi(M, g)$. A *deformation* of φ is a mapping $\omega_t : J \to A^1$, where (i) $J \subset \mathscr{R}$ is a neighborhood of $0 \in \mathscr{R}$, (ii) $\omega_0 = \varphi$, (iii) for each $t \in J$, we have

$$(11) \qquad \mathrm{d}\omega_t(X, Y) = -[\omega_t(X), \omega_t(Y)]\,,$$

(iv) the mapping ω_t is analytic in t.

The form ω_t may be written, in a suitable neighborhood $J' \subset J$ of $0 \in \mathscr{R}$, as

$$(12) \qquad \omega_t = \varphi + \omega_1 t + \omega_2 t^2 + \ldots, \quad \omega_i \in A^1;$$

from (11), we get

$$(13) \qquad \delta\omega_p(X, Y) = -\sum_{i=1}^{p-1} [\omega_i(X), \omega_{p-i}(Y)] \quad \text{for} \quad p = 1, 2, \ldots$$

Thus, the form $\omega_1 = (\mathrm{d}\omega_t/\mathrm{d}t)_{t=0}$ is an infinitesimal deformation of φ.

Proposition 3. *Let the forms $\omega_1, \ldots, \omega_{q-1} \in A^1$ satisfy*

$$(14) \qquad \delta\omega_p(X, Y) = -\sum_{i=1}^{p-1} [\omega_i(X), \omega_{p-i}(Y)] \quad \text{for} \quad p = 1, \ldots, q-1\,.$$

Then the form

$$(15) \qquad \Psi_q(X, Y) = \sum_{i=1}^{q-1} [\omega_i(X), \omega_{q-i}(Y)]$$

is contained in $Z^2_\varphi(M, g)$.

Proof. We have

$$\delta\Psi_q(X, Y, Z) = X\,\Psi_q(Y, Z) - Y\,\Psi_q(X, Z) + Z\,\Psi_q(X, Y) - \Psi_q([X, Y], Z) +$$
$$+ \Psi_q([X, Z], Y) - \Psi_q([Y, Z], X) + [\varphi(X), \Psi_q(Y, Z)] - [\varphi(Y), \Psi_q(X, Z)] +$$
$$+ [\varphi(Z), \Psi_q(X, Y)] =$$
$$= \sum_{i=1}^{q-1} \{[X\,\omega_i(Y), \omega_{q-i}(Z)] + [\omega_i(Y), X\,\omega_{q-i}(Z)] - [Y\,\omega_i(X), \omega_{q-i}(Z)] -$$
$$- [\omega_i(X), Y\,\omega_{q-i}(Z)] + [Z\,\omega_i(X), \omega_{q-i}(Y)] + [\omega_i(X), Z\,\omega_{q-i}(Y)] -$$
$$- [\omega_i([X, Y]), \omega_{q-i}(Z)] + [\omega_i([X, Z]), \omega_{q-i}(Y)] - [\omega_i([Y, Z]), \omega_{q-i}(X)] +$$
$$+ [\varphi(X), [\omega_i(Y), \omega_{q-i}(Z)]] - [\varphi(Y), [\omega_i(X), \omega_{q-i}(Z)]] +$$
$$+ [\varphi(Z), [\omega_i(X), \omega_{q-i}(Y)]]\} =$$

$$= \sum_{i=1}^{q-1} \{[d\omega_i(X, Y), \omega_{q-i}(Z)] - [d\omega_i(X, Z), \omega_{q-i}(Y)] + [d\omega_i(Y, Z), \omega_{q-i}(X)] -$$
$$- [\omega_i(Y), [\omega_{q-i}(Z), \varphi(X)]] - [\omega_{q-i}(Z), [\varphi(X), \omega_i(Y)]] +$$
$$+ [\omega_i(X), [\omega_{q-i}(Z), \varphi(Y)]] + [\omega_{q-i}(Z), [\varphi(Y), \omega_i(Y)]] -$$
$$- [\omega_i(X), [\omega_{q-i}(Y), \varphi(Z)]] - [\omega_{q-i}(Y), [\varphi(Z), \omega_i(X)]]\} =$$
$$= \sum_{i=1}^{q-1} \{[\delta\omega_i(X, Y), \omega_{q-i}(Z)] - [\delta\omega_i(X, Z), \omega_{q-i}(Y)] + [\delta\omega_i(Y, Z), \omega_{q-i}(X)]\} =$$
$$= -\sum_{i=1}^{q-1} \sum_{j=1}^{i-1} \{[[\omega_j(X), \omega_{i-j}(Y)], \omega_{q-i}(Z)] - [[\omega_j(X), \omega_{i-j}(Z)], \omega_{q-i}(Y)] +$$
$$+ [[\omega_j(Y), \omega_{i-j}(Z)], \omega_{q-i}(X)]\} = 0 .$$

Definition 4. A series of the type (12) is called a *formal deformation* of φ if the forms ω_p satisfy (13).

