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ON SOME INVARIANTS OF UNARY ALGEBRAS

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1. PROBLEM

1.0. Notation. If ♦ is a set we denote by |♦| the cardinal number of ♦; similarly, if α is an ordinal then its cardinal number is denoted by |α|. We denote by Ord the class of all ordinals. If α ∈ Ord then we put Wα = {β ∈ Ord; β < α}; further, the least ordinal cofinal with α is denoted by cf α. We denote by N the set of all finite ordinals.

We shall need some simple results concerning ordinals (see [3] and [4]).

(i) If α, β, γ ∈ Ord, α < β then γ + α < γ + β.

(ii) If α, β ∈ Ord, α < β then there is precisely one ξ ∈ Ord such that α + ξ = β. We put ξ = -α + β.

(iii) If α, β ∈ Ord, α ≤ β then α + (-α + β) = β, -α + (α + β) = β.

Indeed, the first equation follows directly by definition of -α + β. If we put ξ = -α + (α + β) then α + ξ = α + β by definition. Then ξ = β follows from the uniqueness of the solution.

(iv) If α, β, γ ∈ Ord, α ≤ β < γ, then -α + β < -α + γ.

Indeed, -α + β ≥ -α + γ would imply β = α + (-α + β) ≥ α + (-α + γ) = γ by (iii) and (i).

(v) If α, β ∈ Ord, α ≤ β < α + ω₀ then -α + β < α + ω₀.

Indeed, -α + β ≥ ω₀ would imply β = α + (-α + β) ≥ α + ω₀ by (iii) which is a contradiction.

(vi) Suppose α, β, δ ∈ Ord, 0 ∈ δ ⊆ Ord, β ≤ γ for each γ ∈ δ, δ > α + (-β + γ) for each γ ∈ δ. Let ε be the least ordinal greater than all γ ∈ δ. Then δ ≥ α + (-β + ε).

Indeed, suppose, on the contrary, δ < α + (-β + ε). Since δ > α + (-β + γ) for at least one γ ∈ δ we have δ ≥ α which implies the existence of -α + δ and -α + (α + (-β + ε)) by (ii). Then -α + δ < -α + (α + (-β + ε)) = -β + ε
by (iv) and (iii). It follows \( \beta + (\alpha + \delta) < \beta + (\beta + \varepsilon) = \varepsilon \) by (i) and (iii).

Thus, there is at least one \( \gamma_0 \in \Gamma \) such that \( \beta + (\alpha + \delta) \leq \gamma_0 \). It follows

\[
-\alpha + \delta = -\beta + (\beta + (\alpha + \delta)) \leq -\beta + \gamma_0 \text{ by (iii) and (iv) which implies } \delta = \alpha + (-\alpha + \delta) \leq \alpha + (\beta + \gamma_0) \text{ by (iii) and (iv) which is a contradiction. Thus, }
\]

\[\delta \geq \alpha + (\beta + \varepsilon).\]

Let \( \infty \notin \text{Ord} \). If \( M \) is an arbitrary set of ordinals then we denote by \( < \) the order relation on \( M \cup \{\infty\} \) such that its restriction \( < \cap (M \times M) \) to \( M \) is the natural order relation of ordinals and that \( \alpha < \infty \) for each \( \alpha \in M \).

If \( \varphi \) is a map of the set \( A \) into the set \( B \), \( \varphi : A \to B \), and \( C \subseteq A, D \subseteq B \) then we put \( \varphi(C) = \{\varphi(x); x \in C\} \); further, we define \( \varphi^{-1}(D) = \{x \in A; \varphi(x) \in D\} \). If \( \varphi : A \to B \) is a map, \( C \subseteq A \), then we denote by \( \varphi \mid C \) the restriction \( \varphi \cap (C \times B) \) of \( \varphi \); it is a map of \( C \) into \( B \).

Let \( A \) be a set, \( f \) a map of \( A \) into \( A \), \( f : A \to A \). Then the ordered pair \( (A, f) \) is called a unary algebra. For a unary algebra \( (A, f) \) we put \( f^0 = id_A, f^{n+1} = ff^n \) for each \( n \in N \). Clearly, \( f^{n+m} = f^m f^n \) for all \( n, m \in N \). A unary algebra \( (A, f) \) is called connected if, for all \( x, y \in A \), there are \( m, n \in N \) such that \( f^m(x) = f^n(y) \). If \( (A, f) \) is a unary algebra and \( x \in A \) an arbitrary element then we put \( [x]_{(A, f)} = \{f^n(x); n \in N\} \).

We denote by \( \cong \) the relation of isomorphism of algebras.

1.1. Definition. Let \( (A, f) \) be a connected unary algebra, \( x \in A \). We put \( Z(x) = \{y \in A; \) there exists an infinite set \( N(y) \subseteq N \) such that \( f^n(x) = y \) for each \( n \in N(y)\}\).

1.2. Lemma. Let \( (A, f) \) be a connected unary algebra. Then the following assertions hold:

(a) If \( x \in A, y = f(x) \) then \( Z(x) = Z(y) \).

(b) If \( x \in A, n \in N, y = f^n(x) \) then \( Z(x) = Z(y) \).

(c) If \( x, y \in A \) then \( Z(x) = Z(y) \).

Proof of (a). Suppose \( x \in A, y = f(x), z \in A \). Then \( z \in Z(x) \) iff there is an infinite set \( M \subseteq N \) such that \( f^n(x) = z \) for each \( n \in M \); we can suppose, without loss of generality, that \( 0 \notin M \). The last condition is equivalent to the condition \( f^n(z) = f^n(f^n(x)) = z \) for each \( n \in M \) which is \( z \in Z(y) \). Thus, \( Z(x) = Z(y) \).

Proof of (b). The assertion (b) follows from (a) by induction.

Proof of (c). If \( x, y \in A \) then there exist \( m, n \in N \) such that \( f^m(x) = f^n(y) \). It follows from (b) that \( Z(x) = Z(f^m(x)) = Z(f^n(y)) = Z(y) \).

1.3. Definition. Let \( (A, f) \) be a connected unary algebra. We put \( Z(A, f) = Z(x) \) where \( x \in A \) is an arbitrary element, \( R(A, f) = |Z(A, f)| \). Then \( Z(A, f) \) is called the cycle and \( R(A, f) \) the range of \( (A, f) \).
1.4. Lemma. Let \((A, f)\) be a connected unary algebra. Then \((Z(A, f), f \mid Z(A, f))\) is a subalgebra of the algebra \((A, f)\).

Proof. If \(x \in Z(A, f)\) then there exists an infinite set \(N(x) \subseteq N\) such that \(x = f^n(x)\) for each \(n \in N(x)\). It follows \(f(x) = f^{n+1}(x)\) for all \(n \in N(x)\) which implies \(f(x) \in Z(f(x)) = Z(A, f)\).

1.5. Lemma. Let \((A, f)\) be a connected unary algebra and suppose \(x, y \in A\). Then

(a) If \(n_1, n_2 \in N, \ n_1 \leq n_2\) are such that \(y = f^{n_1}(x) = f^{n_2}(x)\) then \(y = f^{n_1 + m(n_2 - n_1)}(x)\) for each \(m \in N\).

(b) \(x \in Z(A, f)\) iff there is \(n \in N - \{0\}\) such that \(f^n(x) = x\).

Proof of (a). We put \(n_2 - n_1 = d\); thus, \(f^{n_1 + 0d}(x) = f^{n_1}(x) = y\). Let \(m \in N\) and suppose \(f^{n_1 + md}(x) = y\). Then \(f^{n_1 + (m+1)d}(x) = f^{d + n_1 + md}(x) = f^n(f^{n_1 + md}(x)) = f^d(y) = f^d(f^n(x)) = f^{n_1 + d}(x) = y\).

Proof of (b). Suppose, for \(x \in A\), the existence of \(n \in N - \{0\}\) such that \(f^n(x) = x\); then, by (a), we have \(x = f^{mn}(x)\) for each \(m \in N\). Thus, we have \(x = f^n(x)\) for all \(p \in \{mn; \ m \in N\}\) the latter set being infinite. Thus, \(x \in Z(x) = Z(A, f)\).

The necessity of the condition for \(x \in Z(A, f)\) follows directly from 1.3 and 1.1.

1.6. Lemma. Let \((A, f)\) be a connected unary algebra. Then the following assertions hold:

(a) If \(x \in Z(A, f)\) then \(|Z(A, f)| = \min \{n \in N - \{0\}; f^n(x) = x\}\).

(b) \(R(A, f) < \aleph_0\).

Proof of (a). We put \(d = \min \{n \in N - \{0\}; f^n(x) = x\}\). Since \(x \in Z(A, f)\) we have \(\{x, f(x), \ldots, f^{d-1}(x)\} \subseteq Z(A, f)\), by 1.4. Let us have \(y \in Z(A, f)\). Then \(y \in Z(x)\); thus, there exists \(m \in N\) such that \(f^m(x) = y\). Let \(p, q \in N\) be such numbers that \(m = pd + q, 0 \leq q < d\). Thus, by definition of \(d\) and by 1.5 (a), we have \(f^{pd}(x) = x\) and \(y = f^m(x) = f^p(f^{qd}(x)) = f^q(x)\). Thus, \(y \in \{x, f(x), \ldots, f^{d-1}(x)\}\) and we have \(x, f(x), \ldots, f^{d-1}(x)\) \(= Z(A, f)\). Therefore, \(|Z(A, f)| = d\).

Proof of (b). If \(Z(A, f) = 0\) then \(R(A, f) = 0 < \aleph_0\). If \(Z(A, f) \neq 0\) then there is \(x \in Z(A, f)\) and \(\{n \in N - \{0\}; f^n(x) = x\} \neq 0\) by 1.5 (b). It follows \(R(A, f) = \min \{n \in N - \{0\}; f^n(x) = x\} < \aleph_0\), by (a).

1.7. Definition. Let \((A, f)\) be a connected unary algebra. We put \(A^\infty = \{x \in A; \text{there is a sequence } (x_i)_{i \in N}\) such that \(x_0 = x\) and \(f(x_{i+1}) = x_i\) for each \(i \in N\}\), \(A^0 = \{x \in A; f^{-1}(x) = 0\}\).
Let \( \alpha \in \text{Ord}, \alpha > 0 \) and suppose that the sets \( A^x \) have been defined for all \( x \in W_\alpha \). Then we put \( A^x = \{ x \in A - \bigcup_{x \in W_\alpha} A^\xi; f^{-1}(x) \subseteq \bigcup_{x \in W_\alpha} A^\xi \} \).

**1.8. Lemma.** Let \((A, f)\) be a connected unary algebra, \( \alpha, \beta \in \text{Ord}, \alpha < \beta \). Then \( A^\alpha \cap A^\beta = \emptyset \).

**Proof.** Clearly, \( A^\beta \subseteq A - \bigcup_{x \in W_\beta} A^x \) which implies \( A^\beta \cap A^\alpha \subseteq A^\beta \cap \bigcup_{x \in W_\beta} A^x = \emptyset \).

**1.9. Lemma.** Let \((A, f)\) be a connected unary algebra. Then there is \( \theta \in \text{Ord} \) such that \( A^\theta = \emptyset \).

**Proof.** Let \( v \in \text{Ord} \) be such an ordinal number that \( |A| \leq \aleph_v \). Suppose \( A^\xi \neq \emptyset \) for each \( \xi \in W_{\text{ord}+1} \). Then \( \aleph_{\text{ord}+1} \leq \sum_{\lambda \in W_{\text{ord}+1}} |A^\lambda| = \bigcup_{\lambda \in W_{\text{ord}+1}} |A^\lambda| \leq |A| \leq \aleph_v \) by 1.8 which is a contradiction.

