EPI-ARCHIMEDEAN GROUPS

PAUL CONRAD, Lawrence

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An epi-archimedean group is a lattice-ordered group for which each $l$-homomorphic image is archimedean. Such groups are abelian and have been called hyper-archimedean or para-archimedean. They are at the opposite end of the spectrum from the free abelian $l$-groups.

Each group in the class $\mathcal{E}$ of epi-archimedean groups can be represented as a subdirect sum of reals. Let $\mathcal{S}$ be the class of all $l$-groups which have a representation as a subdirect sum of reals in which each element has finite range. Then $\mathcal{S} \subseteq \mathcal{E}$ and, in fact, an $l$-group belongs to $\mathcal{S}$ if and only if it is an $l$-subgroup of a vector lattice in $\mathcal{E}$ with an order unit. $\mathcal{S}$ is closed with respect to cardinal sums, $l$-subgroups, $l$-homomorphic images and $l$-hulls. If $G \in \mathcal{S}$ then the $v$-hull $G^v$ of $G$ is an $a$-closure of $G$ and $G^v$ is the unique $a$-closure of $G$ in $\mathcal{S}$ (Theorem 5.1). Also each $G \in \mathcal{S}$ has a unique essential closure in $\mathcal{S}$ (Theorem 2.3).

In Section 4 the subdirect sums of integers that have been studied by Specker, Nobeling and others are shown to be those $l$-subgroups of $\Pi \mathbb{Z}$ that are generated by characteristic functions. These can also be characterized as rings of bounded integral functions. Such groups belong to $\mathcal{S}$ and the group generated by the set of all singular elements in an arbitrary $l$-group is such a group. We study these groups and also their $v$-hulls. Epi-archimedean $f$-rings are investigated in Sections 4 and 6.

DEFINITIONS AND NOTATION

Let $G$ be an $l$-group. If $g \in G$ then $G(g)$ will denote the convex $l$-subgroup of $G$ generated by $g$

$$G(g) = \{ x \in G \mid |x| \leq n|g| \text{ for some } n > 0 \}.$$ 

If $A$ is a subset of $G$ then $A'$ will denote the polar of $A$.

$$A' = \{ x \in G \mid |x| \wedge |a| = 0 \text{ for all } a \in A \}.$$
The cardinal sum (product) of a set \( \{G_i \mid i \in I\} \) of \( l \)-groups will be denoted by \( \Sigma G_i (\prod G_i) \) or if \( I \) is finite by \( G_1 \oplus \ldots \oplus G_n \).

A prime subgroup \( M \) of \( G \) is a convex \( l \)-subgroup such that the convex \( l \)-subgroups that contain it form a chain. \( M \) is a minimal prime if and only if \( 0 < g \in M \) implies \( g' \not\in M \). A convex \( l \)-subgroup \( N \) of \( G \) that is maximal without some element \( g \) is prime and the intersection \( N^* \) of all convex \( l \)-subgroups of \( G \) that properly contain \( N \) covers \( N \). \( (N^*, N) \) or just \( N \) is called a value of \( g \). \( N \) is called regular.

An \( l \)-group \( G \) is laterally complete if each disjoint subset of \( G \) has a least upper bound. If \( G \) is a subdirect sum and a sublattice of a cardinal product of totally ordered groups then \( G \) is called representable.

Let \( H \) be an \( l \)-subgroup of \( G \). Then \( G \) is an \( a \)-extension of \( H \) if for each \( 0 < g \in G \) there exists \( 0 < h \in H \) such that \( nh > g \) and \( ng > h \) for some \( n > 0 \). In this case \( g \) and \( h \) are said to be \( a \)-equivalent. \( G \) is an \( a \)-extension of \( H \) if and only if the map \( L \to L \cap H \) is a one to one map of the set of all convex \( l \)-subgroups of \( G \) onto those of \( H \). \( G \) is \( a \)-closed if it admits no proper \( a \)-extensions and an \( a \)-closed \( a \)-extension of \( G \) is called an \( a \)-closure of \( G \).

\( H \) is a large \( l \)-subgroup of \( G \) or \( G \) is an essential extension of \( H \) if for each non-zero convex \( l \)-subgroup \( L \) of \( G \), \( L \cap H \neq 0 \). Note that an \( a \)-extension is an essential extension.

Each archimedean \( l \)-group \( G \) has a unique essential closure \( G^e \) in the class \( \mathcal{A} \) of archimedean \( l \)-groups [13], (i.e., \( G^e \) is an essential extension of \( G \) that admits no proper essential extensions in \( \mathcal{A} \)). Also \( G \) is contained in a unique minimal vector lattice \( G^* \) in \( \mathcal{A} \) called the \( v \)-hull of \( G \). \( G \) is large in \( G^e \) (see [8] and [15]). We shall denote the Dedekind-MacNeille completion of \( G \) by \( G^\wedge \), and the divisible closure or the injective hull of \( G \) by \( G^d \).

Finally \( R \) will denote the additive group of real numbers with the natural order and \( M \prec R \) denotes that \( M \) is a group that is \( o \)-isomorphic to a subgroup of \( R \).

1. EPI-ARCHIMEDEAN \( l \)-GROUPS

The following theorem is basic for the theory developed in this paper.

**Theorem 1.1.** For an \( l \)-group \( G \) the following are equivalent.

1) \( G \) is epi-archimedean.

2) Each proper prime subgroup of \( G \) is maximal and hence minimal.

3) \( G = G(g) \oplus g' \) for each \( g \in G \).

4) If \( 0 < f, g \in G \), then \( [f - (mg \wedge f)] \wedge g = 0 \) for some \( m > 0 \).

5) \( G \) is \( l \)-isomorphic to an \( l \)-subgroup \( G^* \) of \( \Pi_1 \) and for each \( 0 < x, y \in G^* \) there exists an \( n > 0 \) such that \( nx_i > y_i \) for all \( x_i \neq 0 \).
6) If \( 0 < f, g \in G \) then \( f \wedge ng = f \wedge (n + 1)g \) for some \( n > 0 \).
Moreover each representation of an epi-archimedean \( l \)-group as a group of real valued functions must satisfy (5).

The history of this theorem is as follows: AMEMIYA [1] proved 1 \( \iff \) 3 for vector lattices; BAKER [2] proved 1 \( \iff \) 2 for vector lattices; PEDERSEN [19] proved 5 \( \iff \) 2; LUXEMBURG and MOORE [17] proved 1 \( \iff \) 3 for vector lattices; ZANNEN [23] proved 3 \( \iff \) 1 for vector lattices; BIGARD [5] proved 2 \( \iff \) 3; BIGARD, CONRAD and WOLFENSTEIN [7] proved 1 \( \iff \) 2 \( \iff \) 3.

Also conditions on pairs of elements from \( G^+ \) that are equivalent to (4) or (6) were derived by most of these authors.

Proof of Theorem. (1 \( \iff \) 2) \( G \) is archimedean and hence abelian. If \( M \) is a proper prime subgroup of \( G \) then \( G/M \) is an archimedean \( o \)-group and hence \( M \) is maximal.

(2 \( \iff \) 3) If \( G \neq G(g) \oplus g' \) then \( G(g) \oplus g' \subseteq M \subset G \) for some prime subgroup \( M \) which is necessarily minimal. But then \( 0 < g \in M \) and so \( g' \nleq M \), a contradiction.

(3 \( \iff \) 4) \( f = f_1 + f_2 \in G(g) \oplus g' \). Thus \( mg \leq f_1 \) for some \( m > 0 \) and so \( mg \land f = f_1 \). Therefore
\[
[f - (mg \land f)] \land g = f_2 \land g = 0.
\]

(4 \( \iff \) 1) First \( G \) is archimedean. For suppose that \( 0 \leq f, g \in G \) and \( ng \leq f \) for all \( n \). If
\[
0 = [f - (mg \land f)] \land g = (f - mg) \land g
\]
then \( mg = f \land (m + 1)g = (m + 1)g \) and so \( g = 0 \). Now if \( \sigma \) is an \( l \)-homomorphism of \( G \) then clearly \( G\sigma \) also satisfies (4) and so \( G\sigma \) is archimedean.

(1, 2, 3 \( \iff \) 5) \( G \) is abelian and each prime subgroup is maximal. Thus without loss of generality \( G \) is a subdirect sum of reals, \( G \subseteq \prod R_i \). Pick \( 0 < x, y \in G \). \( y = a + b \in G(x) \oplus x' \) and so \( nx > a \) for some \( n > 0 \). Thus \( nx_i > x_i \) provided \( x_i \neq 0 \). Note that we have shown that every representation of \( G \) as a subdirect sum of reals satisfies (5).

(5 \( \iff \) 6) There exists \( n > 0 \) such that \( ng_i > f_i \) for all \( g_i \neq 0 \). Thus
\[
(f \land ng)_i = \begin{cases} f_i & \text{if } g_i > 0 \\ 0 & \text{if } g_i = 0 \end{cases} = (f \land (n + 1)g)_i
\]
and so \( f \land ng = f \land (n + 1)g \).

(6 \( \iff \) 1) First \( G \) is archimedean. For if \( mg \leq f \) for all \( m \) then \( ng = f \land ng = f \land (n + 1)g = (n + 1)g \) and so \( g = 0 \). Next each \( l \)-homomorphic image of \( G \) satisfies (6) and so is archimedean.

Let \( \mathcal{E} \) be the class of all epi-archimedean \( l \)-groups. It follows at once from (2) that an \( l \)-group \( G \) belongs to \( \mathcal{E} \) if and only if \( G \) is representable and each totally ordered \( l \)-homomorphic image is archimedean. One only needs the fact that the minimal prime subgroups of a representable group are normal.
If \( G \in \mathcal{E} \) has an order unit \( u \), then \( u' = 0 \) and so by (3) \( G = G(u) \). Thus \( u \) is a strong order unit. Also each \( l \)-ideal of \( G \in \mathcal{E} \) that contains an order unit is a cardinal summand. It follows from (4) or (6) that \( \mathcal{E} \) is closed with respect to \( l \)-subgroups, \( l \)-homomorphic images and cardinal sums. But \( \prod_{i=1}^{\infty} R_i \notin \mathcal{E} \) and so \( \mathcal{E} \) is not closed with respect to cardinal product. It follows from (2) that \( \mathcal{E} \) is closed with respect to \( a \)-extensions, but Example 7.1 shows that \( G \in \mathcal{E} \) need not have a unique \( a \)-closure.