Proposition 4. *Let $\mathscr{H}^2_\varphi(M, g) = 0$, and let ω_1 be an infinitesimal deformation of φ. Then there exists a formal deformation $\omega_t = \varphi + \omega_1 t + \omega_2 t^2 + \ldots$ of φ.*

Proof. Suppose that we have already constructed the forms $\omega_2, \ldots, \omega_{q-1}$; we have to prove the existence of ω_q satisfying $\delta\omega_q = -\Psi_q$. Because of $\Psi_q \in Z^2_\varphi(M, g)$ and $\mathscr{H}^2_\varphi(M, g) = 0$, we have $\Psi_q \in B^2_\varphi(M, g)$ and the existence of the form ω_q follows.

Be given a Lie group G with the corresponding Lie algebra g. To make the calculations more simple, suppose that $G \subset \mathrm{GL}(N, \mathscr{R})$ for a convenable N; this supposition does not restrict the generality of our considerations. Further, let $\Phi : M \to G$ be a mapping such that

$$(16) \qquad \varphi = g^{-1}\, dg ;$$

of course, here I do suppose the existence of such a mapping. The precise meaning of (16) is as follows: Let $m \in M$, $X \in T_m(M)$, then

$$(17) \qquad \varphi(X) = \Phi(m)^{-1} \,.\, d\Phi_m(X) .$$

Because of $\varphi(X) = g^{-1} \,.\, Xg$, we have $g\, \varphi(X) = Xg$ and

$$Yg \,.\, \varphi(X) + g \,.\, Y\varphi(X) = YXg\,, \quad \text{i.e.,} \quad Y\varphi(X) = g^{-1} \,.\, YXg - \varphi(Y)\,\varphi(X) .$$

Thus the form (16) satisfies (1). This is also obvious from the fact that (16) is the restriction of the Maurer-Cartan form.

Definition 5. The formal deformations (12) and

$$(18) \qquad \tau_t = \varphi + \tau_1 t + \tau_2 t^2 + \ldots$$

of φ are said to be *p-equivalent* $(p = 1, 2, \ldots)$ if there is a mapping $h : M \times J \to G$

(with $J \subset \mathscr{R}$ a neighborhood of $0 \in \mathscr{R}$ and $h(m, 0) = e$) and forms $\psi_{p+1}, \psi_{p+2}, \ldots \in \in A^1$ such that

$$\omega_t = h^{-1}\tau_t h + h^{-1}\, d_M h + \psi_{p+1} t^{p+1} + \psi_{p+2} t^{p+2} + \ldots, \tag{19}$$

d_M denoting the differential satisfying $d_M t = 0$. The formal deformations of φ are *formally equivalent* if they are p-equivalent for $p = 1, 2, \ldots$

Proposition 5. *Let $\Phi : M \to G$ be a mapping inducing the form φ. Let $\mathscr{H}^1_\varphi(M, g) = 0$, and let the formal deformations ω_t, τ_t of φ satisfy $\omega_t - \tau_t \in Z^1_\varphi(M, g)$. Then ω_t and τ_t are formally equivalent.*

Proof. Obviously, it is sufficient to prove the following assertion: Let

$$\begin{aligned} \omega_t &= \varphi + \omega_1 t + \ldots + \omega_p t^p + \omega_{p+1} t^{p+1} + \ldots, \\ \tau_t &= \varphi + \omega_1 t + \ldots + \omega_p t^p + \tau_{p+1} t^{p+1} + \ldots \end{aligned} \tag{20}$$

be formal deformations of φ with $\delta\omega_{p+1} = \delta\tau_{p+1}$ and $\mathscr{H}^1_\varphi(M, g) = 0$; then ω_t and τ_t are $(p + 1)$-equivalent. On M, choose a coordinate neighborhood U with the local coordinates u^i ($i = 1, \ldots, \dim M$). On U, we have

$$\frac{\partial h}{\partial u^i} = h\varkappa_i, \quad \frac{\partial h}{\partial t} = h\varkappa \tag{21}$$

with $\varkappa_i, \varkappa : U \times \mathscr{R} \to g$. The integrability conditions of (21) are

$$\frac{\partial \varkappa_i}{\partial t} - \frac{\partial \varkappa}{\partial u^i} = [\varkappa_i, \varkappa], \quad \frac{\partial \varkappa_i}{\partial u^j} - \frac{\partial \varkappa_j}{\partial u^i} = [\varkappa_i, \varkappa_j]. \tag{22}$$

From $h(u, 0) = e$, we get $\varkappa_i(u, 0) = 0$. Let us write, in U,

$$\omega_t = A_i(u, t)\, du^i, \quad \tau_t = B_i(u, t)\, du^i; \tag{23}$$

we have

$$\frac{\partial^q A_i(u, 0)}{\partial t^q} = \frac{\partial^q B_i(u, 0)}{\partial t^q} \quad \text{for} \quad q = 0, \ldots, p. \tag{24}$$