Thus, there is \( \theta \in \text{Ord} \), \( \theta \in W_{\text{ord}+1} \) such that \( A^\theta = \emptyset \).

**1.10. Lemma.** Let \((A, f)\) be a connected unary algebra. If \( \theta \in \text{Ord} \), \( A^\theta = \emptyset \) then \( A^\lambda = \emptyset \) for each \( \lambda \in \text{Ord} \) with the property \( \lambda \geq \theta \).

**Proof.** We denote by \( V(\lambda) \) the following assertion: \( A^\lambda = \emptyset \).

Then \( V(\theta) \) holds.

Let us have \( \beta \in \text{Ord}, \theta < \beta \), suppose that \( V(\lambda) \) holds for each \( \lambda \) such that \( \theta \leq \lambda < \beta \). Then \( \bigcup_{\lambda \in W_\beta} A^\lambda = \bigcup_{\lambda \in W_\beta} A^\lambda \) which implies \( A^\beta = \{ x \in A - \bigcup_{\lambda \in W_\beta} A^\lambda; f^{-1}(x) \subseteq \bigcup_{\lambda \in W_\beta} A^\lambda \} = \bigcup_{\lambda \in W_\beta} A^\lambda \).

Thus, \( A^\beta = \emptyset \).

The assertion follows by transfinite induction.

**1.11. Definition.** Let \((A, f)\) be a connected unary algebra. Then we denote by \( \text{ord}(A, f) \) the least ordinal \( \theta \) such that \( A^\theta = \emptyset \).

**1.12. Lemma.** Let \((A, f)\) be a connected unary algebra. Then \( A^\infty = A - \bigcup_{x \in W_{\text{ord}(A, f)}} A^x \).

**Proof.** (1) If \( x \in A - \bigcup_{x \in W_{\text{ord}(A, f)}} A^x \) then there is an element \( x' \in f^{-1}(x) \) such that \( x' \in A - \bigcup_{x \in W_{\text{ord}(A, f)}} A^x \). Indeed, if we had \( x' \in \bigcup_{x \in W_{\text{ord}(A, f)}} A^x \) for each \( x' \in f^{-1}(x) \) then we should have \( f^{-1}(x) \subseteq \bigcup_{x \in W_{\text{ord}(A, f)}} A^x \). We denote by \( \text{ord} \) the least ordinal such that \( f^{-1}(x) \subseteq \bigcup_{x \in W_\beta} A^x \). Then \( \theta \leq \text{ord}(A, f) \) and \( x \in A^\theta \) by 1.7 which is a contradiction either with \( A^{\text{ord}(A, f)} = \emptyset \) (in the case \( \theta = \text{ord}(A, f) \)) or with \( x \in A - \bigcup_{x \in W_{\text{ord}(A, f)}} A^x \) (in the case \( \theta < \text{ord}(A, f) \)).
We put \( x_0 = x \) and \( x_{n+1} = x' \) for \( n \in \mathbb{N} \). Then \( f(x_{n+1}) = x_n \) for \( n \in \mathbb{N} \) and \( x \in A^\infty \). Thus \( A - \bigcup_{x \in W_{\mathcal{A}}(A, f)} A_x \subseteq A^\infty \).

(2) Let us have \( x \in A^\infty \cap \bigcup_{x \in W_{\mathcal{A}}(A, f)} A_x \). Then there is a sequence \((x_i)_{i \in \mathbb{N}}\) such that \( x_0 = x \) and \( f(x_i+1) = x_i \) for each \( i \in \mathbb{N} \). By 1.8, there exists precisely one \( x_0 \in W_{\mathcal{A}}(A, f) \) such that \( x_0 \in A^\infty \).

Suppose that we have constructed ordinals \( x_0 > x_1 > \ldots > x_n \) such that \( x_i \in A^{x_i} \) for \( i = 0, 1, \ldots, n \) where \( n \in \mathbb{N} \). Then \( x_{n+1} \in f^{-1}(x_n) \subseteq \bigcup_{x \in W_{\mathcal{A}}(A, f)} A_x \) which implies the existence of \( x_{n+1} \in A^{x_{n+1}} \). Thus, \((x_i)_{i \in \mathbb{N}}\) is an infinite decreasing sequence of ordinals which is a contradiction.

It follows that \( A^\infty \subseteq A - \bigcup_{x \in W_{\mathcal{A}}(A, f)} A_x \).

1.13. Theorem. Let \((A, f)\) be a connected unary algebra. Then \( A = \bigcup_{x \in W_{\mathcal{A}}(A, f)} A_x \) with disjoint summands.

It is a consequence of 1.12 and 1.8.

1.14. Lemma. Let \((A, f)\) be a connected unary algebra. Then \((A^\infty, f|A^\infty)\) is a subalgebra of \((A, f)\).

Proof. Let us have \( x \in A^\infty \). It follows the existence of a sequence \((x_n)_{n \in \mathbb{N}}\) such that \( x_n \in A \), \( x_0 = x \) and \( f(x_{n+1}) = x_n \) for each \( n \in \mathbb{N} \). We put \( f(x) = y = y_0 \), \( y_n = x_{n-1} \) for each \( n \in \mathbb{N} - \{0\} \). Then \( f(y_{n+1}) = y_n \) for each \( n \in \mathbb{N} \) which implies \( f(x) = y \in A^\infty \).

1.15. Lemma. Let \((A, f)\) be a connected unary algebra. Then \( Z(A, f) \subseteq A^\infty \).

Proof. \( Z(A, f) \subseteq A^\infty \) holds if \( Z(A, f) = 0 \). Thus, we can suppose \( Z(A, f) \neq 0 \). Let us have \( x \in Z(A, f) \). Then \( Z(A, f) = Z(x) \) by 1.3. By 1.1, there exists an infinite set \( N(x) \subseteq N \) such that \( f^n(x) = x \) for each \( n \in N(x) \). We denote by \( d \) the least positive element of \( N(x) \). Then \( f^d(x) = x \) and \( f^{md}(x) = x \) for each \( m \in \mathbb{N} \) by 1.5 (a). We put, for each \( n \in \mathbb{N} \), \( x_n = f^{n(2d-1)}(x) \). Then \( f(x_{n+1}) = f(f(x_n+1)) = f^{n(2d-1) + 2d}(x) = f^{n(2d-1)}(f^{2d}(x)) = f^{n(2d-1)}(x) = x_n \) for each \( n \in \mathbb{N} \) and \( x_0 = f^0(x) = x \). Thus, \( x \in A^\infty \).

1.16. Lemma. Let \((A, f)\) be a connected unary algebra, suppose \( \lambda, \mu \in W_{\mathcal{A}}(A, f) \), \( \lambda < \mu \). Then, for each \( x \in A^\mu \), there is an \( x' \in A^\lambda \) and an \( n \in \mathbb{N} - \{0\} \) such that \( f^n(x') = x \).

Proof. Let us have \( x \in A^\mu \). Then there is \( v_1 \in \text{Ord}, \lambda \leq v_1 < \mu \) and \( x_1 \in A^{v_1} \) such that \( f(x_1) = x \). Indeed, if no such \( v_1 \) and \( x_1 \) exist then \( f^{-1}(x) \subseteq \bigcup_{x \in W_{\mathcal{A}}} A_x \). Since \( x \in A - \bigcup_{x \in W_{\mathcal{A}}} A_x \subseteq A - \bigcup_{x \in W_{\mathcal{A}}} A_x \) we have \( x \in A^\lambda \), by 1.7, which contradicts 1.8.
If \( \lambda < \nu_1 \) we construct similarly \( \nu_2 \in \text{Ord} \), \( \lambda \leq \nu_2 < \nu_1 \) and \( x_2 \in A'' \) such that \( f(x_2) = x_1 \). As each decreasing sequence of ordinals is finite we construct, after a finite number of such steps, some ordinals \( \lambda = \nu_n < \nu_{n-1} \ldots < \nu_1 < \mu \) and some elements \( x_i \in A^i \) for \( i = 1, 2, \ldots, n \) such that \( f(x_{i+1}) = x_i \) for \( i = 1, 2, \ldots, n-1 \) and \( f(x_1) = x \). It follows \( f^n(x_n) = x, \) \( x_n \in A^2 \), \( n \neq 0 \) because \( n = 0 \) would imply \( x = x_n \in A^2 \cap A^0 \) which contradicts 1.8.

**1.17. Lemma.** Let \((A, f)\) be a connected unary algebra, \( A^\infty \neq \emptyset \). Then the following assertions hold:

(a) For each \( x \in \bigcup_{x \in W_{\lambda}(A, f)} A^x \) there exists \( n(x) \in \mathbb{N} \) such that \( f^{n(x)}(x) \in A^\infty \).

(b) If \( A - A^\infty \neq \emptyset \) and \( x \in A - A^\infty \) then there is precisely one \( i_0 \in \mathbb{N} - \{0\} \) such that \( f^{i_0-1}(x) \in A - A^\infty, f^{i_0}(x) \in A^\infty \).

(c) If \( A - A^\infty \neq \emptyset \) then there is at least one \( x \in A - A^\infty \) such that \( f(x) \in A^\infty \).

**Proof of (a).** We take \( y \in A^\infty \). Then there are \( m, n \in \mathbb{N} \) such that \( f^m(y) \in A^\infty \). By 1.14, we have \( f^n(y) \in A^\infty \) and we obtain the first assertion.

**Proof of (b).** By 1.12 and (a), for each \( x \in A - A^\infty \), there exists \( n(x) \in \mathbb{N} \) such that \( f^{n(x)}(x) \in A^\infty \). It follows by 1.13 that \( n(x) > 0 \). Thus, in the set of natural numbers \( i, 0 < i \leq n(x), \) there is the least element \( i_0 \) such that \( f^{i_0}(x) \in A^\infty \). Clearly, \( i_0 > 0 \) and \( f^{i_0}(x) \in A - A^\infty \).

If \( i > i_0 \) then \( i - 1 \geq i_0 \) and \( f^{i-1}(x) = f^{i_0-1}(f^{i_0}(x)) \in A^\infty \) as \( f^{i_0}(x) \in A^\infty \) and \( (A^\infty, f \mid A^\infty) \) is a subalgebra of \((A, f)\) by 1.14. Thus, \( f^{i-1}(x) \notin A - A^\infty \).

If \( i < i_0 \) then \( f^i(x) \notin A^\infty \) on the basis of the minimality of \( i_0 \).

Thus, \( i_0 \) is the only element \( i \in \mathbb{N} - \{0\} \) such that \( f^{i-1}(x) \in A - A^\infty, f^i(x) \in A^\infty \).

**Proof of (c).** We take an arbitrary \( z \in A - A^\infty \). By (b), there is precisely one \( i_0 \in \mathbb{N} - \{0\} \) such that \( f^{i_0-1}(z) \in A - A^\infty, f^{i_0}(z) \in A^\infty \). We put \( x = f^{i_0-1}(z) \). Then \( x \in A - A^\infty, f(x) = f^{i_0}(z) \in A^\infty \).

**1.18. Definition.** Let \((A, f)\) be a connected unary algebra. We define a map \( S(A, f) : A \to \text{Ord} \cup \{\infty\} \) by the condition \( S(A, f)(x) = \kappa \) for each \( x \in A^\kappa, \kappa \in W_{\lambda}(A, f) \cup \{\infty\} \). \( S(A, f)(x) \) is called the degree of \( x \).

**1.19. Lemma.** Let \((A, f)\) be a connected unary algebra. Then the following assertions hold:

(a) If \( x \in A \) is such element that \( S(A, f)(x) \neq \infty \) then \( S(A, f)(f^n(x)) \geq S(A, f)(x) + n \) for each \( n \in \mathbb{N} \).