If \( G \) is an epi-archimedean \( l \)-subgroup of an \( l \)-group \( H \) then it follows from (4) that there is a maximal \( l \)-subgroup of \( H \) that contains \( G \) and is epi-archimedean.

In Section 3 we show that each \( l \)-group \( H \) admits a largest epi-archimedean \( l \)-ideal. This is the epi-archimedean kernel of \( H \) introduced by Martínez [20].

Suppose that each proper \( l \)-homomorphic image of the \( l \)-group \( G \) is archimedean and let \( K \) be the intersection of all the non-zero \( l \)-ideals of \( G \). Then there are three possibilities.

I. If \( K = G \) then \( G \) is \( l \)-simple.

II. If \( K = 0 \) then \( G \) is epi-archimedean.

Proof. Let \( \{L_\alpha \mid \alpha \in A\} \) be the set of non-zero \( l \)-ideals of \( G \). Then there is a natural \( l \)-isomorphism of \( G \) into the abelian group \( \prod_{\alpha \in A} G/L_\alpha \) and so \( G \) is abelian. Thus if \( P \neq 0 \) is a prime subgroup of \( G \) then \( G/P \) is epi-archimedean and totally ordered and hence \( G/P \vartriangleleft R \). Thus each prime subgroup is a maximal \( l \)-ideal and so \( G \) is epi-archimedean.

III. If \( 0 \neq K \neq G \) then \( G \) is an extension of \( K \) by an epi-archimedean \( l \)-group \( B \cong G/K \).

The following are examples of Case III.

a) \( G \) is a lexicographic extension of an \( l \)-simple \( l \)-group \( K \) by \( R \). In particular, the lexicographic extension of \( R \) by \( R \) is such a group.

b) Let \( G \) be the wreath product of \( Z \) by \( Z \). Then \( G \) is an extension of \( K \) by \( Z \), where \( K \) is the direct product of a countable number of copies of \( Z \). Let \( K \) have the cardinal order and let \( G \) be the lexicographic extension of \( K \) by \( Z \).

c) Let \( G \) be the restricted wreath product of \( Z \) by \( Z \). Then \( G \) is an extension of \( K \) by \( Z \), where \( K \) is the direct sum, of a countable number of copies of \( Z, K = \bigoplus Z_i \). Order \( K \) lexicographically and let \( G \) be the lexicographic extension of \( K \) by \( Z \). Note that \( G \) is an \( o \)-group with one proper normal convex subgroup.

We have the following special cases.

Corollary. If \( G \) is representable then \( G \) is epi-archimedean or \( G \) is an \( o \)-group with one or no proper normal convex subgroup.

Proof. The intersection of all prime subgroups of \( G \) equals 0 and each minimal prime subgroup is normal. Thus if 0 is not prime it follows that \( K = 0 \).
Corollary. If $G$ is abelian then $G$ is epi-archimedean or $G$ is an $o$-group with rank 2.

Lemma A. (a) If $G$ is an $l$-subgroup of $\Pi_{i}R_{i}$ and for each $0 < g \in G$ there exists $0 < r, s \in R$ such that $r < g_{i} < s$ for each $g_{i} \neq 0$ then $G \in \mathcal{E}$.

(b) If $G \in \mathcal{E}$ has an order unit $u$ then there exists an $l$-isomorphism $\tau$ of $G$ into $\Pi_{i}R_{i}$ with $u\tau = (1, 1, 1, \ldots)$. Moreover, each such representation $G\tau$ of $G$ satisfies (a).

(c) If $G \in \mathcal{E}$ and $G$ is an $f$-ring with no nilpotent elements then there exists a ring $l$-isomorphism $\tau$ of $G$ into $\Pi_{i}R_{i}$. Moreover, each such representation $G\tau$ of $G$ satisfies (a).

Proof. (a) For $0 < x, y \in G$ there exist $0 < r, s \in R$ such that $r < x_{i}$ for all $x_{i} \neq 0$ and $y_{i} < s$ for all $i$. Pick an integer $n$ such that $nr > s$. Then $nx_{i} > y_{i}$ for all $x_{i} \neq 0$ and so by (5) $G \in \mathcal{E}$.

(b) By Theorem 1.1 we may assume $G = G(u) \subseteq \Pi_{i}R_{i}$ and $u = (1, 1, 1, \ldots)$. If $0 < g \in G$ then by (5) there exist positive integers $m, n$ such that $ng_{i} > u_{i} = 1$ for all $g_{i} \neq 0$ and $m = mu_{i} > g_{i}$ for all $i$. Thus $m > g_{i} > 1/n$ for all $g_{i} \neq 0$.

(c) Each prime $l$-ideal $M$ of $G$ is minimal and hence the join of principal polars. Thus $M$ is a ring ideal and so $G/M$ is $o$-isomorphic to a subring of $R$. Let $\{M_{i} \mid i \in I\}$ be the set of all prime ideals. Then there exists a ring $l$-isomorphism of $G$ into $\Pi_{i}G/M_{i}$ and hence into $\Pi_{i}R_{i}$.

So (without loss of generality) we assume that $G$ is an $l$-subring of $\Pi_{i}R_{i}$ and consider $0 < g \in G$. By (5) there exist positive integers $m$ and $n$ such that $mg > g^{2}$ and $ng_{i} > g_{i}$. Thus $m > g_{i} > 1/n$ for all $g_{i} \neq 0$.

We have not been able to answer the following questions.

Does each $G \in \mathcal{E}$ have a representation that satisfies (a)?

Find an example of $G \in \mathcal{E}$ that is not contained in an epi-archimedean $f$-ring with zero radical.

Suppose $G \subseteq \Pi_{i}R_{i}$ satisfies (a). Does the $l$-subring of $\Pi R_{i}$ generated by $G$ belong to $\mathcal{E}$?

Baker [2] defines an element $g$ in $\Pi R_{i}$ to be a step function if $g$ has finite range. Let $\mathcal{F}$ be the class of all $l$-groups $G$ which have a representation as real valued step functions. Then by (5) $\mathcal{F} \subseteq \mathcal{E}$ and by Example 7.1 $\mathcal{F} \neq \mathcal{E}$.

Now clearly $\mathcal{F}$ is closed with respect to $l$-subgroups and cardinal sums. We shall show that $\mathcal{F}$ is closed with respect to $l$-homomorphic images and $v$-hulls and that the $v$-hull $G^{v}$ of $G \in \mathcal{F}$ is the unique $a$-closure of $G$ in $\mathcal{F}$.

Also both $\mathcal{F}$ and $\mathcal{E}$ are closed with respect to divisible hulls. For if $G \in \mathcal{F}$ then we may assume that $G$ is an $l$-subgroup of step functions in $\Pi R_{i}$ and so $G \subseteq G^{d} \subseteq \Pi R_{i}$.

If $x \in G^{d}$ then $nx \in G$ for some $n > 0$ and hence $x$ must be a step function. If $G \in \mathcal{E}$
then we may assume that $G \subseteq G^d \subseteq \Pi R_i$. If $0 < x, y \in G^d$ then $nx, ny \in G$ for some $n > 0$ and hence by (5) there exists $m > 0$ such that $mnx_i > ny_i$ for all $x_i \neq 0$. Therefore $mx_i > y_i$ for all $x_i \neq 0$ and so by (5) again $G \in \mathcal{S}$.

**Proposition 1.2.** If $G$ is an epi-archimedean $l$-subgroup of $\Pi R_i$ that contains the long constants then $G$ consists of step functions and so belongs to $\mathcal{S}$.

**Proof.** If $0 < g \in G$ then by Lemma A there exists $0 < r, s \in R$ such that $r < g_i < s$ for all $g_i \neq 0$. Suppose (by way of contradiction) that $g$ has infinite range. Then $\{g_i \mid i \in I \text{ and } g_i \neq 0\}$ is an infinite subset of the compact set $[r, s]$ and so has a limit point $a$ in $[r, s]$. Let $\bar{a}$ be the long constant $(a, a, a, \ldots)$. Each open interval of $R$ that contains $a$ must contain a component $g_i \neq a$ or $0$ of $g$.

Case I. A sequence of the $g_i$ converge to $a$ from below. Then $(\bar{a} - g) \vee 0$ has a sequence of strictly positive components that converge to zero which contradicts Lemma A.

Case II. A sequence of the $g_i$ converge to $a$ from above. Then $(g - \bar{a}) \vee 0$ has a sequence of strictly positive components that converge to zero.

**Corollary I.** Let $G$ be an epi-archimedean vector lattice with an order unit $u$. Then there exists an $l$-isomorphism $\tau$ of $G$ into $\Pi R_i$ with $u \tau = (1, 1, 1, \ldots)$ and $G \tau$ consists of step functions and so $G \in \mathcal{S}$.

**Corollary II.** An $l$-group $G$ belongs to $\mathcal{S}$ if and only if $G$ is an $l$-subgroup of an epi-archimedean vector lattice $H$ with a unit.

**Proof.** If $G \in \mathcal{S}$ then we may assume that $G \subseteq \Pi R_i$ and each $g$ in $G$ is a step function. Thus $G$ is an $l$-subgroup of the group $H$ of all step functions in $\Pi R_i$. The converse follows from Corollary I.

An $l$-group $G$ is locally $\mathcal{S}$ (locally $\mathcal{S}$) if each $G(g)$ belongs to $\mathcal{S}(\mathcal{S})$. Clearly locally $\mathcal{S}$ implies locally $\mathcal{S}$ and it follows from (4) that $G$ is locally $\mathcal{S}$ if and only if $G \in \mathcal{S}$. By Corollary I each epi-archimedean vector lattice is locally $\mathcal{S}$.