Consider the mappings $h : M \times J \to G$ such that

$$\frac{\partial^\alpha h(u, 0)}{\partial t^\alpha} = 0 \quad \text{for} \quad \alpha = 1, \ldots, p, \quad \text{i.e.,} \tag{25}$$

$$\frac{\partial^\alpha \varkappa(u, 0)}{\partial t^\alpha} = 0 \quad \text{for} \quad \alpha = 0, \ldots, p - 1.$$

Further, consider the equation

$$(26) \qquad h(u, t)\, A_i(u, t) = B_i(u, t)\, h(u, t) + \varkappa_i(u, t)\,.$$

We get

$$\sum_{\alpha=0}^{p+1} \binom{p+1}{\alpha} \frac{\partial^{p-\alpha+1} h(u, t)}{\partial t^{p-\alpha+1}} \frac{\partial^\alpha A_i(u, t)}{\partial t^\alpha} =$$

$$= \sum_{\alpha=0}^{p+1} \binom{p+1}{\alpha} \frac{\partial^{p-\alpha+1} B_i(u, t)}{\partial t^{p-\alpha+1}} \frac{\partial^\alpha h(u, t)}{\partial t^\alpha} + \frac{\partial^{p+1} \varkappa_i(u, t)}{\partial t^{p+1}}\,,$$

i.e., taking regard of (24) and (25),

$$(27) \qquad \frac{\partial^{p+1} h(u, 0)}{\partial t^{p+1}} A_i(u, 0) + \frac{\partial^{p+1} A_i(u, 0)}{\partial t^{p+1}} =$$

$$= \frac{\partial^{p+1} B_i(u, 0)}{\partial t^{p+1}} + B_i(u, 0) \frac{\partial^{p+1} h(u, 0)}{\partial t^{p+1}} + \frac{\partial^{p+1} \varkappa_i(u, 0)}{\partial t^{p+1}}\,.$$

From (21_2) and (22_1), we obtain

$$\frac{\partial^{p+1} h}{\partial t^{p+1}} = \sum_{\alpha=0}^{p} \binom{p}{\alpha} \frac{\partial^\alpha h}{\partial t^\alpha} \frac{\partial^{p-\alpha} \varkappa}{\partial t^{p-\alpha}}\,,$$

$$\frac{\partial^{p+1} \varkappa_i}{\partial t^{p+1}} - \frac{\partial^{p+1} \varkappa}{\partial t^p\, \partial u^i} = \sum_{\alpha=0}^{p} \binom{p}{\alpha} \left[\frac{\partial^\alpha \varkappa_i}{\partial t^\alpha}, \frac{\partial^{p-\alpha} \varkappa}{\partial t^{p-\alpha}} \right]$$

and

$$\frac{\partial^{p+1} h(u, 0)}{\partial t^{p+1}} = \frac{\partial^p \varkappa(u, 0)}{\partial t^p}\,, \quad \frac{\partial^{p+1} \varkappa_i(u, 0)}{\partial t^{p+1}} = \frac{\partial^{p+1} \varkappa(u, 0)}{\partial t^p\, \partial u^i}\,.$$

The equation (27) may be rewritten as

$$(28) \qquad \frac{\partial^{p+1} A_i(u, 0)}{\partial t^{p+1}} - \frac{\partial^{p+1} B_i(u, 0)}{\partial t^{p+1}} = \frac{\partial^{p+1} \varkappa(u, 0)}{\partial u^i\, \partial t^p} + \left[A_i(u, 0)\,, \frac{\partial^p \varkappa(u, 0)}{\partial t^p} \right],$$

i.e.,

$$(29) \qquad \omega_{p+1} - \tau_{p+1} = \delta v\,.\,(p+1)!$$

valid now over all of M; here,

$$(30) \qquad v = \frac{\partial^p \varkappa(u, 0)}{\partial t^p}\,.$$

From $\delta(\omega_{p+1} - \tau_{p+1}) = 0$ and $\mathscr{H}^1_\varphi(M, g) = 0$ there follows the existence of a $v \in A^0$ satisfying (29); obviously, there is a mapping $h : M \times J \to G$ satisfying $h(u, 0) = e$,

(25) and (30). By means of this mapping, we substitute τ_t by a formally equivalent deformation

(31) $$\tau'_t = \varphi + \omega_1 t + \ldots + \omega_{p+1} t^{p+1} + \tau'_{p+2} t^{p+2} + \ldots$$

using (26). Clearly, $\delta\omega_t = \delta\tau'_t$.

Now, it is easy to see the validity of the following

Theorem 2. *Let $\Phi : M \to G$ be a mapping inducing the form φ, and let $\mathscr{H}^2_\varphi(M, g) = 0$. Then $\mathscr{H}^1_\varphi(M, g)$ is the parameter space of the set of formally non-equivalent formal deformations of φ. If $\mathscr{H}^1_\varphi(M, g) = 0$, then each formal deformation of φ is formally equivalent to φ, φ being considered as the formal deformation $\tau_t = \varphi$ of itself.*

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