(b) If \( x \in A, \kappa \in W_{\lambda}(A, f) \) are arbitrary elements then \( \left| A^\kappa \cap [x]_{\lambda(A, f)} \right| \leq 1 \).
Proof of (a). If \( n = 0 \) then \( S(A, f) (f^0(x)) = S(A, f) (x) \). Let \( n \in \mathbb{N} \) and suppose \( S(A, f) (f^n(x)) \geq S(A, f) (x) + n \). We put \( \alpha = S(A, f) (f^{n+1}(x)) \). If \( \alpha = \infty \) then \( \alpha > S(A, f) + n + 1 \). If \( \alpha < \infty \) then \( f^{n+1}(x) \in A^\alpha \) and \( f^n(x) \in f^{-1}(f^{n+1}(x)) \subseteq \bigcup_{\beta \in A^\alpha} A^\beta \). Thus, \( S(A, f) (f^n(x)) < \alpha \) and \( \alpha \geq S(A, f) (f^n(x)) + 1 \geq S(A, f) (x) + n + 1 \). We have proved the assertion (a).

Proof of (b). Suppose, on the contrary, \( |A^x \cap [x]_{(A, f)}| \geq 2 \); let \( y, z \in A^x \cap [x]_{(A, f)}, y \neq z \). Then there is \( n \in \mathbb{N} - \{0\} \) such that either \( f^n(y) = z \) or \( f^n(z) = y \).

In the first case, we have \( x = S(A, f) (z) \geq S(A, f) (f^n(y)) \geq S(A, f) (y) + n > S(A, f) (y) = x \), by (a), which is a contradiction. Similarly, the second case leads to a contradiction. We have proved the assertion (b).

1.20. Lemma. Let \((A, f), (A_*, f_*)\) be unary connected algebras, \( \varphi : A \to A_* \) an isomorphism of \((A, f)\) onto \((A_*, f_*)\). Then \( \vartheta(A, f) = \vartheta(A_*, f_*) \), \( \varphi(A^x) = A_*^x \) for each \( x \in W_{\vartheta(A, f)} \cup \{\infty\} \) and \( \varphi(Z(A, f)) = Z(A_*, f_*) \).

Proof. For each \( \alpha \in \text{Ord} \) we denote by \( V(\alpha) \) the following assertion: \( \varphi(A^x) = A_*^x \).

The following conditions are equivalent:

(i) \( x \in A^0 \)
(ii) \( f(y) = x \) for no \( y \in A \)
(iii) \( f_*(z) = \varphi(x) \) for no \( z \in A_* \)
(iv) \( \varphi(x) \in A_*^0 \).

Indeed, (i) and (ii) are equivalent by 1.7 and (iii) and (iv), too. If \( f(y) = x \) for no \( y \in A \) and there is \( z \in A_* \) such that \( f_*(z) = \varphi(x) \) then \( f(\varphi^{-1}(z)) = \varphi^{-1}(f_*(z)) = \varphi^{-1}(\varphi(x)) = x \) because \( \varphi^{-1} \) is an isomorphism; we have a contradiction. Thus, (ii) implies (iii) and, similarly, (iii) implies (ii).

It follows that \( V(0) \) holds.

Let \( \beta > 0 \) be an ordinal, suppose that \( V(\gamma) \) holds for each \( \gamma < \beta \). It follows \( \varphi(\bigcup_{x \in W_\beta} A^x) = \bigcup_{x \in W_\beta} A_*^x \).

The following conditions are equivalent:

(i) \( x \in A^\beta \)
(ii) \( x \in A - \bigcup_{x \in W_\beta} A^x, f^{-1}(x) \subseteq \bigcup_{x \in W_\beta} A^x \)
(iii) \( \varphi(x) \in A_* - \bigcup_{x \in W_\beta} A_*^x, f_*^{-1}(\varphi(x)) \subseteq \bigcup_{x \in W_\beta} A_*^x \)
(iv) \( \varphi(x) \in A_*^\beta \).
Indeed, (i) and (ii) are equivalent by 1.7 and (iii) and (iv), too. If $x \in A - \bigcup \limits_{\alpha \in \omega} A^x$ then $\varphi(x) \in \varphi(A - \bigcup \limits_{\alpha \in \omega} A^x) = \varphi(A) - \varphi(\bigcup \limits_{\alpha \in \omega} A^x) = A^* - \bigcup \limits_{\alpha \in \omega} A^*_\alpha$ by induction hypothesis because $\varphi$ is a bijection. If $f^{-1}(x) \subseteq \bigcup \limits_{\alpha \in \omega} A^x$ then each $y$ with the property $f(y) = x$ is in $\bigcup \limits_{\alpha \in \omega} A^x$. Let us have an arbitrary $z \in f_{\alpha}^{-1}(\varphi(x))$. Then $f_{\alpha}(z) = \varphi(x)$ and $f(\varphi^{-1}(z)) = \varphi^{-1}(f_{\alpha}(z)) = \varphi^{-1}(\varphi(x)) = x$ because $\varphi^{-1}$ is an isomorphism. It follows $\varphi^{-1}(z) \in \bigcup \limits_{\alpha \in \omega} A^x$ which implies $z \in \varphi(\bigcup \limits_{\alpha \in \omega} A^x) = \bigcup \limits_{\alpha \in \omega} A^*_\alpha$. Thus, $f_{\alpha}^{-1}(\varphi(x)) \subseteq \bigcup \limits_{\alpha \in \omega} A^*_\alpha$.

We have proved that (ii) implies (iii). Similarly, (iii) implies (ii).

Thus, the validity of $V(y)$ for all $\gamma < \beta$ implies that of $V(\beta)$.

We have $\varphi(A^x) = A^*_\alpha$ for each $\alpha \in \text{Ord}$. Especially, $A^* = \emptyset$ iff $A^*_\alpha = \emptyset$. It follows $\varepsilon(A,f) = \varepsilon(A^*_\alpha,f_{\alpha})$.

If $x \in A^\omega$ then there is a sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in \mathbb{N}$. It follows $\varphi(x_0) = \varphi(x)$ and $f_{\alpha}(\varphi(x_{i+1})) = \varphi(f(x_{i+1})) = \varphi(x_i)$ for each $i \in \mathbb{N}$. Thus, $\varphi(x) \in A^\omega_\alpha$. Similarly, $x \in A$, $\varphi(x) \in A^\omega_\alpha$ imply $x \in A^\omega$. We have $\varphi(A^\omega) = A^\omega_\alpha$.

We have proved $\varphi(A^\omega) = A^\omega_\alpha$ for each $\alpha \in W_{\varepsilon(A,f)} \cup \{\omega\}$.

If $x \in Z(A,f)$ then there is $n \in \mathbb{N} - \{0\}$ such that $f^n(x) = x$ by 1.5 (b). It follows $f_{\alpha}^n(\varphi(x)) = \varphi(f^n(x)) = \varphi(x)$. Thus, $\varphi(x) \in Z(A^*_\alpha,f_{\alpha})$ by 1.5 (b). Similarly, $x \in A$, $\varphi(x) \in Z(A^*_\alpha,f_{\alpha})$ imply $x \in Z(A,f)$.

We have proved $\varphi(Z(A,f)) = Z(A^*_\alpha,f_{\alpha})$.

1.21. Remark. Let $(A,f)$ be a connected unary algebra. Then the ordinal $\varepsilon(A,f)$ and the cardinals $|A^\omega|$, $\alpha \in W_{\varepsilon(A,f)} \cup \{\omega\}$ and $R(A,f)$ are preserved under isomorphisms, i.e. they are invariant, by 1.20.

If $(A,f)$, $(B,g)$ are connected unary algebras then the numbers $R(A,f)$, $R(B,g)$ and functions $S(A,f)$, $S(B,g)$ enable to construct all homomorphisms of $(A,f)$ into $(B,g)$. Thus, a very natural problem arises:

1.22. Problem. Let $A$ be a set, $R \in \mathbb{N}$, $S : A \to \text{Ord} \cup \{\infty\}$ a map. Find necessary and sufficient conditions for the existence of a complete unary operation $f$ on $A$ such that $(A,f)$ is connected and $R(A,f) = R$, $S(A,f) = S$.

2. AUXILIARY CONSTRUCTION

2.1. Definition. Let $(A,f)$ be a connected unary algebra with the property $A^\omega \neq \emptyset$. Then $(A,f)$ is called an $\infty$-algebra.

2.2. Definition. Let $(A,f)$ be an $\infty$-algebra. Then we put $E(A,f) = f^{-1}(A^\omega) - A^\omega$. 

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2.3. **Lemma.** Let \((A, f)\) be an \(\omega\)-algebra. Then the following assertions hold:

(a) \(E(A, f) \neq \emptyset\) iff \(A - A^\omega \neq \emptyset\).

(b) If \(x \in A - A^\omega\) then there is precisely one \(n_0 \in N\) such that \(f^{n_0}(x) \in E(A, f)\).

(c) If \(\mathcal{G}(A, f) > 0\) is an isolated ordinal then \(\emptyset \neq A^{\mathcal{G}(A, f) - 1} \subseteq E(A, f)\).

**Proof of (a).** The necessity of the condition is clear.

Let us have \(A - A^\omega \neq \emptyset\). Then, by 1.17 (c), there is \(x \in A - A^\omega\) such that \(f(x) \in E(A, f)\). Thus, \(x \in E(A, f)\).

**Proof of (b).** The existence of precisely one \(n_0 \in N\) with the property \(f^{n_0}(x) \in E(A, f)\) is equivalent to the existence of precisely one \(n_0 \in N\) with the properties \(f^{n_0}(x) \notin A^\omega\), \(f^{n_0 + 1}(x) \in A^\omega\) which is equivalent to the existence of precisely one \(i_0 \in N - \{0\}\) such that \(f^{i_0 - 1}(x) \notin A^\omega\), \(f^{i_0}(x) \in A^\omega\). The last assertion holds according to 1.17 (b).

**Proof of (c).** \(A^{\mathcal{G}(A, f) - 1} \neq \emptyset\) follows from the definition of \(\mathcal{G}(A, f)\). If \(x \in A^{\mathcal{G}(A, f) - 1}\) then \(S(A, f)(f(x)) > S(A, f)(x) = \mathcal{G}(A, f) - 1\) by 1.19 (a). It follows \(S(A, f)(f(x)) = +\infty\) which implies \(f(x) \in A^\omega\). It follows \(x \in f^{-1}(A^\omega)\) and we have \(A^{\mathcal{G}(A, f) - 1} \subseteq f^{-1}(A^\omega)\). Further, \(A^{\mathcal{G}(A, f) - 1} \cap A^\omega = \emptyset\) by 1.13. It follows \(A^{\mathcal{G}(A, f) - 1} \subseteq f^{-1}(A^\omega) - A^\omega = E(A, f)\).

2.4. **Definition.** Let \((A, f)\) be a non empty connected unary algebra. Then it is called a **cone** if \(f(A^\omega) = A^{\omega + 1}\) for each \(\omega \in W(\mathcal{G}(A, f))\) such that \(\omega + 1 = \mathcal{G}(A, f)\).

2.5. **Examples.** 1. A connected unary algebra \((A, f)\) such that \(A^\omega = A \neq 0\) is a cone.

2. The unary algebra \((N, f)\) where \(f(n) = n + 1\) for each \(n \in N\) is a cone.

3. If \((m_n)_{n \in N}\) is a non-increasing sequence of cardinals such that \(m_n \neq 0\) for each \(n \in N\) and that there is \(n_0 \in N\) with the property \(m_{n_0} = 1\) then there is a cone \((B, g)\) such that \(|B^\omega| = m_n\) for each \(n \in N\).