Now Bernau [3] and Baker [2] both give an example of an epi-archimedean vector lattice that does not belong to $\mathcal{S}$. Therefore locally $\mathcal{S}$ does not imply $\mathcal{S}$. Thus any elementwise definition of $\mathcal{S}$ must involve an infinite number of elements; otherwise locally $\mathcal{S}$ implies $\mathcal{S}$.

Finally note that Example 7.1 shows that an epi-archimedean $l$-group with an order unit need not belong to $\mathcal{S}$. Let $G$ be as in this example. Then since $G$ has an order unit so does its $v$-hull $G^v$. Thus if $G^v \in \mathcal{S}$ then $G^v \in \mathcal{S}$ and hence so does $G$, a contradiction. Thus the $v$-hull of an epi-archimedean group need not be epi-archimedean.
Theorem 1.3. \( \mathcal{S} \) is closed with respect to cardinal sums, \( l \)-subgroup, \( l \)-homomorphic images and \( v \)-hulls.

Proof. The first two are clear. Let \( K \) be an \( l \)-ideal of \( G \in \mathcal{S} \). Then without loss of generality \( G \) is an \( l \)-subgroup of the group \( H \) of all step functions in \( \Pi R \). Let

\[
K_\mu = \bigcap\{L \mid K \subseteq L \text{ and } L \text{ is an } l \text{-ideal of } H\}.
\]

This is the \( l \)-ideal of \( H \) that is generated by \( K \) and \( K_\mu \cap G = K \). Moreover,

\[
\frac{G}{K} = \frac{G}{K_\mu \cap G} \cong \frac{K_\mu + G}{K_\mu} \subseteq \frac{H}{K_\mu}
\]

but \( H/K_\mu \) is an epi-archimedean vector lattice with an order unit and so belongs to \( \mathcal{S} \).

It follows from a result of BLEIER [8] that the \( v \)-hull \( G^v \) of \( G \) is the intersection of all the \( l \)-subspaces of \( H \) that contain \( G \).

Another proof. \( G \subseteq G_\mu \subseteq (G_\mu)^\wedge \subseteq \Pi R \) and \( G^v \) is the \( l \)-subspace of \( (G_\mu)^\wedge \) generated by \( G \). Clearly \( G \subseteq H \cap (G_\mu)^\wedge \) an \( l \)-subspace of \( (G_\mu)^\wedge \). Thus \( G^v \subseteq H \) and so \( G^v \in \mathcal{S} \).

Corollary. If \( G \in \mathcal{S} \) has a unit then \( G \in \mathcal{S} \) if and only if the \( v \)-hull of \( G \) belongs to \( \mathcal{S} \).

Proof. If \( u \) is a unit for \( G \) then it is also a unit for \( G^v \) and so by Corollary I of Proposition 1.2 \( G^v \in \mathcal{S} \) implies \( G \in \mathcal{S} \) and so \( G \in \mathcal{S} \). Conversely if \( G \in \mathcal{S} \) then so does \( G^v \) and hence \( G^v \in \mathcal{S} \).

Lemma. If \( 0 \neq A \) is a subgroup of an archimedean o-group \( B \) and \( \alpha \) is an o-isomorphism of \( A \) into \( R \) then there exists a unique extension of \( \alpha \) to an o-isomorphism of \( B \) into \( R \).

This follows from the fact that each o-isomorphism of a subgroup of \( R \) into a subgroup of \( R \) is a multiplication by a positive real number.

Lemma. If \( G \) is an \( l \)-subgroup of an \( l \)-group \( H \) and \( M \) is a regular subgroup of \( G \) then \( M = G \cap N \) for a regular subgroup \( N \) of \( H \).

Proof. Let \( Y \) be the convex \( l \)-subgroup of \( H \) generated by \( M \). Then \( Y \cap G = M \). Now \( M \) is maximal without some \( g \in G \) and so \( g \notin Y \). Thus \( Y \subseteq N \) a value of \( g \) in \( H \). \( N \cap G \supseteq M \) and \( g \notin N \cap G \) a convex \( l \)-subgroup of \( G \). Therefore \( N \cap G = M \).
**Proposition 1.4.** If $G$ is a large $l$-subgroup of $H \in \mathcal{E}$ and $\tau$ is an $l$-isomorphism of $G$ into $\Pi R_i$ then there exists an extension of $\tau$ to an $l$-isomorphism of $H$ into $\Pi R_i$.

**Proof.** We may assume (without loss of generality) that $G_i = \{ g \in G \mid (g\tau)_i = 0 \} \neq 0$ for each $i \in I$. Thus each $G_i$ is a regular subgroup of $G$. Pick $H_i$ regular in $H$ such that $H_i \cap G = G_i$. Then

$$\frac{G}{G_i} = \frac{G}{H_i \cap G} \simeq \frac{H_i + G}{H_i} \subseteq \frac{H}{H_i}$$

and since $H \in \mathcal{E}$, $H/H_i \lt R$. The map $g \rightarrow (g\tau)_i$ is an $l$-homomorphism of $G$ into $R_i$ with kernel $G_i$. Thus

$$H_i + g \rightarrow H_i \cap G + g = G_i + g \rightarrow (g\tau)_i$$

is an $o$-isomorphism of $(H_i + G)/H_i$ into $R_i$ and so there exists a unique extension to an $o$-isomorphism $\alpha_i$ of $H/H_i$ into $R_i$. Now for $h \in H$ and $g \in G$ we consider the maps

$$h \rightarrow (\ldots, h_i + h_i, \ldots) \in \Pi H/H_i \rightarrow (\ldots, (H_i + h) \alpha_i, \ldots) \in \Pi R_i$$

$$g \rightarrow (\ldots, h_i + g, \ldots) \rightarrow (\ldots, (g\tau)_i, \ldots) = g\tau.$$ 

Thus we have extended $\tau$ to an $l$-homomorphism of $H$ into $\Pi R_i$ with kernel $\bigcap H_i$. Now $(\bigcap H_i) \cap G = \bigcap (H_i \cap G) = \bigcap G_i = 0$ and since $G$ is large in $H$ we have $\bigcap H_i = 0$. Thus this extended map is an $l$-isomorphism of $H$ into $\Pi R_i$.

**Corollary.** Each $G \in \mathcal{E}$ admits an essential closure $H$ in $\mathcal{E}$, and if $G \subseteq \Pi R_i$ then $G \subseteq H \subseteq \Pi R_i$.

We shall show in section 7 that $H$ need not be unique even if $G \in \mathcal{S}$, but each $G \in \mathcal{S}$ has a unique essential closure in $\mathcal{S}$.

Suppose that $G$ is archimedean and has a strong unit $u$ then there exists an $l$-isomorphism $\tau$ of $G$ such that

$$G\tau \subseteq \Pi R_i \quad \text{and} \quad u\tau = (1, 1, 1, \ldots)$$

**Proposition 1.5.** $G \in \mathcal{S}$ if and only if each $g\tau$ is a step function in this representation.

**Proof.** ($\leftrightarrow$) Trivial.

($\rightarrow$) The $v$-hull $H$ of $G$ is an essential extension of $G$ and so by Proposition 1.4 $\tau$ can be extended to an $l$-isomorphism $\varphi$ of $H$ into $\Pi R_i$. Now the long constant belongs to $H_\varphi$ and so $H_\varphi$ consists of step functions and hence so does $G\tau$ (see Proposition 1.2).
2. THE ESSENTIAL CLOSURE OF $G \in \mathcal{F}$

If $B$ is an essential extension of an $l$-group $A$ and $u$ is an order unit in $A$ then $u$ is also an order unit in $B$. For suppose (by way of contradiction) that $0 < b \in B$ and $b \wedge u = 0$. Then $b^* \wedge u^* = 0$, where the polar operations are in $B$, and since $b^* \wedge A \neq 0$ we have $a \wedge u = 0$ for some $0 < a \in A$, a contradiction.

**Proposition 2.1.** If $G \in \mathcal{F}$ then there exists an essential extension of $G$ in $\mathcal{F}$ that contains an order unit.

**Proof.** We may assume that $G$ is an $l$-subgroup of

$$H = \text{all step functions in } \Pi R_i.$$ 

Let $W$ be the $l$-subspace of $H$ generated by $G$ and $u = (1, 1, \ldots)$ and let $B$ be an $l$-ideal of $W$ that is maximal with respect to $B \cap G = 0$. Then

$$G \cong \frac{B \oplus G}{B} \subseteq \frac{W}{B} \in \mathcal{F}$$

and $B + u$ is a unit in $W/B$. Now if $J/B$ is a non-zero $l$-ideal of $W/B$ then $J \supseteq B$ and hence $J \cap G = 0$. Thus $W/B$ is an essential extension of $(B \oplus G)/B$.

**Corollary 1.** If $G \in \mathcal{F}$ and $G$ is an $l$-subgroup of $\Pi R_i$ then there exists $w \in \Pi R_i$ such that each $w_i > 0$ and for which $Gw = \{gw \mid g \in G\}$ consists of step functions.

**Proof.** Let $K$ be an essential extension of $G$ in $\mathcal{F}$ that contains an order unit $u$. By Proposition 1.4 we may assume that $G \subseteq K \subseteq \Pi R_i$ and we may also assume that $G_i = \{g \in G \mid g_i \neq 0\} \neq 0$ for each $i \in I$. Since $u$ is a strong order unit for $K$ we have $u_i > 0$ for each $i \in I$.

Let $w$ be the multiplicative inverse of $u$ in the ring $\Pi R_i$. The map $x \to xw$ is an $l$-automorphism of the group $\Pi R_i$ and $Kw \in \mathcal{F}$ and contains $(1, 1, 1, \ldots)$. Thus by Proposition 1.5 $Kw$ consists of step functions and hence so does $Gw$.

**Corollary II.** If $G$ is an $l$-subgroup of $\Pi R_i$ then so is $K = G + \Sigma R_i$. Moreover, if $G \in \mathcal{F}$ or $\mathcal{E}$ then so does $K$.

**Proof.** Since $\Sigma R_i$ is an $l$-ideal of $\prod R_i$ it follows that $K$ is an $l$-subgroup of $\Pi R_i$. If $0 < x, y \in K$ then they differ in only a finite number of places from elements in $G$ and so it follows from (5) of Theorem 1.1 that if $G \in \mathcal{E}$ so does $K$.