Indeed, we take mutually disjoint sets \(B_n\) such that \(|B_n| = m_n\) for each \(n \in N\). We put \(B = \bigcup_{n \in N} B_n\). For an arbitrary \(n \in N\), we take an arbitrary surjection \(g_n : B_n \to B_{n+1}\); such a surjection exists because the sequence \((m_n)_{n \in N}\) is non-increasing. We define the map \(g : B \to B\) in such a way that \(g \mid B_n = g_n\). Then \((B, g)\) is a unary algebra. Clearly, \(|B_n| = 1\) for each \(n \geq n_0\). If \(x, y \in B\) then there are \(m, n \in N\) such that \(x \in B_m, y \in B_n\). There is \(p \in N, p \geq \max \{m, n, n_0\}\). Then \(g^{p-m}(x) \in B_p, g^{p-n}(y) \in B_p\). Since \(|B_p| = 1\) it follows \(g^{p-m}(x) = g^{p-n}(y)\). Thus, \((B, g)\) is connected. Clearly, \(B^\omega = B_n\) for each \(n \in N\) and \(g(B^\omega) = g(B_n) = g_n(B_n) = B_{n+1} = B^{\omega + 1}\) which implies that \((B, g)\) is a cone such that \(|B^\omega| = m_n\) for each \(n \in N\).
2.6. Lemma. Let \((A, f)\) be a cone. Then \(\exists(A, f) \geq \omega_0\).

Proof. (1) Let \(x \in A, n \in \mathbb{N}\) be such element that \(S(A, f) (x) + n \in W_{5(A, f)}\). We put \(S(A, f) (x) = x\); then \(x \in A^x\). By 2.4, we have \(f^n(x) \in A^{x+n}\) which implies \(S(A, f) (f^n(x)) = x + n = S(A, f) (x) + n\).

(2) Let \(x \in \text{Ord}, x \geq \omega_0\); we prove that \(A^x = 0\). Indeed, suppose, on the contrary, \(y \in A^x\). Let us have \(\lambda \in \text{Ord}, \lambda < \omega_0\). Then \(\lambda < x\) and, by 1.16, there exist \(z \in A^z\) and \(n \in \mathbb{N} - \{0\}\) such that \(f^n(z) = y\). By (1), we obtain \(x = S(A, f) (y) = S(A, f) (f^n(z)) = S(A, f) (z) + n = \lambda + n < \omega_0\) which is a contradiction.

Thus, \(\exists(A, f) = \min \{x \in \text{Ord}; A^x = 0\} \leq \omega_0\).

2.7. Definition. Let \(\{(A_i, f_i); i \in I\}\) be a non empty system of mutually disjoint \(\infty\)-algebras. Let \((B, g)\) be a cone which is disjoint with all \(\infty\)-algebras \((A_i, f_i), i \in I\). Let \(\varphi : \bigcup_{i \in I} E(A_i, f_i) \rightarrow B\) be an arbitrary map. (If \(\bigcup_{i \in I} E(A_i, f_i) = \emptyset\) then \(\varphi = \emptyset\).

Then \(\bigcup_{i \in I} (A_i, f_i) \oplus (B, g)\) denotes a unary algebra \((C, h)\) such that \(C = B \cup \bigcup_{i \in I} (A_i - A_i^\infty)\) and that, for each \(x \in C\),

\[
h(x) = \begin{cases} 
    f_i(x) & \text{if } x \in (A_i - A_i^\infty) - E(A_i, f_i) \text{ for some } i \in I \\
    \varphi(x) & \text{if } x \in \bigcup_{i \in I} E(A_i, f_i) \\
    g(x) & \text{if } x \in B 
\end{cases}
\]

2.8. Remark. Let \((C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g)\) be a unary algebra defined in 2.7. If \(x \in A_i - A_i^\infty\) for some \(i \in I\) then \(h^{-1}(x) = f_i^{-1}(x)\).

2.9. Lemma. Let \((C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g)\) be a unary algebra defined in 2.7. Then \((C, h)\) is a connected unary algebra.

Proof. (1) Let \(x \in C\) be arbitrary. Then there is \(m \in \mathbb{N}\) such that \(h^m(x) \in B\). Indeed, if \(x \in B\) then we have nothing to prove.

If \(x \in A_i - A_i^\infty\) for some \(i \in I\) then, by 2.3 (b), there is precisely one \(m \in \mathbb{N}\) such that \(f_i^m(x) \in E(A_i, f_i)\). It follows, for \(n \in \mathbb{N}, n < m\), that \(f_i^n(x) \notin A_i^\infty\), since \(f_i^n(x) \in A_i^\infty\) would imply \(f_i^n(x) = f_i^{m-n}(f_i^n(x)) \in A_i^\infty\) by 1.14 which is a contradiction as \(A_i^\infty \cap E(A_i, f_i) = \emptyset\). Thus, \(0 \leq n < m\) implies \(f_i^n(x) \in A_i - A_i^\infty - E(A_i, f_i)\). It follows \(h^n(x) = f_i^n(x)\) for each \(n\), \(0 \leq n < m\) and especially \(h^{m-1}(x) = f_i^{m-1}(x) \in A_i - A_i^\infty - E(A_i, f_i)\) which implies \(h^n(x) = f_i^n(x) \in E(A_i, f_i)\) and \(h^{m+1}(x) = h(h^m(x)) = h(f_i^m(x)) = \varphi(f_i^m(x)) \in B\).

(2) Let us have \(x, y \in C\). Then there are \(n, m \in \mathbb{N}\) such that \(h^n(x) \in B, h^m(y) \in B\) by (1). Since \((B, g)\) is connected there are \(p, q \in \mathbb{N}\) such that \(g^p(h^n(x)) = g^q(h^m(y))\) which implies \(h^{p+n}(x) = g^p(h^n(x)) = g^q(h^m(y)) = h^{q+m}(y)\). Thus, \((C, h)\) is in connected.
2.10. Lemma. Let \((C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g)\) be a unary algebra defined in \(i \in I\) (p 2.7. Then the following assertions hold:

(a) If \(i \in I\) and \(x \in \text{Ord}\) then \(A_i^x = C^x \cap A_i\).

(b) \(C^\infty = B^\infty\).

(c) \(Z(C, h) = Z(B, g)\).

(d) Putting \(I(x) = \{i \in I; x < \vartheta(A_i, f_i)\}\) for each \(x \in W_{\vartheta(C, h)}\) we have \(C^x \subseteq (B - B^\infty) \cup \bigcup_{i \in I(x)} (A_i - A_i^\infty)\).

(e) We put \(\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i)\). If \(\vartheta_I \leq \lambda < \vartheta(C, h)\) then \(A_i \subseteq B - B^\infty\).

(f) If \(i \in I\) then \(\vartheta(A_i, f_i)\) is the least ordinal greater than \(S(C, h)(x)\) for all \(x \in E(A_i, f_i)\).

(g) If \(x \in C - C^\infty\) and there exists \(i \in I\) such that \(0 \neq h^{-1}(x) = E(A_i, f_i)\) then \(S(C, h)(x) = \vartheta(A_i, f_i)\).

(h) If \(i \in I\), \(x \in \text{Ord}\) and \(C^x \subseteq A_i - A_i^\infty\) then \(C^x = A_i^x\).

Proof of (a). Let \(i \in I\) be an arbitrary element. If \(A_i - A_i^\infty = \emptyset\) then \(W_{\vartheta(A_i, f_i)} = \emptyset\).

It follows \(A_i^x = \emptyset\) and \(C^x \cap A_i \subseteq C \cap A_i \subseteq A_i - A_i^\infty = \emptyset\).

Thus, we can suppose \(A_i - A_i^\infty \neq \emptyset\). We have \(A_i^0 = C^0 \cap A_i\) because \(x \in A_i^0\) iff \(x \in A_i\) and \(f_i^{-1}(x) = \emptyset\); by 2.8, it is equivalent to \(x \in A_i\) and \(h^{-1}(x) = \emptyset\) which means \(x \in C^0 \cap A_i\).

Let us have \(\lambda \in \text{Ord}, \lambda > 0\) and suppose \(A_i^x = C^x \cap A_i\) for each \(x \in W_\lambda\). Then

\[
\bigcup_{x \in W_\lambda} A_i^x = \bigcup_{x \in W_\lambda} (C^x \cap A_i) = A_i \cap \left( \bigcup_{x \in W_\lambda} C^x \right)
\]

and

\[
A_i - \bigcup_{x \in W_\lambda} A_i^x = \left( A_i \cap C \right) - \left( A_i \cap \left( \bigcup_{x \in W_\lambda} C^x \right) \right) = A_i \cap \left( C - \bigcup_{x \in W_\lambda} C^x \right).
\]

It follows that, for \(x \in C\), the following assertions are mutually equivalent:

(i) \(x \in A_i^1\)

(ii) \(x \in A_i - \bigcup_{x \in W_\lambda} A_i^x, f_i^{-1}(x) \subseteq \bigcup_{x \in W_\lambda} A_i^x\)

(iii) \(x \in A_i - \bigcup_{x \in W_\lambda} A_i^x, h^{-1}(x) \subseteq \bigcup_{x \in W_\lambda} A_i^x\)

(iv) \(x \in A_i, x \in C - \bigcup_{x \in W_\lambda} C^x, h^{-1}(x) \subseteq A_i \cap \left( \bigcup_{x \in W_\lambda} C^x \right)\)

(v) \(x \in A_i, x \in C - \bigcup_{x \in W_\lambda} C^x, h^{-1}(x) \subseteq \bigcup_{x \in W_\lambda} C^x\)

(vi) \(x \in A_i \cap C^\lambda\).
Indeed, (i) and (ii) are equivalent by 1.7, (v) and (vi), too. Clearly, \( x \in C, x \in A_i \) implies \( x \in A_i - A_{\alpha_i} \) which implies \( h^{-1}(x) = f_{i-1}^{-1}(x) \) by 2.8. Thus, (ii) and (iii) are equivalent. Since \( h^{-1}(x) = f_{i-1}^{-1}(x) \subseteq A_i \) (iv) and (v) are equivalent. The equivalence of (iii) and (iv) follows by (\(*)\) and (\(\ast\ast\)).

We have proved \( A_i = C^\lambda \cap A_i \). The assertion (a) follows by transfinite induction.

**Proof of (b).** Let us have \( x \in C^\infty \). Then there is a sequence \((x_k)_{k \in \mathbb{N}}\) such that \( x_0 = x \) and \( h(x_{k+1}) = x_k \) for each \( k \in \mathbb{N} \). If \( x_k \in B \) for all \( k \in N \) then \( g(x_{k+1}) = h(x_{k+1}) = x_k \) for all \( k \in N \) which implies \( x \in B^\infty \). If there is \( k \in N \) such that \( x_k \notin B \) then \( x_k \in A_i - A_{i_k} \) for some \( i \in I \). Clearly, for each \( l \geq k \), we have \( x_l \in A_i \). Thus, for all \( l \in N, l \geq k \), we obtain \( f_i(x_{l+1}) = h(x_{l+1}) = x_l \). It follows \( x_k \in A_i^\infty \) which is a contradiction. Thus, \( x \in B^\infty \) and \( C^\infty \subseteq B^\infty \).

If \( x \in B^\infty \) then there is a sequence \((x_k)_{k \in \mathbb{N}}\) such that \( x_0 = x \) and \( g(x_{k+1}) = x_k \) for each \( k \in N \). It follows \( h(x_{k+1}) = x_k \) for each \( k \in N \). Thus \( x \in C^\infty \).