If $G \in \mathcal{F}$ then by Corollary 1 there is a $w \in \Pi R_i$ such that each $w_i > 0$ and $Gw$ consists of step functions. Clearly $(\Sigma R_i)w = \Sigma R_i$ and hence $Kw$ also consists of step functions. Therefore $K \in \mathcal{F}$.
If $G \in \mathcal{S}$ has a basis then ([11], p. 3.15) there is an $l$-isomorphism $\sigma$ of $G$ such that

$$\Sigma_i T_i \subseteq G\sigma \subseteq \Pi_i R_i,$$

where $0 \neq T_i \subseteq R$

and by Corollary I we may assume that $G\sigma$ consists of step functions. It follows that the set of all step functions in $\Pi R_i$ is the essential closure of $G\sigma$ in $\mathcal{S}$.

Let $X$ be a Stone space (that is, a compact extremely disconnected Hausdorff topological space) and let $S(X)$ be the group of all step functions in the $l$-group $C(X)$ of all continuous real valued functions on $X$. Then $S(X)$ is the subspace of $C(X)$ that is generated by the characteristic functions on the clopen subsets of $X$ and $C(X)$ is an essential extension of $S(X)$.

**Proposition 2.2.** $S(X)$ is essentially closed in $\mathcal{E}$ and hence in $\mathcal{S}$.

**Proof.** Suppose that $S(X) \subseteq K \in \mathcal{E}$, where $K$ is an essential extension of $S(X)$. Now $S(X)$ and $K$ have the same Boolean algebra of polars [13] and hence the same associated Stone space, namely $X$. Thus (see [4]) we can embed $K$ into $C(X)$ so that $(1, 1, 1, \ldots)$ is mapped onto itself. This induces the identity map on $S(X)$ and so we may assume that

$$S(X) \subseteq K \subseteq C(X).$$

Here we use the fact that $(1, 1, 1, \ldots)$ is also an order unit for $K$ and hence a strong order unit. Now by Proposition 1.2 it follows that $K$ consists of step functions and so $K = S(X)$.

**Theorem 2.3.** Each $G \in \mathcal{S}$ has a unique essential closure in $\mathcal{S}$ namely the $l$-group $S(X)$ of all step functions in $C(X)$, where $X$ is the Stone space associated with the Boolean algebra of polars of $G$. Moreover $S(X)$ is essentially closed in $\mathcal{E}$.

**Proof.** Let $H$ be an essential extension of $G$ in $\mathcal{S}$. By Proposition 2.1 there is an essential extension $K$ of $H$ in $\mathcal{S}$ that has a unit $u$. Also the $v$-hull of $K$ belongs to $\mathcal{S}$, and so we may assume that $K$ is a vector lattice. Thus we can embed $K$ into $C(X)$ so that $u = (1, 1, 1, \ldots)$ and so that $C(X)$ is an essential extension of $K$.

$$G \subseteq H \subseteq K \subseteq C(X).$$

Thus by Proposition 1.2 $K$ consists of step functions and so

$$G \subseteq H \subseteq K \subseteq S(X).$$

Thus $S(X)$ is an essential extension of $G$ that is essentially closed in $\mathcal{S}$.

Now suppose that $T$ is an essential closure of $G$ in $\mathcal{S}$. Then clearly $T$ is a vector lattice with an order unit $u$ and so there is an $l$-isomorphism $\sigma$ of $T$ onto a large subgroup of $C(X)$ such that $u\sigma = (1, 1, 1, \ldots)$. Then as above $S(X)$ is an essential extension of $T\sigma$ and so $T\sigma = S(X)$. 

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Note that $S(X) \in \mathcal{S}$ but $S(X)^\wedge = C(X) \not\in \mathcal{S}$ unless $S(X) = C(X)$. Thus $\mathcal{S}$ and $\mathcal{S}$ are not closed with respect to Dedekind-MacNeille completions. For $S(X)$ is dense in $C(X)$ and so $S(X)^\wedge$ is the $l$-ideal generated by $S(X)$ which is $C(X)$ (see [10]).

3. SOME PROPERTIES OF EPI-ARCHIMEDEAN $l$-GROUPS

Most of the theory in this section is not new, but the proofs given here are shorter than those in print.

**Proposition 3.1.** If $G$ is a laterally complete epi-archimedean $l$-group then $G \simeq T_1 \oplus \cdots \oplus T_s$ where each $T_i \subseteq R$.

**Proof.** Suppose (by way of contradiction) that $a_1, a_2, \ldots$ is an infinite disjoint subset of $G$ then so is $a_1, 2a_2, 3a_3, \ldots$. Let $x = \bigvee a_k$ and $y = \bigvee k a_k$. Then clearly $x$ and $y$ do not satisfy (5) of Theorem 1.1, a contradiction. Thus $G$ has a finite basis and so $G \simeq T_1 \oplus \cdots \oplus T_s$.

**Proposition 3.2.** If $G$ is an epi-archimedean $l$-group, $2G = G$ and each countable bounded disjoint subset has a least upper bound, then $G \simeq \Sigma T_\lambda$, where each $T_\lambda \subseteq R$. Thus if $G$ also has an order unit then $G \simeq T_1 \oplus \cdots \oplus T_n$.

**Proof.** It suffices to show that each $G(g)$ has a finite basis; for then $G$ has a basis and so we may assume

$$\Sigma T_\lambda \subseteq G \subseteq \Pi T_\lambda$$

and since each $G(g)$ has a finite basis it follows that $G = \Sigma T_\lambda$.

If $G(g)$ does not have a finite basis then there exists a countable disjoint subset $g_1, g_2, \ldots$ of $G(g)$. Since each $g_k$ is divisible by 2 we may assume that $g_k \leq g$ for all $k$ and, hence, without loss of generality, $g = \bigvee g_k$. Now let $h = \bigvee (1/2^k) g_k$. Then $h$ is a unit in $G(g)$ and hence a strong unit, but clearly $nh = \bigvee (n/2^k) g_k \not\leq g$ for any $n$, which contradicts (5) of Theorem 1.1.

**Remarks.** We can replace the hypothesis epi-archimedean by archimedean and each order unit in each $G(g)$ is a strong order unit.

If $G$ is the $l$-group of all bounded sequences of integers then $G \in \mathcal{S}$ and each bounded disjoint subset has a least upper bound. Thus the hypothesis $2G = G$ cannot be dispensed with. Note that $(1, 0, 0, \ldots), (0, 1/2, 0, 0, \ldots), (0, 0, 1/2^2, 0, 0), \ldots$ has no least upper bound in $G^2$; so we cannot use the divisible hull of $G$. 202
Corollary I. If $G$ is a laterally complete epi-archimedean vector lattice then 
$G = R_1 \oplus \cdots \oplus R_n$.
This is also a corollary of Proposition 3.1.

Corollary II. (Bigard, Bernau). If $G$ is a $\sigma$-complete epi-archimedean vector lattice
then $G = \Sigma R_\lambda$. Thus if $G$ has a unit then $G = R_1 \oplus \cdots \oplus R_n$.

Proposition 3.3. (Bernau). If $G$ is an epi-archimedean vector lattice with countable
dimension as a real vector space then $G = \Sigma G(f_i)$ and so $G \in \mathcal{S}$.

Proof. Let $g_1, g_2, \ldots$ be a positive basis for the vector space $G$ and let
$f_1 = g_1$
$f_2 = b_2$, where $g_2 = a_2 + b_2 \in G(f_1) \oplus G(f_1)'$

\vdots

$f_{n+1} = b_{n+1}$, where $g_{n+1} = a_{n+1} + b_{n+1} \in G(f_1 + \cdots + f_n) \oplus G(f_1 + \cdots + f_n)'$

Then $f_1, f_2, \ldots$ are disjoint and $g_1, \ldots, g_n \in G(f_1 + \cdots + f_n) = G(f_1) \oplus \cdots \oplus G(f_n)$. Now $x \in G$ is a linear combination of a finite number of the $g_i$, say $g_1, \ldots, g_n$ and so $x \in G(f_1) \oplus \cdots \oplus G(f_n)$. Thus $G = \Sigma G(f_i)$. By Corollary I to Proposition 1.2 each $G(f_i) \in \mathcal{S}$ and hence $G \in \mathcal{S}$.

Baker [2] and Bernau [3] both show that an epi-archimedean vector lattice with
uncountable dimension need not belong to $\mathcal{S}$.

Proposition 3.4. (Bigard) An $l$-group $G$ is epi-archimedean if and only if $G$ is
($l$-isomorphic to) a group of real valued functions on a topological space $X$ with
pointwise addition and order and such that

a) $G$ separates points, and

b) the support of each $g \in G$ is compact and open.

Proof. ($\rightarrow$) Let $E$ be the set of all maximal $l$-ideals of $G$ and let $\tau$ be the natural
$l$-isomorphism of $G$ into $\Pi_{P \in \mathcal{E}} G/P$

$g \tau = (\ldots, P + g, \ldots)$. 

For each $g \in G$ let $\sigma(g) = \{ P \in \mathcal{E} | g \notin P \}$ the support of $g$. The $\sigma(g)$ form a basis of open sets for a topology on $E$. This is the hull kernel topology on $E$. If $P_1 \neq P_2$ then $(P_1 \setminus P_2) \cap G = \emptyset$ and so $G$ separates points.

Suppose that $\sigma(g) = \bigcup \sigma(g_\lambda)$ for $g, g_\lambda \in G$. If $g \notin \bigvee G(g_\lambda)$ then $g \notin P \supseteq \bigvee G(g_\lambda)$ for some value $P$ of $g$. Thus $P \in \sigma(g) = \bigcup \sigma(g_\lambda)$ and so $g_\lambda \notin P$ for some $\lambda$, a contradiction.
Thus \( g \in \bigvee G(g,\lambda) \) and so \( g \in G(g,\lambda_1) + \ldots + G(g,\lambda_n) \). But then \( \sigma(g) \subseteq \sigma(g,\lambda_1) \cup \ldots \cup \sigma(g,\lambda_n) \) and so \( \sigma(g) \) is compact.

(\( \leftarrow \)) \( G \) is an \( l \)-group of functions on \( X \) with compact open support. For \( 0 < f, g \in G \) and \( n > 0 \) let

\[
V_n = \{ x \in \sigma(g) \mid ng(x) > f(x) \} = \sigma(g) \cap \sigma((ng - f)^+) \]

which is open. Now \( \sigma(g) = \bigcup V_n \) and since \( \sigma(g) \) is compact

\[
\sigma(g) = V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_k} .
\]

Let \( m = \text{maximum of } n_1, n_2, \ldots, n_k \). Then \( mg(x) > f(x) \) for all \( x \in \sigma(g) \) and so by Theorem 1.1 \( G \) is epi-archimedean.

**Remarks.** The topology on \( E \) is Hausdorff. The set of *all* functions on \( X \) with compact open support need not be an \( l \)-group.

We next discuss the epi-archimedean kernel of an \( l \)-group \( C \). This concept and theory are due to JORGE MARTINEZ [20]. We have removed his hypothesis that \( G \) be representable. Recall that a *value* of \( g \in G \) is a regular subgroup \( G_\gamma \) of \( G \) such \( g \in G^\gamma \setminus G_\gamma \). Let

\[
E = \{ g \in G \mid \text{each value of } g \text{ is a minimal prime} \}
\]

\[\mathcal{N} = \text{set of all prime subgroup of } G \text{ that are not minimal}.\]

**Theorem 3.5.** (Martinez) \( E = \bigcap \mathcal{N} \) and so \( E \) is a convex \( l \)-subgroup of \( G \) that is invariant under all \( l \)-automorphisms of \( G \). Moreover, \( E \) is epi-archimedean and contains each convex \( l \)-subgroup of \( G \) that is epi-archimedean; \( E \) is the epi-archimedean kernel of \( G \).

**Proof.** If \( g \in E \) and \( N \in \mathcal{N} \) then \( g \in G \); otherwise \( g \) has a value that contains \( N \). Thus \( E \subseteq \bigcap \mathcal{N} \). Conversely if \( g \in \bigcap \mathcal{N} \) and \( g \in G^\gamma \setminus G_\gamma \) then clearly \( G_\gamma \) is minimal and so \( \bigcap \mathcal{N} \subseteq E \).

Therefore \( E = \bigcap \mathcal{N} \) a convex \( l \)-subgroup of \( G \) and if \( \tau \) is an \( l \)-automorphism of \( G \) then \( \tau \) induces a permutation on the set \( \mathcal{N} \) and hence \( E\tau = E \). If \( P \) is a prime subgroup of \( G \) that does not contain \( E \) then clearly \( P \) is minimal and so \( P \cap E \) is a minimal prime in \( E \). Since each prime in \( E \) is of this form (see [11] Theor. 1.14) it follows by Theorem 1.1 that \( E \) is epi-archimedean.

Finally consider \( 0 < g \in K \) an epi-archimedean convex \( l \)-subgroup of \( G \) and suppose (by way of contradiction) that \( g \in G^\gamma \setminus G_\gamma \) where \( \gamma \) is not minimal. Then \( G_\gamma \supset G_\beta \). Pick \( 0 < x \in G^\beta \setminus G_\beta \). Then by replacing \( x \) by \( g \wedge x \) we may assume that \( g \leq x \) and so \( x \in K \). But then \( K \supset G_\gamma \cap K \supset G_\beta \cap K \) and so \( K \) is not epi-archimedean, a contradiction. Thus each value of \( g \) is a minimal prime and so \( K \subseteq E \).
4. SPECKER GROUPS

Let $B$ be the group of all bounded functions in $\Pi IZ_i$. If $g \in \Pi Z_i$ then $S(g)$ will denote the support of $g$

$$S(g) = \{i \in I | g_i \ne 0\}$$

and if $X \subseteq I$ then $\chi_X$ will denote the characteristic function on $X$.

$$(\chi_X)_i = 1 \text{ if } i \in X \text{ and } 0 \text{ otherwise}.$$ 

Each $0 \ne g \in B$ has a unique representation

$$g = n_1 \chi_{X_1} + \ldots + n_k \chi_{X_k}$$

where the $n_i$ are distinct non-zero integers and the $X_i$ are disjoint subsets of $I$.

The next proposition is more or less implicit in [21] but this formulation and proof is due to LASZLO FUCHS.

4.1. For a subgroup $G$ of $B$ the following are equivalent.

a) $g = n_1 \chi_{X_1} + \ldots + n_k \chi_{X_k} \in G$ implies $\chi_{X_i} \in G$ for $i = 1, \ldots, k$, where of course this is the unique representation of $g$.

b) $g \in G$ implies $\chi_{S(g)} \in G$.

c) $G$ is pure in $B$ and a subring of $B$.

A subgroup $G$ of $B$ that satisfies a), b), and c) is called a Specker group.

Proof. (a $\rightarrow$ b) Clear, since $\chi_{S(g)} = \chi_{X_1} + \ldots + \chi_{X_k}$.

(b $\rightarrow$ a) We use induction on $k$.

$$(n_1 - n_k) \chi_{X_1} + \ldots + (n_{k-1} - n_k) \chi_{X_{k-1}} = g - n_k \chi_{S(g)} \in G.$$ 

Thus by induction $\chi_{X_1}, \ldots, \chi_{X_{k-1}} \in G$ and so since

$$n_k \chi_{X_k} = g - n_1 \chi_{X_1} - \ldots - n_{k-1} \chi_{X_{k-1}} \in G$$

we have $\chi_{X_k}$ also belongs to $G$.

(a $\rightarrow$ c) We first show that if $g = n_1 \chi_{X_1} + \ldots + n_k \chi_{X_k} \in B$ and $mg \in G$ for some $m \ne 0$ then $g \in G$ and so $G$ is pure.

$$mg = mn_1 \chi_{X_1} + \ldots + mn_k \chi_{X_k}.$$ 

Thus by a) the $\chi_{X_i}$ belong to $G$ and so $g \in G$. 205
Now $G$ is generated by characteristic functions. Thus it suffices to show that if $\chi_x, \chi_y \in G$ then $\chi_x \chi_y \in G$. For then it follows that $G$ is closed with respect to multiplication. Note that $\chi_x \chi_y = \chi_{x \cap y}$, and

$$\chi_x + \chi_y = \chi_{(x \cup y) \setminus (x \cap y)} + 2\chi_{x \cap y}.$$ 

Thus by a) $\chi_{x \cap y} \in G$.

(c $\Rightarrow$ a) If $g = n\chi_x \in G$ then since $G$ is pure $\chi_x \in G$. Now consider $g = n_1\chi_{x_1} + \ldots + n_k\chi_{x_k}$ and use induction on $k$.

$$g^2 - n_kg = (n_1^2 - n_kn_1)\chi_{x_1} + \ldots + (n_{k-1}^2 - n_kn_{k-1})\chi_{x_{k-1}}$$

and $g^2 - n_kg \in G$ since $G$ is a ring. Thus by induction $\chi_{x_1}, \ldots, \chi_{x_{k-1}} \in G$. But

$$n_k\chi_{x_k} = g - n_1\chi_{x_1} - \ldots - n_{k-1}\chi_{x_{k-1}}$$

and so by purity again it follows that $\chi_{x_k} \in G$.

Note that the group of all bounded continuous functions from a topological space $X$ into $\mathbb{Z}$ is Specker. Also the intersection of Specker groups is Specker and the join of a chain of Specker groups is Specker.

Clearly a Specker group is generated by characteristic functions. In the next proposition we make use of the cardinal order of $\prod_{i} \mathbb{Z}_i$ and the fact that $B$ is an $l$-ideal of $\prod_{i} \mathbb{Z}_i$.

4.2. For a subgroup $G$ of $\prod_{i} \mathbb{Z}_i$ that is generated by its set $S$ of characteristic functions the following are equivalent.

a) $G$ is Specker.

b) $G$ is an $l$-subgroup of $B$.

c) $S$ is closed with respect to multiplication.

d) $S$ is closed with respect to $\wedge$.

Proof. If $x, y \in S$ then $xy = x \wedge y$ and hence (c) and (d) are equivalent and clearly (b) implies (d).

(a $\Rightarrow$ b) If $g = n_1\chi_{x_1} + \ldots + n_k\chi_{x_k} \in G$ then the $\chi_{x_i} \in G$ and so it follows that $g \wedge 0 \in G$.

(d $\Rightarrow$ a) If $0 \neq g \in G$ then $g = m_1\chi_{Y_1} + \ldots + m_i\chi_{Y_i}$ where the $m_i$ are integers and the $\chi_{Y_i} \in S$. Here we do not assume that the $Y_i$ are disjoint subsets of $I$.

$$\chi_{Y_1 \cap Y_2} = \chi_{Y_1 \cap Y_2} = \chi_{Y_1} \wedge \chi_{Y_2} \in S.$$ 

Thus $\chi_{Y_1} - \chi_{Y_1 \cap Y_2} = \chi_{Y_1 \setminus Y_2} \in G$ and so we have

$$m_1\chi_{Y_1} + m_2\chi_{Y_2} = m_1\chi_{Y_1 \setminus Y_2} + (m_1 + m_2)\chi_{Y_1 \cap Y_2} + m_2\chi_{Y_2 \setminus Y_1}.$$ 

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It follows that $g$ has a representation

$$g = n_1x_1 + \ldots + n_kx_k$$

where the $n_i$ are distinct non-zero integers, the $X_i$ are disjoint subsets of $I$ and each $x_i \in S$.

Note that each Specker group belongs to $\mathcal{S}$. Also if $L$ is an $l$-ideal of a Specker group $G$ then clearly $L$ satisfies (b) of 4.1 and so $L$ is also Specker.

4.3. Each $l$-ideal $L$ of a Specker group $G$ is a ring ideal.

Proof. Since $G \in \mathcal{S}$, $G = G(g) \oplus g'$ for each $g \in G$. So each $G(g)$ is a ring ideal, but $L$ is the join of a directed (by inclusion) set of such $G(g)$ and so $L$ is a ring ideal.