We have proved \( C^\infty = B^\infty \).

**Proof of (c).** Let us have \( x \in Z(C, h) \). Then there is \( n \in N - \{0\} \) such that \( h^n(x) = x \), by 1.5 (b). Then \( Z(C, h) \subseteq C^\infty = B^\infty \subseteq B \) by (b) and 1.15 which implies \( x \in B \) and \( [x^n]_{(C, h)} \subseteq B \). Thus, \( h^n(x) = g^n(x) \) which implies \( x \in Z(B, g) \), by 1.5 (b).

Suppose \( x \in Z(B, g) \). Then there is \( n \in N - \{0\} \) such that \( g^n(x) = x \), by 1.5 (b). We have \( h^n(x) = g^n(x) = x \) which implies \( x \in Z(C, h) \).

Thus, \( Z(C, h) = Z(B, g) \).

**Proof of (d).** Let us have \( i \in I - I(x) \). By (a), it follows \( C^\alpha \cap A_i = A_i^\alpha = \emptyset \) because \( \alpha \geq \mathfrak{g}(A_i, f_i) \). By (b), we have \( C^\alpha \subseteq C - C^\infty = (B - B^\infty) \cup \bigcup_{i \in I(\alpha)} (A_i - A_{i}^\lambda) \) which implies (d).

**Proof of (e).** We have \( C^\lambda \subseteq (B - B^\infty) \cup \bigcup_{i \in I(\lambda)} (A_i - A_i^\lambda) \) by (d) where \( I(\lambda) = \{i \in I; \lambda < \mathfrak{g}(A_i, f_i)\} \). Since \( \mathfrak{g}(A_i, f_i) \leq \mathfrak{g}_i \leq \lambda \) for each \( i \in I \) we have \( I(\lambda) = \emptyset \) and \( C^\lambda \subseteq B - B^\infty \).

**Proof of (f).** Since \( E(A_i, f_i) \subseteq \bigcup_{\lambda \in W_{\mathfrak{g}(A_i, f_i)}} A_i^\lambda \), then, for each \( x \in E(A_i, f_i) \), there is \( \lambda \in W_{\mathfrak{g}(A_i, f_i)} \) such that \( x \in A_i^\lambda \subseteq C^\lambda \) by (a). It follows \( S(C, h) \langle x \rangle = \lambda < \mathfrak{g}(A_i, f_i) \).

Suppose the existence of \( \beta \in \text{Ord}, \beta < \mathfrak{g}(A_i, f_i) \) such that \( S(C, h) \langle x \rangle < \beta \) for each \( x \in E(A_i, f_i) \). Then there is \( y \in A_i^\beta = A_i \cap C^\beta \) by (a). Then \( y \in A_i - A_{i}^\alpha \). By 2.3 (b), there is precisely one \( n \in N \) such that \( f_i^n(y) \in E(A_i, f_i) \). Clearly, \( f_i(y) \in A_i - A_{i}^\alpha \) for \( j = 0, 1, \ldots, n \). It follows \( h^n(y) = f_i^n(y) \) and \( \beta = S(C, h)(y) \leq S(C, h)(y) + n \leq S(C, h)(h^n(y)) = S(C, h)(f_i^n(y)) < \beta \) by 1.19 (a), which is a contradiction.

Thus, \( \mathfrak{g}(A_i, f_i) \) is the least ordinal greater than \( S(C, h)(x) \) for all \( x \in E(A_i, f_i) \).
Proof of (g). Let \( y \in E(A_i, f_i) \) be arbitrary. Then \( x = h(y) \) which implies \( S(C, h)(x) = S(C, h)(h(y)) > S(C, h)(y) \) by 1.19 (a). It follows \( S(C, h)(x) \geq \mathcal{A}(A_i, f_i) \) by (f).

Suppose \( S(C, h)(x) > \mathcal{A}(A_i, f_i) \). Then there are \( z \in C, n \in N - \{0\} \) such that \( S(C, h)(z) = \mathcal{A}(A_i, f_i) \) and \( h^n(z) = x \), by 1.16. We put \( t = h^{-1}(z) \). Then \( h(t) = h^n(z) = x \) which implies \( t \in E(A_i, f_i) \). It follows \( \mathcal{A}(A_i, f_i) = S(C, h)(z) \leq S(C, h)(z) + n - 1 \leq S(C, h)(h^{-1}(z)) = S(C, h)(t) < \mathcal{A}(A_i, f_i) \) by 1.19 (a) and (f) which is a contradiction.

Thus, \( S(C, h)(x) = \mathcal{A}(A_i, f_i) \).

Proof of (h). We have \( C^x = C^x \cap (A_i - A_i^x) \subseteq C^x \cap A_i = A_i^* \subseteq C^x \) by (a). It follows \( C^x = A_i^* \).

2.11. Definition. Let \( \emptyset \neq M \subseteq \text{Ord} \), \( \alpha \in \text{Ord} \). Then we put \( M \leq \alpha \) if \( \beta \leq \alpha \) for each \( \beta \in M \).

2.12. Lemma. Let \( (C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g) \) be a unary algebra defined in 2.7.

We put \( \mathcal{A}_I = \sup_{i \in I} \mathcal{A}(A_i, f_i) \), then \( C^{\mathcal{A}_I} \subseteq B - B^\infty \) and we put

\[
\mathcal{A}_I = \begin{cases} 
\min \{ n \in W_{(B, g)} ; B^n \cap C^{\mathcal{A}_I} \neq \emptyset \} & \text{if } C^{\mathcal{A}_I} \neq \emptyset \\
\mathcal{A}(B, g) & \text{if } C^{\mathcal{A}_I} = \emptyset .
\end{cases}
\]

If \( m \in W_{(B, g)} \), \( m \geq n^* \) then \( S(C, h)(B^m) \leq \mathcal{A}_I + (m - n^*) \).

Proof. \( C^{\mathcal{A}_I} \subseteq B - B^\infty \) by 2.10 (e).

Let us have \( m \in W_{(B, g)} \), \( m \geq n^* \). Then \( n^* \leq m < \mathcal{A}(B, g) \). We denote by \( V(m) \) the following assertion: \( S(C, h)(B^m) \leq \mathcal{A}_I + (m - n^*) \).

Then \( V(n^*) \) holds: Suppose, on the contrary, the existence of \( y_0 \in B^{n^*} \) such that \( S(C, h)(y_0) > \mathcal{A}_I \). By 2.10 (b) \( S(C, h)(y_0) \neq \infty \). By 1.16, there is \( z \in C^{\mathcal{A}_I} \) and \( n_0 \in N - \{0\} \) such that \( h^{n_0}(z) = g^{n_0}(z) = y_0 \) which implies \( n^* \leq \mathcal{A}(B, g)(z) < S(B, g)(z) + n_0 \leq S(B, g)(g^{n_0}(z)) = S(B, g)(y_0) = n^* \) by 1.19 (a) which is a contradiction. Thus, \( S(C, h)(B^{n^*}) \leq \mathcal{A}_I \).

Let us have \( k \in W_{(B, g)} \), \( k \geq n^* \). Suppose that \( V(k) \) holds and that \( k + 1 \in W_{(B, g)} \). Let us have \( y \in B^{k+1} \). Then \( h^{-1}(y) \in B^k \cup \bigcup_{i \in I} E(A_i, f_i) \) because \( (B, g) \) is a cone.

By 2.10 (f), we have \( S(C, h)(E(A_i, f_i)) < \mathcal{A}(A_i, f_i) \leq \mathcal{A}_I \leq \mathcal{A}_I + (k - n^*) \) for each \( i \in I \). The validity of \( V(k) \) means \( S(C, h)(B^k) \leq \mathcal{A}_I + (k - n^*) \). It follows \( S(C, h)(h^{-1}(y)) \leq \mathcal{A}_I + (k - n^*) \). According to the definition of \( S(C, h) \) we obtain \( S(C, h)(y) \leq \mathcal{A}_I + (k + 1 - n^*) \) which is \( V(k + 1) \).

It follows by induction that \( V(m) \) holds for each \( m \in W_{(B, g)} \) with the property \( m \geq n^* \).
2.13. Theorem. Let \((C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g)\) be a unary algebra defined in 2.7, let \(\mathcal{I}\) and \(n^*\) be defined by 2.12. Then

\[ \mathcal{I}(C, h) = \mathcal{I} + (-n^* + \mathcal{I}(B, g)) \.

Proof. (1) Suppose \(n^* = \mathcal{I}(B, g)\).

If \(i \in I, x \in W_{\mathcal{I}(A_i, f_i)}\) then \(0 \neq A_i^x = C^x \cap A_i\) by 2.10 (a) which implies \(C^x \neq 0\). It follows \(\mathcal{I}(C, h) > x\) for each \(x \in W_{\mathcal{I}(A_i, f_i)}\). It follows \(\mathcal{I}(C, h) \geq \mathcal{I}(A_i, f_i)\) which implies \(\mathcal{I}(C, h) \geq \mathcal{I}_I\).

Suppose \(\mathcal{I}(C, h) > \mathcal{I}_I\). By 1.10, there is \(x \in C\) such that \(S(C, h)(x) = \mathcal{I}_I\) which implies \(C^{\mathcal{I}_I} \neq 0\). It is a contradiction with the fact \(n^* = \mathcal{I}(B, g)\).

Thus, \(\mathcal{I}(C, h) = \mathcal{I}_I\).

(2) Suppose \(n^* < \mathcal{I}(B, g)\).

Then \(C^{\mathcal{I}_I} = 0\) and there exists at least one \(x \in B^{**}\) such that \(S(C, h)(x) = \mathcal{I}_I\). Let us have \(n \in W_{\mathcal{I}(B, g)}, n \geq n^*\). By 1.19 (a), we have \(S(C, h)(h^{n-n^*}(x)) \geq \mathcal{I}_I + (n - n^*)\). Since \((B, g)\) is a cone and \(x \in B^{**} \subseteq B\) we have \(h^{n-n^*}(x) = g^{n-n^*}(x) \in \mathcal{I}(B^{**}) = B^n\). Thus, by 2.12 we have \(S(C, h)(B^n) \geq \mathcal{I}_I + (n - n^*)\) which implies \(S(C, h)(h^{n-n^*}(x)) = \mathcal{I}_I + (n - n^*)\). It follows \(S(C, h)(h^{n-n^*}(x)) = \mathcal{I}_I + (n - n^*)\).

Thus, for each \(n \in W_{\mathcal{I}(B, g)}, n \geq n^*\), we have \(\mathcal{I}(C, h) = S(C, h)(h^{n-n^*}(x)) = \mathcal{I}_I + (n - n^*) = \mathcal{I}_I + (-n^* + n)\) which implies \(\mathcal{I}(C, h) \geq \mathcal{I}_I + (-n^* + \mathcal{I}(B, g))\) by 1.0 (vi).

Suppose \(\mathcal{I}(C, h) > \mathcal{I}_I + (-n^* + \mathcal{I}(B, g))\). We put \(x = \mathcal{I}_I + (-n^* + \mathcal{I}(B, g))\).