**Theorem.** (Nobeling [21]) If $G \in H$ are Specker groups then $H = G \oplus F$ where $F$ is a free abelian group with characteristic basis.

Laszlo Fuchs (unpublished) and Paul Hill [16] have derived simpler proofs of this remarkable result.

Actually, as we now show, Specker groups occur quite naturally, in the theory of $l$-groups. Recall that an element $s$ in an $l$-group $H$ is singular if $s > 0$ and

$$0 \leq g < s \text{ implies } g \land (s - g) = 0 \text{ for each } g \in H,$$

and let $S$ be the set of all singular elements in $H$. Then in [10] it is shown that:

4.4. The subgroup $[S]$ of $H$ generated by $S$ is an abelian $l$-ideal of $H$.

4.5. There exists an $l$-isomorphism $\tau$ of $[S]$ onto a subdirect sum of $\Pi_iZ_i$ and for each such mapping $\tau$, $[S]\tau$ is Specker and hence a subring of $\Pi_iZ_i$.

Proof. In [10] it is shown that $\tau$ exists and for each $s \in S$, $\tau s$ is characteristic. Thus $[S]\tau$ is Specker by (b) of 4.2.

It follows that a group $G$ is (isomorphic to) a Specker group if and only if there exists a set $S$ of generators of $G$ and a lattice order for $G$ in which each $s \in S$ is singular. An $l$-group $G$ is $l$-isomorphic to a Specker group if and only if $G$ is generated as a group by singular elements.

4.6. If $\tau$ is an $l$-homomorphism of $[S]$ then $[S]\tau$ is also an $l$-group that is generated by singular elements as a group.

Proof. It is shown in [10] that if $s \in S$ then $\tau s = 0$ or $\tau s$ is singular.
Definition. An *S-group* is an *l*-group *G* that is generated (as a group) by singular elements. Such a group *G* is free abelian, belongs to $\mathcal{S}$ and each *l*-homomorphic image of *G* is also an *S*-group. A subgroup *H* of *G* that is generated by its set *T* of singular elements is an *l*-subgroup and hence an *S*-group if and only if *T* is closed with respect to $\wedge$.

4.7. Let $G = [S]$ be an *S*-group. Then there exists a unique multiplication on *G* so that it is a ring for which $st = s \wedge t$ for all $s, t \in S$. Moreover *G* is an *f*-ring with zero radical, each *l*-ideal of *G* is a ring ideal and each *l*-homomorphism of the group *G* is a ring homomorphism.

Proof. We may assume that *G* is an *l*-subgroup of $\prod Z_i$ and each $s \in S$ is characteristic. Since *G* is Specker it is a subring of $\prod Z_i$ and so $st = s \wedge t$ for $s, t \in S$.

Now suppose that $\cdot$ and $*$ are multiplications for *G* so that it is a ring for both and

$$s \cdot t = s \wedge t = s * t \quad \text{for all} \quad s, t \in S.$$  

Then

$$(ms) \cdot (nt) = mn(s \cdot t) = mn(s * t) = (ms) * (nt)$$

for all $m, n \in Z$ and it follows that $g \cdot h = g * h$ for all $g, h \in G$.

It follows from [14] that this multiplication on *G* has a unique extension to the $v$-hull $G^v$ of *G*.

Note that if *u* is an order unit in an *S*-group $G = [S]$ then $\chi_{S(u)}$ is an order unit and a singular element.

4.8. If $G = [S]$ is an *S*-group and $s \in S$ is an order unit for *G* then the multiplication in 4.7 is the unique multiplication so that *G* is an *f*-ring with identity *s*.

Proof. Clearly $s = \bigvee S$. Thus if $a \in S$ then $sa = s \wedge a = a$ and so $s$ is the identity in the above multiplication. In [12] it is shown that there is at most one such multiplication.

Corollary. If *G* is an *l*-group with a singular element *u* as a strong order unit then *G* is an *S*-group and there is a unique multiplication on *G* so that it is an *f*-ring with identity *u*.

Proof. Let *S* be the set of all singular elements of *G*. Then $[S]$ is an *l*-ideal of *G* that contains a strong order unit of *G* and so $G = [S]$.

Suppose that $G = [S]$ is an *S*-group with no order unit. Then without loss of generality $G$ is a subdirect sum and a subring of $\prod Z_i$. Let

$$H = G \oplus Z(1, 1, 1, \ldots).$$
Then $H$ is an $I$-subgroup of $\Pi Z_i$. In fact $H$ is an $S$-group with $G$ as an $I$-ideal and with $(1, 1, 1, \ldots)$ as a unit. Or one can define $H$ by

$$H = Z \oplus G$$

and let $H^+$ be the subsemigroup of $H$ generated by all the elements of the form

$$(n, 0), (n, -s), (0, s), (0, 0) \text{ where } 0 < n \in Z \text{ and } s \in S.$$  

Here $(1,0)$ is an order unit; in fact $(1, 0)$ is the join of all the singular elements in $G$. Then by 4.8 $H$ is an $f$-ring with identity $(1, 0)$ and, of course, this is just the standard way of adjoining an identity to the ring $G$.

Let $F$ be the group of all functions in $\Pi R_i$ with finite range. Each $0 \neq g \in F$ has a unique representation

$$g = a_1\chi_{X_1} + \ldots + a_k\chi_{X_k}$$

where the $a_i$ are distinct non-zero reals and the $X_i$ are disjoint subsets. The proofs of the next two propositions are almost identical with the proofs of 4.1, 4.2 and 4.3 and we shall omit them.

4.9. For a subspace $G$ of $F$ the following are equivalent.

a) $g = a_1\chi_{X_1} + \ldots + a_k\chi_{X_k} \in G$ implies each $\chi_{X_i} \in G$.

b) $g \in G$ implies $\chi_{\Lambda(g)} \in G$.

c) $G$ is a subring of $F$.

d) $G$ is generated as a subspace of $F$ by a set of characteristic functions and $G$ is an $I$-subgroup of $F$.

e) $G$ is generated as a subspace of $F$ by a set $S$ of characteristic functions and $S$ is closed with respect to $\wedge$.

A subspace $G$ of $F$ that satisfies a)–e) will be called a Specker space.

4.10. Each $l$-ideal of a Specker space is a ring ideal and, of course, a Specker space.

4.11. If $H$ is an $l$-subgroup of $\Pi R_i$ consisting of step functions and $u = (1, 1, 1, \ldots) \in H$ then $H$ satisfies condition a) of 4.9. Thus if $G$ is an epi-archimedean vector lattice with order unit $u$ then $G$ is ($l$-isomorphic to) a Specker space and there exists a unique multiplication so that $G$ is an $f$-ring with identity $u$.

Proof. Each $0 < h \in H$ has a unique representation

$$h = a_1\chi_{X_1} + \ldots + a_k\chi_{X_k}$$

where $0 < a_1 < a_2 < \ldots < a_k$ are real numbers and the $X_i$ are disjoint subsets of $I$. 

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Pick positive integers \( m \) and \( n \) so that \( n a_{k-1} < m < n a_k \). Then

\[(nh - mu) \vee 0 = t \chi_{X_k} \text{ where } 0 < t = n a_k - m.\]

Now pick a positive integer \( q > 1/t \). Then

\[\chi_{X_k} = qt \chi_{X_k} \wedge u \in G.\]

Thus each of the \( \chi_{X_i} \) belongs to \( G \).

Let \( \tau \) be an \( l \)-isomorphism of \( G \) into \( \Pi_f R_f \) such that \( \mu \tau = (1, 1, 1, \ldots) \). By Proposition 1.2 \( G \tau \) consists of step functions and so by the above is a Specker space. Thus \( G \tau \) is a subring of \( \Pi_f R_f \) with identity \( \mu \tau \). Finally it is shown in [12] that there exists at most one multiplication so that \( G \) is an \( f \)-ring with identity \( u \).

Note that if \( G \) is an epi-archimedean vector lattice and \( 0 < g \in G \), then \( G(g) \) satisfies 4.11. Thus “locally” \( G \) is an \( f \)-ring with no nilpotent elements.

**4.12. (Bleier).** If \( G \) is an epi-archimedean vector lattice that is finitely generated as a vector lattice then \( G \simeq \sum_{i=1}^{n} R_i. \)

**Remark.** Note that Example 7.1 shows that an epi-archimedean \( l \)-group generated by two elements need not belong to \( \mathcal{F} \).

**Proof.** If \( g_1, \ldots, g_n \) generate \( G \) then clearly \( G = G(u) \) where \( u = |g_1| + \ldots + |g_n| \).

So by 4.11 we may assume that \( G \) is a Specker subspace of \( \Pi_f R_f \). Then \( G \) is generated by a finite number of characteristic functions and in fact by a finite number of disjoint characteristic functions.

**Corollary.** A finitely generated Specker space is \( l \)-isomorphic to \( \sum_{i=1}^{n} R_i. \)

**Corollary.** Each finitely generated epi-archimedean vector lattice is generated by two elements.

**Proof.** \( \sum_{i=1}^{n} R_i \) is generated by \((1, 1, 1, \ldots)\) and \((1, 2, 3, \ldots, n)\).

**Corollary.** \( R \oplus R \) is the free epi-archimedean vector lattice on the one generator \((1, -1)\). There is no free epi-archimedean vector lattice on more than one generator.

**Proof.** Suppose that \( F \) is a free epi-archimedean lattice on two generators, then we may assume that \( F = \sum_{i=1}^{n} R_i \) for some \( n > 0 \). Then there must be a linear \( l \)-homomorphism of \( F \) onto \( \sum_{i=1}^{n+1} R_i \) since \( \sum_{i=1}^{n+1} R_i \) is generated by two elements, but this is impossible.
Let $G \subseteq \Pi_i R_i$ be a Specker space generated by a set $S$ of characteristic functions. Then $[S]$ is a Specker group and so $[S]$ is an $l$-subgroup of $\Pi_i Z_i$ and $G$ is the $v$-hull of $[S]$. Conversely let $[S] \subseteq \Pi_i Z_i$ be a Specker group and let $G$ be the subspace of $\Pi_i R_i$ generated by $[S]$. Then $G$ consists of all real linear combinations of the elements of $S$. Since $S$ is closed with respect to multiplication it follows that $G$ is a Specker space and the $v$-hull of $[S]$.

4.13. For a vector lattice $G$ the following are equivalent.

a) $G \in \mathcal{S}$ and $G$ is an $f$-ring with no nilpotent elements.

b) $G$ is the $v$-hull of an $S$-group.

c) $G$ is $I$-isomorphic to a Specker space.

Proof. We have shown that b) and c) are equivalent and clearly c) implies a). The fact that a) implies c) follows from the next proposition.

4.14. If $G \in \mathcal{S}$ is an $f$-ring with no nilpotent elements then $G$ can be embedded as a ring into a cardinal product of reals and each such representation consists of step functions. Thus if $G$ is an $f$-algebra then it is $I$-isomorphic to a Specker space.

Proof. By Lemma A we can embed $G$ as an $f$-ring into $\Pi_i R_i$. Suppose (by way of contradiction) that $0 < g = (..., g_i, ...) \in G$ has infinite range. By Corollary I of Proposition 2.1 there exists $w \in \Pi R_i$ such that $w_i > 0$ for all $i$ and $Gw$ consists of step functions. Now there is an infinite subset $J$ of $I$ for which the $g_j$ are all distinct and each $g_j w_j = k$, a constant. Thus $g_j^2 w_j = k g_j$ and so $g^2 w$ is not a step function, a contradiction.

5. THIS SECTION CONSISTS OF THE FOLLOWING THEOREM

Theorem 5.1. If $G \in \mathcal{S}$ then $G^v$ is an $a$-closure of $G$. Moreover, $G^v$ is the unique $a$-closure of $G$ in $\mathcal{S}$. In particular $G \in \mathcal{S}$ is a closed if and only if $G$ is a vector lattice.

Proof. Case I. $G$ has an order unit $u$. By Proposition 1.5 we may assume that $G$ is an $l$-subgroup of $\Pi_i R_i$ consisting of step functions and containing $(1, 1, 1, ...)$. By 4.11 $G$ satisfies condition a) of 4.9. Thus $G^v$ is the Specker space generated by the set $S$ of characteristic functions in $G$. If $0 < h \in G^v$ then $h = h_1 \chi_{X_1} + ... + h_k \chi_{X_k}$ where the $h_i$ are non-zero reals and the $X_i$ are disjoint subsets of $I$. Thus $g = \chi_{X_i} + ... + \chi_{X_k} \in G$ and clearly $G^v(h) = G^v(g)$ and so $G^v$ is an $a$-extension of $G$.

If $H$ is an $a$-extension of $G^v$ then $H \in \mathcal{S}$ and so by Proposition 1.4 we may assume that $G \subseteq G^v \subseteq H \subseteq \Pi_i R_i$, and by Proposition 1.2 $H$ consists of step functions. Consider

$$0 < h = h_1 \chi_{X_1} + ... + h_k \chi_{X_k} \in H$$
where the $X_i$ are disjoint subsets of $I$ and $0 < h_1 < \ldots < h_k$. By 4.11 the $\chi_{X_i} \in H$. Now there exists $0 < g \in G^\circ$ such that $H(g) = H(\chi_{X_k})$. In particular, $\chi_{X_k} = \chi_{S(g)} \in G^\circ$ and since $G^\circ$ is a vector space, $h_k \chi_{X_k} \in G^\circ$. Thus $h \in G^\circ$ and so $G^\circ$ is $a$-closed.

Case II. $G$ does not contain an order unit. $G^\circ \subseteq (G^\circ)^\wedge$ and so if $0 < h \in G^\circ$ then $h < g$ for some $g \in G$.

$$G = G(g) \uplus g' \quad \text{and} \quad G^\circ = G^\circ(g) \uplus g^*.$$  

Now $h \in G^\circ(g) = G(g)^\circ$ and $g$ is a unit for $G(g)$. Thus by Case I $G^\circ(g)$ is an $a$-extension of $G(g)$ and so $h$ is $a$-equivalent to an element in $G$. Therefore $G^\circ$ is an $a$-extension of $G$.

Now suppose that $H$ is an $a$-extension of $G^\circ$ and consider $0 < h \in H$. Then $H(h) = H(g)$ for some $0 < g \in G^\circ$ and $H \in \mathcal{G}$. Thus,

$$H = H(g) \uplus g^* \quad \text{and} \quad G^\circ = G^\circ(g) \uplus g^*$$

where $\#$, $*$ are the polar operations in $H$ and $G^\circ$ respectively. Now $H(g)$ is an $a$-extension of $G^\circ(g)$ and $G^\circ(g)$ is a vector lattice in $\mathcal{G}$ with an order unit $g$. Thus by Case I $G^\circ(g) = H(g)$ and so $h \in G^\circ(g) \subseteq G^\circ$. Therefore $G^\circ$ is $a$-closed.

Thus we have shown that if $G \in \mathcal{G}$ then $G^\circ$ is an $a$-closed $a$-extension of $G$. Now let $K$ be an $a$-closure of $G$ in $\mathcal{G}$. Then $K$ is a vector lattice and without loss of generality

$$G \subseteq G^\circ \subseteq \Pi_I R_I \quad \text{and} \quad G \subseteq K \subseteq \Pi_I R_I.$$  

Thus $G^\circ \cap K$ is a vector lattice that contains $G$ and so $G^\circ \cap K = G^\circ$. Therefore $G \subseteq G^\circ \subseteq K$ and so $G^\circ = K$.

6. $\textit{EPI-ARCHIMEDEAN } f\textit{-RINGS}$

Suppose that $G \in \mathcal{G}$ is an $f$-ring with no nilpotent elements and let $X$ be the Stone space associated with the Boolean algebra of polars of $G$. By the embedding theorem of Bernau [4] $G$ is ($l$-isomorphic to) a large subring of the ring $D(X)$ of continuous extended real valued functions on $X$. Here each $f \in D(X)$ is real on a dense open subset of $X$.

**Lemma 6.1.** $G \subseteq S(X)$.

**Proof.** $0 < g \in G$ is real on a dense open subset $Y$ of $X$. Then $G(g)$ is a subring of $C(Y)$ and so by 4.14 $G(g) \subseteq S(Y)$. Therefore $g$ is a real valued step function in $D(X)$ and hence $G \subseteq S(X)$.

**Theorem 6.2.** If $G \in \mathcal{G}$ is an $f$-ring with no nilpotent element, then $S(X)$ is the essential ring closure of $G$ in $\mathcal{G}$. 

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Proof. Let $H$ be an essential $f$-ring extension of $G$ in $\mathcal{S}$. Then clearly $H$ has no nilpotent elements, and the Stone space associated with $H$ is $X$ (see [13]). Thus by the Lemma we can embed $H$ into $S(X)$ as an $f$-ring.

In order to get rid of the hypotheses that $G$ contains no nilpotent elements we need the following concept. An $l$-group $G$ is strongly projectable ("SP-group") if

$$G = C \oplus C'$$

for each polar $C$ of $G$.

In [14] it is shown that each archimedean $l$-group $G$ admits a unique SP-hull $G^{SP}$. Thus $G^{SP}$ is the minimal essential extension of $G$ that is an SP-group.

Proposition 6.3. a) If $G \in \mathcal{S}$ then $G^{SP} \in \mathcal{S}$.

b) If $G \in \mathcal{S}$ then $G^{SP} \in \mathcal{S}$.

Proof. a) For each polar $C$ of $G$, $G/C' \in \mathcal{S}$ and so each $G_C$ used in the construction of $G^{SP}$ belongs to $\mathcal{S}$ (see [14] Theorem A). Now $G^{SP}$ is a direct limit of these $G_C$ and since we have a two element characterization of the groups in $\mathcal{S}$ (Theorem 1.1 (4) or (6)) the direct limit $G^{SP}$ must also be epi-archimedean.

b) We show that the essential closure $S(X)$ of $G$ in $\mathcal{S}$ is an SP-group. Then $G^{SP}$ is the intersection of all $l$-subgroups of $S(X)$ that contain $G$ and are SP-group and so $G^{SP} \in \mathcal{S}$.

Let $T$ be a polar in $S(X)$ and let

$$X_T = \{x \in X \mid t(x) \neq 0 \text{ for some } t \in T\}.$$

Then the closure $Y$ of $X_T$ is clopen and

$$T = \{s \in S(X) \mid \text{ the support of } s \text{ is contained in } Y\}.$$

Therefore

$$S(X) = S(Y) \oplus S(X \setminus Y) = T \oplus S(X \setminus Y).$$

Corollary 1. If $G \in \mathcal{S}$ is an $f$-ring then $G^{SP}$ is also an epi-archimedean $f$-ring. Thus the radical of $G^{SP}$ is a cardinal summand

$$G^{SP} = \text{rad } G^{SP} \oplus H$$

where $H$ is an $f$-ring with no nilpotent element and rad $G^{SP}$ has the zero multiplication.

Corollary II. If $G \in \mathcal{S}$ is an $f$-ring then the $f$-ring essential closure of $G$ in $\mathcal{S}$ is of the form

$$S(Y) \oplus S(W)$$
where \( Y \) and \( W \) are Stone spaces. \( \mathbb{S}(W) \) has the natural multiplication and \( \mathbb{S}(Y) \) has the zero multiplication.

The proofs of these corollaries follow from the Proposition and from the theory in Section 6 of [14].

7. EXAMPLES AND OPEN QUESTIONS

Example 7.1. A divisible epi-archimedean \( l \)-group \( G \) with an order unit such that \( G \notin \mathcal{S} \). Let \( H = \prod_{i=1}^{\infty} R_i \) and let
\[
G = \{ h \in H \mid \text{there exist rationals } r, s \text{ such that } h_i = r(\pi + 1/i) + s \text{ for almost all } i \}.
\]

Clearly \( G \) is a divisible subgroup of \( H \) and \( G \supseteq \sum_{i=1}^{\infty} R_i \).

a) \( G \) is an \( l \)-subgroup of \( H \). For consider \( g \in G \) where \( g_i = r(\pi + 1/i) + s \) for almost all \( i \). It suffices to show that almost all the \( g_i \) are positive or almost all of them are negative. For then \( g \vee 0 \in G \) or \( g \vee 0 \in \sum R_i \subseteq G \) and so \( G \) is an \( l \)-subgroup of \( H \).