By 1.10, there exists \(y \in C^x\). Since \(x > \mathcal{I}_I\), there is \(z \in C^{\mathcal{I}_I}\) and \(\forall n \in N - \{0\}\) such that \(h^{\mathcal{I}_I}(z) = y\), by 1.16. It follows \(C^x \subseteq B - B^{**}\), \(C^{\mathcal{I}_I} \subseteq B - B^{**}\) by 2.10 (e). It follows the existence of \(m \in W_{\mathcal{I}(B, g)}\) such that \(y \in B^m\). Since \(z \in B\) we have \(g^m(z) = y\) which implies \(S(B, g)(y) = S(B, g)(g^m(z)) \geq S(B, g)(z) + n\) by 1.19 (a). Clearly, \(z \in C^{\mathcal{I}_I}\) implies \(n^* \leq S(B, g)(z) < S(B, g)(y) = m\). By 2.12, we have \(\mathcal{I}_I + (-n^* + \mathcal{I}(B, g)) = \mathcal{I}_I + (-n^* + \mathcal{I}(B, g)) = \mathcal{I}_I + (n^* + m)\). It follows \(-n^* + \mathcal{I}(B, g) = -\mathcal{I}_I + (\mathcal{I}_I + (-n^* + \mathcal{I}(B, g))) \leq -\mathcal{I}_I + (\mathcal{I}_I + (-n^* + m)) = -n^* + m\) by 1.0 (iii) and (iv) which implies \(\mathcal{I}(B, g) = n^* + (-n^* + \mathcal{I}(B, g)) \leq n^* + (-n^* + m) = m\) by 1.0 (iii) and (i). Thus, \(\mathcal{I}(B, g) \leq m\) which is a contradiction.

It follows \(\mathcal{I}(C, h) = \mathcal{I}_I + (-n^* + \mathcal{I}(B, g))\).

3. NECESSARY CONDITIONS

3.1. Lemma. Let \((A, f)\) be a connected unary algebra. If \(|A^\infty| < \aleph_0\) then \(Z(A, f) = A^\infty\) and \(R(A, f) = |A^\infty|\).

Proof. By 1.15 we have \(Z(A, f) \subseteq A^\infty\).
Let us suppose $|A| < \aleph_0$. We prove $A^\infty \subseteq Z(A, f)$. It holds if $A^\infty = 0$. Thus, we can suppose $A^\infty \neq 0$. Let us have $x \in A^\infty$. Then there is a sequence $(x_i)_{i \in N}$ such that $f(x_{i+1}) = x_i$ for each $i \in N$ and $x_0 = x$. Clearly, $x_i \in A^\infty$ for each $i \in N$. From the finiteness of $A^\infty$, it follows the existence of $i, j \in N, i < j$, such that $x_i = x_j$. We prove by an easy induction that $f^n(x_i) = x$ for each $n \in N$. It follows $f'(x_i) = x = f'(x_j)$. We put $d = j - i > 0$. By 1.5 (b), we have $x \in Z(A, f)$ because $f'(x_i) = f'(f'(x_i)) = f'(x_i) = x$.

We have proved $Z(A, f) = A^\infty$ which implies $R(A, f) = |A^\infty|$. 

3.2. Lemma. Let $(A, f)$ be a connected unary algebra, suppose $\lambda, \mu \in W_{\theta(A, f)}$, $\lambda < \mu$. Then the following assertions hold:

(a) If $x, y \in A^\mu, x' \in A^\lambda, m, n \in N - \{0\}, f^n(x') = x, f^n(x') = y$ then $x = y$.

(b) If $\varphi : A^\mu \to A^\lambda$ is a map such that, for each $x \in A^\mu$, there exists $n(x) \in N - \{0\}$ with the property $f^n(x) = x$ then $\varphi$ is injective.

Proof of (a). Let us have $x, y \in A^\mu, x' \in A^\lambda, m, n \in N - \{0\}, f^n(x') = x, f^n(x') = y$. Suppose $m \geq n$. Then $x = f^m(x') = f^{m-n}(f^n(x')) = f^{m-n}(y)$. Thus, $f^{m-n}(y) = x \in A^\mu, f^0(y) = y \in A^\mu$ which implies $x = y$ by 1.19 (b).

Proof of (b). Suppose that $\varphi : A^\mu \to A^\lambda$ is such a map that, for each $x \in A^\mu$, there exists $n(x) \in N - \{0\}$ with the property $f^n(x) = x$. Let $s, t \in A^\mu$ be such elements that $\varphi(s) = \varphi(t)$. Then there exist $n(s), n(t) \in N - \{0\}$ such that $s = f^n(s), f^n(t) = f^n(t) = f^n(t) = f^n(t) = f^n(t)$. Then, by (a), we have $s = t$ and (b) holds.

3.3. Lemma. Let $(A, f)$ be a connected unary algebra, suppose $\lambda, \mu \in W_{\theta(A, f)}$, $\lambda \leq \mu$. Then $|A^\mu| \leq |A^\lambda|$. 

Proof. By 1.16, there exists a map $\varphi : A^\mu \to A^\lambda$ such that, for each $x \in A^\mu$, there is $n(x) \in N - \{0\}$ such that $f^n(x) = x$. By 3.2 (b), this map is injective. Thus $|A^\mu| \leq |A^\lambda|$. 

3.4. Lemma. Let $(A, f)$ be a connected unary algebra and $\alpha$ a limit ordinal with the property $\alpha \leq \theta(A, f)$. If $(A, f)$ is no $\infty$-algebra suppose $\alpha < \theta(A, f)$. Then $|A^\alpha| \geq \longest \alpha$ for each $x \in W_a$.

Proof. If $\alpha = 0$ then we have nothing to prove as $W_a = \emptyset$.

Suppose $\alpha > 0$.

(1) Suppose first $\alpha \neq \theta(A, f)$. Then $x \in A^\alpha$ implies $f^{-1}(x) \subseteq \bigcup_{x \in W_a} A^\alpha$, $x \in A - \bigcup_{x \in W_a} A^\alpha$. 

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Let $\kappa \in W$ be an arbitrary ordinal. Then there is an ordinal $\lambda \in W$, $\lambda > \kappa$ and an element $y \in A^\lambda$ such that $f(y) = x$. Indeed, if such $\lambda$, $y$ do not exist then there is an ordinal $\mu \in W$ such that $f^{-1}(x) \subseteq A^\mu$. Further, $x \in A - \bigcup_{\nu \in W_\mu} A^\nu \subseteq A - \bigcup_{\nu \in W_\mu} A^\nu$ which implies $x \in A^\mu$ in contradiction to 1.8.

(2) Let $(x_v)_{v \in W_{ctx}}$ be an arbitrary increasing sequence of ordinals such that $\sup x_v = \alpha$. By (1), there is an ordinal $\mu_0 \in W$, $\mu_0 > \kappa$ and an element $x_{\mu_0} \in A^{\mu_0}$ such that $f(x_{\mu_0}) = x$.

Let $\eta \in W_{ctx}$ be an arbitrary ordinal and suppose that we have constructed, for each ordinal $\nu < \eta$, an ordinal $\mu_\nu$ such that $x_\nu < \mu_\nu < \alpha$ and an element $x_{\mu_\nu} \in A^{\mu_\nu}$ in such a way that $(\mu_\nu)_{\nu \in W_\eta}$ is an increasing sequence. Then $\sup \mu_\nu < \alpha$ because $\eta < \text{cf } \alpha$ and $\text{cf } \alpha$ is the least ordinal cofinal with $\alpha$. Thus, we can take an ordinal $\mu_\eta \in W$ such that $\mu_\eta > \kappa$, $\mu_\eta > \sup \mu_\nu$, and an element $x_{\mu_\eta} \in A^{\mu_\eta}$ such that $f(x_{\mu_\eta}) = x$, by (1).

By transfinite induction, we obtain an increasing sequence of ordinals $(\mu_\nu)_{\nu \in W_{ctx}}$ and a sequence of elements $(x_{\mu_\nu})_{\nu \in W_{ctx}}$ such that $f(x_{\mu_\nu}) = x$ for each $\nu \in W_{ctx}$. Further, we have $\alpha = \sup x_\nu = \sup \mu_\nu = \alpha$; thus, $\sup \mu_\nu = \alpha$.

(3) Let $\kappa \in W$ be arbitrary. By 1.16, for each $\nu \in W_{ctx}$ such that $\mu_\nu > \kappa$ there exists $y_\nu \in A^\kappa$ and $n_\nu \in N - \{0\}$ such that $f^{n_\nu}(y_\nu) = x_{\mu_\nu}$. We put $X = \{x_{\mu_\nu}; \nu \in W_{ctx}, \kappa < \mu_\nu\}$. Clearly, $\nu, \nu' \in W_{ctx}, \nu \neq \nu'$, $\nu < \mu_\nu, \nu' < \mu_\nu$, imply $x_{\mu_\nu} \neq x_{\mu_\nu'}$ because $(\mu_\nu)_{\nu \in W_{ctx}}$ is an increasing sequence. It implies $x_{\mu_\nu} \neq x_{\mu_\nu'}$ by 1.8. Suppose $y_\nu = y_{\nu'}$, $n_\nu \leq n_{\nu'}$. We put $d = n_{\nu'} - n_\nu$ and we have $x_{\mu_\nu'} = f^{n_{\nu'}}(y_{\nu'}) = f^{n_{\nu'}}(y_\nu) = f^{d}(f^{n_\nu}(y_\nu)) = f^{d}(x_{\mu_\nu}).$ Then, for $d = 0$, we have $x_{\mu_\nu} = x_{\mu_\nu'}$, which is a contradiction. Thus, $d > 0$ and $x_{\mu_\nu'} = f^{d}(x_{\mu_\nu}) = f^{d-1}(f(x_{\mu_\nu})) = f^{d-1}(x)$ which implies $x = f(x_{\mu_\nu'}) = f^{d}(x)$. It follows from $d > 0$ that $x \in Z(A, f)$ by 1.5 (b); thus $x \in A^\omega$ by 1.3 and 1.15 which contradicts 1.13. Thus, $y_\nu \neq y_{\nu'}$.

We have proved that there exists an injection of $X$ into $A^\kappa$. Clearly, $|X| = |\text{cf } \alpha|$. Thus $|A^\kappa| \geq |\text{cf } \alpha|$.

(4) Suppose now $\alpha = \mathfrak{B}(A, f)$. By our hypothesis, $(A, f)$ is an $\infty$-algebra. Let $(B, g)$ be a cone such that $B = B^0 \cup B^\omega$ where $|B^0| = 1$, $|B^\omega| = 1$. Let $\varphi : E(A, f) \rightarrow B^0$ is the only map of $E(A, f)$ onto $B^0$. We put $I = \{1\}$, $A_1 = A, f_1 = f$. $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g)$.

By 2.9, $(C, h)$ is a connected unary algebra. We define $\mathfrak{B}_I, n^*$ by 2.12. Clearly, $\mathfrak{B}_I = \mathfrak{B}(A, f)$. If $x \in B^0$ then $S(C, h)(x) = \mathfrak{B}(A, f)$ by 2.10 (g) and we have $x \in C^r$ which implies $n^* = 0$. Clearly, $\mathfrak{B}(B, g) = 1$. It follows $\mathfrak{B}(C, h) = \mathfrak{B}(A, f) + 1$ by 2.13. Thus, $\alpha = \mathfrak{B}(A, f) < \mathfrak{B}(C, h)$. For each $\kappa \in W$, we have $|C^\kappa| \geq |\text{cf } \alpha|$ by (1), (2), (3). By 2.10 (d), we have $C^\kappa \subseteq (B - B^\omega) \cup (A - A^\omega) = B^0 \cup (A - A^\omega)$. As we have seen, $S(C, h)(x) = \mathfrak{B}(A, f)$ for $x \in B^0$. It follows $C^\kappa \subseteq A - A^\omega$. By 2.10 (h), we have $C^\kappa = A^\omega$. Thus $|A^\omega| \geq |\text{cf } \alpha|$.
3.5. Lemma. Let \((A,f)\) be a non empty connected unary algebra which is not an \(\infty\)-algebra. Then the following assertions hold:

(a) \(\mathcal{A}(A,f)\) is a limit ordinal cofinal with \(\omega_0\).