If \( r = 0 \) then \( g_i = s \) for almost all \( i \). If \( r > 0 \) then \( r(\pi + 1/i) + s < r(\pi + 1/j) + s \) for all \( i > j \). Thus if \( r(\pi + 1/j) + s < 0 \) for some \( j \) then almost all the \( g_i \) are negative and otherwise almost all \( g_i \) are positive. If \( r < 0 \) then \( r(\pi + 1/i) + s > r(\pi + 1/j) + s \) for all \( j > i \). Thus if \( r(\pi + 1/i) + s > 0 \) for some \( i \) then almost all \( g_i \) are positive and otherwise almost all \( g_i \) are negative.

b) \( G \in \mathcal{S} \). If \( 0 < g \in G \) then clearly the \( g_i \) are bounded from above and since \( \lim g_i = r\pi + s \geq 0 \) it follows that the \( g_i \neq 0 \) are bounded away from zero. Thus by Lemma \( A \) \( G \in \mathcal{S} \).

c) It follows from Proposition 1.5 that the \( l \)-subgroup of \( G \) generated by \( (1, 1, 1, \ldots) \) and \( (\pi + 1, \pi + 1/2, \pi + 1/3, \ldots) \) does not belong to \( \mathcal{S} \) and hence \( G \notin \mathcal{S} \).

This is a limiting example in many ways.

1) \( G^r \notin \mathcal{S} \) and so \( G^r \) is not an \( a \)-extension of \( G \). For if \( G^r \in \mathcal{S} \) then by Proposition 1.2 \( G^r \in \mathcal{S} \) and hence \( G \in \mathcal{S} \).

2) \( G_i = \{ g \in G \mid g_i = 0 \} \) for \( i = 1, 2, \ldots \) and \( \sum R_i \) are the prime \( l \)-ideals of \( G \).

Proof. Clearly the \( G_i \) are maximal \( l \)-ideals and since they are polars they are also minimal primes. Next the map
\[
\sum R_i + g \rightarrow r\pi + s
\]
where \( g_i = r(\pi + 1/i) + s \) for almost all \( i \), is an \( o \)-isomorphism of \( G/\sum R_i \) onto \( Q\pi + Q \) and so \( \sum R_i \) is also a maximal \( l \)-ideal. Now if \( M \neq G \) is a prime \( l \)-ideal of \( G \)

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and if for each $i$ there is a $0 < g \in G$ such that $g_i > 0$ then $M = \Sigma R_i$. Otherwise $M \subseteq G_i$ for some $i$ and hence $M = G_i$.

Next let $E$ be the $l$-group of all eventually constant sequences of rational numbers. Then $E \in \mathcal{S}$ and it follows from (2) that $G$ is an $a$-extension of $E$. Therefore

3) There is an $a$-closure (essential closure) of $E$ that belongs to $\mathcal{E}$ but not $\mathcal{S}$ and also an $a$-closure (essential closure) of $E$ in $\mathcal{S}$. $E^a$ is the $a$-closure of $E$ in $\mathcal{S}$.

4) Let $K$ be an $a$-closure of $E$ that is not in $\mathcal{S}$. Then $K \in \mathcal{E}$ and $K$ is not a vector lattice. Also $K^a$ is not an $a$-extension of $K$.

Proof. If $K$ is a vector lattice then by Proposition 1.2 $K \in \mathcal{S}$.

Example 7.2. An $f$-ring $G$ that belongs to $\mathcal{E}$ but not $\mathcal{S}$.

Let $l_1, l_2, \ldots$ be a sequence of positive rationals that converge to $\pi$ and let $G$ be the $l$-subring of $\prod_{i=1}^{\infty} R_i$ that is generated by $l = (l_1, l_2, \ldots)$, $(1, 1, 1, \ldots)$ and $\sum_{i=1}^{\infty} R_i$. Then $G$ consists of $\Sigma R_i +$ polynomial in $l$ with integral coefficients. For if $0 \neq f(x) \in \mathbb{Z}[x]$ then $f(\pi) \neq 0$ and so $f(\pi)$ and $f(l_i)$ agree in sign for almost all $i$. It follows from this that $G$ is an $l$-subring of $\Pi R_i$ and that it satisfies Lemma A and hence belongs to $\mathcal{E}$.

If $G \in \mathcal{S}$ then so does $G^a$ but then it follows from Proposition 1.5 that this representation of $G$ consists of functions with finite range.

Theorem 1.1 asserts that an $l$-group is epi-archimedean if and only if the set $P$ of proper prime subgroups is trivially ordered with respect to inclusion. In general, $P$ is a root system (that is, a po set such that the elements above any fixed element form a chain).

A maximal chain in $P$ will be called a root.

If $G$ is an $l$-group and $E$ is the epi-archimedean kernel and if $G/E$ is also epi-archimedean then [20] each root in $P$ has length at most 2. The next example shows that the converse is false.
Example 7.3. For the free vector lattice $G$ on two generators $P$ looks like

![Diagram](https://via.placeholder.com/150)

one for each point on the unit circle (see [2]). But Bleier (Tulane Dissertation 1971) showed that $G$ contains no $l$-ideals that are invariant under all $l$-automorphisms and hence $E = 0$.

Example 7.4 (CHAMBLESS) The group $C(X, Z)$ of all continuous integral valued functions on a compact Hausdorff space belongs to $\mathcal{S}$. This is because the range of such a function is a compact subset of $Z$ and hence finite.

Example 7.5. The group $G$ of eventually constant sequences of reals belongs to $\mathcal{S}$ but is not an SP-group. Note that $G$ is a Specker space.

Example 7.6. $G = \Pi Z_i$ for all $i \in [0, 1]$ has the property that for each maximal $l$-ideal $C$, $G/C$ is cyclic (see [10]).

The following conjecture is due to Jorge Martinez. If $G$ is an epi-archimedean $l$-group and a subdirect sum of integers then is $G/C$ cyclic for each prime subgroup $C$ of $G$? We show that the answer is no.

1) If the conjecture holds for a particular $l$-group $G$ then it holds for each $l$-subgroup $H$ of $G$.

Proof. Let $P$ be a prime subgroup of $H$. Then there exists a prime $C$ in $G$ such that $C \cap H = P$. Thus

$$\frac{H}{P} = \frac{H}{(C \cap H)} \simeq \left(\frac{C + H}{C} \subseteq G/C\right) \text{ cyclic}.$$  

2) If $0 < s$ is a singular element in the $l$-group $H$ and $(H^a, H_a)$ is a value of $s$, then $H_a \ll H^a$ and $H^a/H_a$ is cyclic.

Proof. $H(s)$ is abelian and so $H_a \cap H(s) \ll H(s)$. Thus $H_a \ll H^a$ (see [11]). Now $H_a + s$ is singular in the archimedean $o$-group $H^a/H_a$ and so $H^a/H_a$ is cyclic.

3) If $C$ is a prime subgroup in the epi-archimedean $l$-group $G$, $s$ is singular and $s \notin C$, then $G/C$ is cyclic.

Proof. $(G, C)$ is a value of $s$.

4) The conjecture is true for the group $G$ of all bounded functions in $\Pi Z_i$.

Proof. If $C$ is a proper prime subgroup of $G$ then $(1, 1, 1, \ldots) \in G \setminus C$. 

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Example 7.7. Let $H$ be the $l$-group of all step functions in $\prod_{i=1}^{\infty} R_i$ and let $\alpha$ be the $l$-automorphism of $\prod R_i$ obtained by multiplication by the element $(1, 2, 3, \ldots)$. Let $G$ be the subgroup of $H\alpha$ consisting of all the integral valued functions. Clearly $G$ is an $l$-subgroup of $H\alpha$ and hence of $\prod_{i=1}^{\infty} Z_i$ and $G \in \mathcal{F}$. Now we construct a prime subgroup $C$ of $G$ such that $G/C$ is not cyclic.

$(0, 1/2, 0, 1/2, 0, \ldots) \in H$ maps onto $x = (0, 1, 0, 2, 0, 3, 0, 4, \ldots)$

and

$(0, 0, 0, 1/4, 0, 0, 1/4, 0, \ldots) \in H$ maps onto $y = (0, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0, \ldots)$ etc.

Next choose the following subsets of $N = 1, 2, 3, \ldots$

$$\{1, 3, 5, 7, 9, 11, \ldots\}$$

$$\{1, 2, 3; 5, 6, 7; 9, 10, 11; \ldots\}$$

$$\{1, 2, 3, 4, 5, 6, 7; 9, 10, 11, 12, 13, 14, 15; \ldots\}$$

These are contained in a dual ultra filter $\mathcal{F}$ of the set of all proper subsets of $N$. Let $C$ be the set of all functions in $G$ whose support belongs to this ultrafilter. Then (see [10]) $C$ is prime, but $C + x > C + y > \ldots$ and so $G/C$ is not cyclic. For suppose that $x = y \mod C$, then

$$x - y = (0, 1, 0, 1, 0, 3, 0, 2, 0, 5, 0, 3, 0, 7, 0, 4, \ldots) \in C$$

but this is impossible since

$$(1, 0, 3, 0, 5, 0, 7, \ldots) \in C$$

and this means that $C$ contains a strong order unit of $G$.

Open questions.
1) If $G \in \mathcal{E}$ and $G$ is a subdirect sum of integers then does $G \in \mathcal{F}$?
2) Does each $G \in \mathcal{E}$ have a representation that satisfies (a) of Lemma A?
3) Find an example of $G \in \mathcal{E}$ that is not contained in an epiarchimedean $f$-ring with no nilpotent elements.
4) Suppose that $G$ is an $l$-subgroup of $\prod R_i$ that satisfies (a) of Lemma A. Does the $l$-subring of $\prod R_i$ generated by $G$ belong to $\mathcal{E}$?
5) Is a vector lattice in $\mathcal{E}$ $a$-closed?
References


Author's address: The University of Kansas, Lawrence, Kansas 660 44, U.S.A.