(b) If \(\lambda \in W_{\mathcal{A}(A,f)}\) is such an ordinal that \(|A^\lambda| < \aleph_0\) then there is such an ordinal \(\mu \in W_{\mathcal{A}(A,f)}\) that \(|A^\mu| = 1\).

Proof of (a). Suppose that \(\mathcal{A}(A,f)\) is an isolated ordinal. Then there is \(x \in A\) such that \(S(A,f)(x) = \mathcal{A}(A,f) - 1\). By 1.19 (a), we have \(S(A,f)(x) < S(A,f)(f(x))\). If \(S(A,f)(f(x)) \in \text{Ord}\) then \(S(A,f)(f(x)) \geq \mathcal{A}(A,f)\) which is impossible. Thus, \(S(A,f)(f(x)) = \infty\) which contradicts the hypothesis \(A^\infty = \emptyset\). Thus, \(\mathcal{A}(A,f)\) is a limit ordinal.

Let \(x \in A\) be such an element that \(S(A,f)(x) = 0\); such an element exists because \(\mathcal{A}(A,f) > 0\). For each \(x \in W_{\mathcal{A}(A,f)}\) there is an element \(y_x \in A\) such that \(S(A,f)(y_x) = x\). Since \((A,f)\) is connected there are \(n_x, n_y \in \mathbb{N}\) such that \(f^{n_x}(x) = f^{n_y}(y_x)\). By 1.19 (a), we have \(S(A,f)(f^{n_x}(x)) = S(A,f)(f^{n_y}(y_x)) \geq \mathcal{A}(A,f)\). Thus \((S(A,f)(f^{n_x}(x)))_{n \in \mathbb{N}}\) is a sequence of the type \(\omega_0\) such that \(\mathcal{A}(A,f)\) is cofinal with this sequence.

Proof of (b). Let us have \(\lambda \in W_{\mathcal{A}(A,f)}, |A^\lambda| < \aleph_0\). If \(|A^\lambda| = 1\) then we have nothing to prove. Suppose \(|A^\lambda| \geq 2\), let \(x, y \in A^\lambda\) be such elements that \(x \neq y\). As \((A,f)\) is connected there are \(n, m \in \mathbb{N} - \{0\}\) such that \(f^n(x) = f^m(y) = z\). Since \(A^\infty = \emptyset\) there is \(\lambda_1 \in W_{\mathcal{A}(A,f)}, \lambda_1 > \lambda\) such that \(z \in A^{\lambda_1}\). By 1.16, there is a map \(\varphi : A^{\lambda_1} \to A^\lambda\) such that \(\varphi(z) = x\) and that, for each \(t \in A^{\lambda_1}\), there is \(k \in N - \{0\}\) such that \(f^k(\varphi(t)) = t\). By 3.2 (b), this map is injective.

We prove that \(y \neq \varphi(A^{\lambda_1})\). Suppose, on the contrary, the existence of \(z' \in A^{\lambda_1}\) with the property \(\varphi(z') = y\). Then there is \(p \in N - \{0\}\) such that \(f^p(y) = f^p(\varphi(z')) = z' \in A^{\lambda_1}\). We have \(f^m(y) = z \in A^{\lambda_1}\). It follows \(z = z'\) by 3.2 (a) which implies \(x = \varphi(z) = \varphi(z') = y\) which is a contradiction. Thus \(\varphi : A^{\lambda_1} \to A^\lambda\) is not a surjection. Since \(A^\lambda\) is a finite set we have \(|A^\lambda| > |A^{\lambda_1}|\).

We proceed similarly with the set \((A^{\lambda_1}, \lambda_1 \in W_{\mathcal{A}(A,f)}\) as \(A^{\lambda_1}\) is a finite set. Since \(\mathcal{A}(A,f)\) is a limit ordinal, we obtain, after a finite number of steps, an ordinal \(\mu \in W_{\mathcal{A}(A,f)}\) such that \(|A^\mu| = 1\).

3.6. Definition. Let \(\alpha \in \text{Ord}\) and suppose that \((m_x)_{x \in W_\alpha \cup \{\infty\}}\) is a sequence of cardinals. We put

\[
\text{crit}((m_x)_{x \in W_\alpha \cup \{\infty\}}) = \begin{cases} W_\alpha \cup \{\alpha\} & \text{if } m_\infty \neq 0 \\ W_\alpha & \text{if } m_\infty = 0. \end{cases}
\]

3.7. Definition. Let \(\Gamma \subseteq \text{Ord} \cup \{\infty\}\) and suppose that \((m_x)_{x \in \Gamma}\) is a sequence of cardinals. This sequence is called \textit{suitable} if the following conditions are satisfied:

(1) \(\Gamma = W_\alpha \cup \{\infty\}\) for some \(\alpha \in \text{Ord}\), the sequence \((m_x)_{x \in W_\alpha}\) is non-increasing and \(m_x \neq 0\) for each \(x \in W_\alpha\) .
(2) If \( m_{\infty} = 0 \) then (a) \( \alpha \) is a limit ordinal cofinal with \( \omega_0 \), (b) the existence of \( \lambda \in W \) with the property \( m_{\lambda,0} < \aleph_0 \) implies the existence of \( \mu \in W \) with the property \( m_{\mu} = 1 \).

(3) For an arbitrary limit ordinal \( \mu \in \text{crit} (m_{x})_{x \in W_{\infty}} \) and for an arbitrary \( \lambda \in W_{\mu} \) we have \( m_{\lambda,0} \geq |\text{cf} \, \mu| \).

3.8. Theorem. Let \((A, f)\) be a non empty connected unary algebra. Then the following assertions hold:

(a) If \(|A_{\infty}| < \aleph_0\) then \( R(A, f) = |A_{\infty}| \).

(b) The sequence \((|A_{\infty}|)_{x \in \mathcal{W}_{\mathcal{A}, f}^{\infty}}\) is suitable.

Proof. (a) follows by 3.1. The property (1) of 3.7 follows by definition of \( \mathcal{E}(A, f) \) and by 3.3, the property (2) of 3.7 follows by 3.5 and the property (3) of 3.7 follows by 3.4.

3.9. Lemma. Let \( \alpha \in \text{Ord} \), let \((m_{x})_{x \in W_{\alpha}}\) be a suitable sequence of cardinals with the property \( m_{\infty} = 1 \). If \( \beta \in W_{\alpha} \) then \((m_{x})_{x \in W_{\alpha}}\) is a suitable sequence with the property \( m_{\infty} = 1 \).

Proof. The sequence \((m_{x})_{x \in W_{\alpha}}\) satisfies the condition (1) of 3.7. The condition (2) is satisfied trivially as \( m_{\infty} = 1 \). If \( \mu \in \text{crit} (m_{x})_{x \in W_{\alpha}} \) then \( \mu \leq \beta \) which implies \( \mu \in \text{crit} (m_{x})_{x \in W_{\alpha}} \). Thus, for each limit ordinal \( \mu \in \text{crit} (m_{x})_{x \in W_{\alpha}} \) and each \( \lambda \in W_{\mu} \) we have \( m_{\lambda,0} \geq |\text{cf} \, \mu| \) which is (3) of 3.7.

3.10. Lemma. Let \( \alpha \in \text{Ord} \), let \((m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}}\) be a suitable sequence of cardinals such that \( m_{\infty} = 0 \). We put \( m_{x}^{'} = m_{x} \) for each \( x \in W_{\alpha} \), \( m_{\infty}^{'} = 1 \). Then \((m_{x}^{'})_{x \in \mathcal{W}_{\alpha}^{\infty}}\) is a suitable sequence for each \( \beta \in W_{\alpha} \).

Proof. The condition (1) of 3.7 is satisfied by the sequence \((m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}}\), the condition (2) of 3.7 is satisfied trivially as \( m_{\infty}^{'} = 1 \). Clearly, \( \beta \in W_{\alpha} \) implies \( \beta \in \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} \) which implies \( \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} = W_{\beta} \cup \{\beta\} \subseteq W_{\alpha} = \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} \). If \( \mu \in \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} \) is a limit ordinal and \( \lambda \in W_{\mu} \) then \( \mu \in \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} \) which implies \( |m_{\lambda,0}^{'}| = |m_{\lambda,0}| \leq |\text{cf} \, \mu| \). Thus, the condition (3) of 3.7 is satisfied by the sequence \((m_{x}^{'})_{x \in \mathcal{W}_{\alpha}^{\infty}}\).

3.11. Lemma. Let \( \alpha \in \text{Ord} \), let \((m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}}\) be a suitable sequence of cardinals such that \( m_{\infty} \neq 0 \). We put \( m_{x}^{'} = m_{x} \) for each \( x \in W_{\alpha} \), \( m_{\infty}^{'} = 1 \). Then \((m_{x}^{'})_{x \in \mathcal{W}_{\alpha}^{\infty}}\) is a suitable sequence.

Proof. \((m_{x}^{'})_{x \in \mathcal{W}_{\alpha}^{\infty}}\) satisfies obviously the condition (1) and (2) of 3.7. Clearly, \( \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} = W_{\alpha} \cup \{\alpha\} = \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} \). Thus, if \( \mu \in \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} \) is a limit ordinal and \( \lambda \in W_{\mu} \) then \( \mu \in \text{crit} (m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}} \) and \( m_{\lambda,0}^{'} = m_{\lambda,0} \geq |\text{cf} \, \mu| \). Thus, \((m_{x})_{x \in \mathcal{W}_{\alpha}^{\infty}}\) satisfies the condition (3) of 3.7.
4. SUFFICIENT CONDITIONS

4.1. Lemma. Let \((C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g)\) be a unary algebra defined in 2.7. We put \(\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i)\). We suppose that \(\emptyset = B^0 \subseteq C^{\vartheta_I}\). Then the following conditions hold:

(a) \(n^* = 0\) where \(n^*\) is defined according to 2.12, \(\vartheta(C, h) = \vartheta_I + \vartheta(B, g)\) and, if we put \(n(x) = -\vartheta_I + x\) for each \(x\) with the property \(\vartheta_I \leq x < \vartheta(C, h)\) then \(\{n(x); \vartheta_I \leq x < \vartheta(C, h)\} = W_{\vartheta(B, g)}\).

(b) \(C^\vartheta = B^{\vartheta(x)}\) for each \(x, \vartheta_I \leq x < \vartheta(C, h)\).

Proof of (a). If \(\emptyset \neq B^0 \subseteq C^{\vartheta_I}\) then \(n^* = 0\) by 2.12. It follows \(\vartheta(C, h) = \vartheta_I + (-n^* + \vartheta(B, g)) = \vartheta_I + \vartheta(B, g)\) by 2.13.

Further, if \(\vartheta_I \leq x < \vartheta(C, h)\) then \(n(x) = -\vartheta_I + x < -\vartheta_I + \vartheta(C, h) = \vartheta(B, g)\).

On the other hand, if \(n < \vartheta(B, g)\) then \(n = -\vartheta_I + (\vartheta_I + n)\) where \(\vartheta_I \leq \vartheta_I + n < \vartheta_I + \vartheta(B, g) < \vartheta(C, h)\).

Proof of (b). (1) For each \(m < \vartheta(B, g)\), we have \(S(C, h)(B^m) \subseteq \vartheta_I + (-n^* + + m) = \vartheta_I + m\) by 2.12 and (a). Further, \(x \in B^m\) implies the existence of \(y \in B^0 \subseteq C^{\vartheta_I}\) such that \(h^n(y) = g^n(y) = x\) because \((B, g)\) is a cone. It follows \(S(C, h)(x) = S(C, h)(h^n(y)) \geq S(C, h)(y) + m = \vartheta_I + m\) by 1.19 (a). Thus, \(S(C, h)(B^m) = \vartheta_I + m\) for each \(m < \vartheta(B, g)\). It implies, for each \(x, \vartheta_I \leq x < \vartheta(C, h)\), that \(B^{\vartheta(x)} \subseteq C^{\vartheta_I + n(x)} = C^x\) by (a).

(2) \(\vartheta_I \leq x < \vartheta(C, h)\) implies \(C^x \subseteq B - B^\vartheta = \bigcup_{n \in W_{\vartheta(B, g)}} B^n\) by 2.10 (c). It implies \(C^x = C^x \cap \bigcup_{n \in W_{\vartheta(B, g)}} B^n = \bigcup_{n \in W_{\vartheta(B, g)}} (C^{\vartheta_I + n(x)} \cap B^n) = C^{\vartheta_I + n(x)} \cap B^{\vartheta(x)}\) by (a) and (1). It follows \(C^x \subseteq B^{\vartheta(x)}\).

Thus, we have \(C^x = B^{\vartheta(x)}\) for each \(x, \vartheta_I \leq x < \vartheta(C, h)\) by (1) and (2).

4.2. Lemma. Let \((A, f)\) be an \(\infty\)-algebra such that \(\vartheta(A, f) > 0\) is an isolated ordinal, \((B, g)\) a cone disjoint with \((A, f)\) such that \(B^0 \neq \emptyset\), \(|B^0| \leq |A^{\vartheta(A, f) - 1}|\). Then the following assertions hold:

(a) There exists a surjection \(\psi : A^{\vartheta(A, f) - 1} \to B^0\) which is a restriction of a surjection \(\varphi : E(A, f) \to B^0\).

We put \(I = \{1\}, A_1 = A, f_1 = f\) and let \((C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g)\) be a unary algebra defined in 2.7.

(b) \(C, h\) is a connected unary algebra.

(c) \(\vartheta(C, h) = \vartheta(A, f) + \vartheta(B, g)\).

(d) \(C^x = A^x\) for each \(x < \vartheta(A, f)\), \(C^x = B^{\vartheta(x)}\) for each \(x, \vartheta(A, f) \leq x < \vartheta(C, h)\) where \(n(x)\) is defined according to 4.1 (a), \(C^\infty = B^\infty\).
Proof of (a). Since \( \mathcal{H}(A, f) > 0 \) is an isolated ordinal then \( 0 \notin A^{\mathcal{H}(A, f)^{-1}} \subseteq E(A, f) \) by 2.3 (c) and since \( |A^{\mathcal{H}(A, f)^{-1}}| \geq |B^0| \) then there is a surjection \( \psi : A^{\mathcal{H}(A, f)^{-1}} \to B^0 \) which is a restriction of a surjection \( \varphi : E(A, f) \to B^0 \).

Proof of (b). \((C, h)\) is a connected unary algebra by 2.9.

Proof of (c). If \( x \in B^0 \) then there exists \( z \in A^{\mathcal{H}(A, f)^{-1}} \) such that \( h(z) = \psi(z) = \varphi(z) = x \). Since \( A^{\mathcal{H}(A, f)^{-1}} \subseteq C^{\mathcal{H}(A, f)^{-1}} \) by 2.10 (a) we have \( S(C, h)(z) = \mathcal{H}(A, f) - 1 \). It follows \( S(C, h)(x) = S(C, h)(h(z)) > S(C, h)(z) = \mathcal{H}(A, f) - 1 \) by 1.19 (a) because \( x \notin B^\infty = C^\infty \) with regard to 2.10 (b). Since \( g^{-1}(x) = 0 \) we have \( h^{-1}(x) = \varphi^{-1}(x) \subseteq A - A^\infty = \bigcup_{x \in W_{A, f}(C, h)} A^x \subseteq \bigcup_{x \in W_{A, f}(C, h)} C^x \) by 2.10 (a). Further, \( x \in C - \bigcup_{x \in W_{A, f}(C, h)} C^x \) because \( S(C, h)(x) \geq \mathcal{H}(A, f) \). It follows \( x \in C^{\mathcal{H}(A, f)} \) which is \( x \in C^{\mathcal{H}(A, f)} \) because \( \mathcal{H} = \sup \mathcal{H}(A_i, f_i) = \mathcal{H}(A, f) \). We have proved \( B^0 \subseteq C^{\mathcal{H}} \).

It implies \( \mathcal{H}(C, h) = \mathcal{H} + \mathcal{H}(B, g) = \mathcal{H}(A, f) + \mathcal{H}(B, g) \) by 4.1 (a).

Proof of (d). We have proved \( B^0 \subseteq C^{\mathcal{H}} \). It follows \( B = B^\infty \cup \bigcup_{x \in W_{A, f}(C, h)} B^x = C^\infty \cup \bigcup_{x \in W_{A, f}(C, h)} C^{\mathcal{H}(A, f)+x} \) by 2.10 (b), 4.1 (b) and 4.1 (a). Thus, \( C^x \subseteq A - A^\infty \) for each \( x < \mathcal{H}(A, f) \) which implies \( C^x = A^x \) for each \( x < \mathcal{H}(A, f) \) by 2.10 (h).

Further, \( C^\infty = B^{\infty(x)} \) for each \( x \), \( \mathcal{H}(A, f) \leq x < \mathcal{H}(C, h) \) by 4.1 (b).

Finally, \( C^\infty = B^\infty \) follows by 2.10 (b).

4.3. Lemma. Let \( \{(A_i, f_i); i \in I\} \) be a set of mutually disjoint \( \infty \)-algebras such that \( \mathcal{H}(A_i, f_i) > 0 \) for each \( i \in I \). We put \( \mathcal{H}_I = \sup_{i \in I} \mathcal{H}(A_i, f_i) \) and \( I(x) = \{i \in I; x \in W_{A_i, f_i}\} \) for each \( x < \mathcal{H}_I \). We suppose that, for each \( x < \mathcal{H}_I \), there is a cardinal \( m_x \geq \max \{|I_i, N_0\} \) such that \( |A_i^x| = m_x \) for each \( i \in I(x) \). Let \((B, g)\) be a cone disjoint with all \( \infty \)-algebras \((A_i, f_i)\) such that \( |B - B^\infty| = |I| \). Then the following assertions hold:

(a) There exists a surjection \( \varphi : \bigcup_{i \in I} E(A_i, f_i) \to B - B^\infty \) such that, for each \( i \in I \):

\( x \in B - B^\infty \), there is (precisely one) \( i \in I \) such that \( \varphi^{-1}(x) = E(A_i, f_i) \).

Let \( (C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g) \) be a unary algebra defined in 2.7.

(b) \((C, h)\) is a connected unary algebra.

(c) Let \( I = W_\alpha \) for some limit ordinal \( \alpha \) and suppose that \( B - B^\infty + B^0 \) implies \( \alpha = \omega_0 \) and \( \varphi(E(A_i, f_i)) = B^1 \) for each \( i \in I \). If \( \langle \mathcal{H}(A_i, f_i) \rangle_{i \in I} \) is an increasing sequence then \( \mathcal{H}(C, h) = \mathcal{H}_I \).

(c') If there is an \( \infty \)-algebra \((A, f)\) such that \( (A_i, f_i) \cong (A, f) \) for each \( i \in I \) then \( \mathcal{H}(C, h) = \mathcal{H}(A, f) + \mathcal{H}(B, g) \).

(d) \( |C^x| = m_x \) for each \( x < \mathcal{H}_I \), \( C^\infty = B^\infty \).

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(d') If there is an \(\infty\)-algebra \((A, f)\) such that \((A, f_i) \cong (A, f)\) for each \(i \in I\) then \(\mathcal{G}_I < \mathcal{G}(C, h)\) and, for each \(x\), \(\mathcal{G}_I \leq x < \mathcal{G}(C, h)\), \(C^x = B^{(\infty)}\) where \(n(x)\) is defined according to 4.1 (a).

Proof of (a). If \(i \in I\) then \(\mathcal{G}(A_i, f_i) > 0\) which implies \(A_i - A_i^\circ \neq 0\); it follows \(E(A_i, f_i) \neq \emptyset\) by 2.3 (a). Let \(\psi : I \to B - B^\infty\) be a bijection; we put \(\varphi(i) = \psi(i)\) for each \(i \in I\) and \(t \in E(A_i, f_i)\) Then \(\varphi : \bigcup_{i \in I} E(A_i, f_i) \to B - B^\infty\) is a surjection with the property: for each \(x \in B - B^\infty\), there is (precisely one) \(i \in I\) such that \(\varphi^{-1}(x) = E(A_i, f_i)\).

Proof of (b). \((C, h)\) is connected unary algebra by 2.9.

Proof of (c). (1) We put \(\{e_i\} = \varphi(E(A_i, f_i))\) for each \(i \in I\). If \(x \in B - B^\infty\) is arbitrary then, by (a), there is (precisely one) \(i \in I\) such that \(e_i = x\). Thus, \(B - B^\infty = \{e_i; i \in I\}\).

(2) We prove that \(\mathcal{S}(A, f) (e_i) = \mathcal{G}(A_i, f_i)\) for each \(i \in I = W_x\). Indeed, if \(B - B^\infty = B^0\) then, for each \(i \in I\), \(e_i \in B^0\) and \(h^{-1}(e_i) = \varphi^{-1}(e_i) = E(A_i, f_i)\) by 2.7. Thus, \(S(C, h) (e_i) = \mathcal{G}(A_i, f_i)\) by 2.10 (g).

We suppose that \(B - B^\infty \neq B^0\). Then \(I = W_{e_0}\) and \(\{e_i\} = B^I\) for each \(i \in I\). The assertion \(S(C, h) (e_i) = \mathcal{G}(A_i, f_i)\ (i \in I)\) will be proved by induction.

If \(i = 0\) then \(e_0 \in B^0\) and, by 2.7 and 2.10 (g), \(S(C, h) (e_0) = \mathcal{G}(A_0, f_0)\).

Let \(i \in I - \{0\}\) and suppose \(S(C, h) (e_{i-1}) = \mathcal{G}(A_{i-1}, f_{i-1})\). By 2.7, we have \(h^{-1}(e_{i-1}) = E(A_{i-1}, f_{i-1}) \cup B^{i-1} = E(A_{i-1}, f_{i-1}) \cup \{e_{i-1}\}\). By 2.10 (f), it follows \(\mathcal{G}(A_i, f_i) = \min \{\alpha \in \text{Ord}; \alpha > S(C, h) (E(A_i, f_i))\}\) (see 2.11). Further, \((\mathcal{G}(A_i, f_i))_{i \in I}\) is increasing and it implies \(\mathcal{G}(A_i, f_i) > \mathcal{G}(A_{i-1}, f_{i-1}) = S(C, h) (e_{i-1})\). Thus, \(\mathcal{G}(A_i, f_i) = \min \{\alpha \in \text{Ord}; \alpha > S(C, h) (E(A_i, f_i) \cup \{e_{i-1}\})\} = S(C, h) (e_i)\).

(3) By (1) and (2), there exists, for each \(x \in B - B^\infty\), \(i \in I\) such that \(S(C, h) (x) = \mathcal{G}(A_i, f_i)\). Further, we have \(\mathcal{G}(A_i, f_i) \neq \mathcal{G}_I\) for each \(i \in I\) because \((\mathcal{G}(A_i, f_i))_{i \in I}\) is increasing and \(|I| \geq \aleph_0\). Thus, \(S(C, h) (x) \neq \mathcal{G}_I\) for each \(x \in B - B^\infty\). It follows \(C^{\mathcal{G}_I} = 0\) and \(n^* = \mathcal{G}(B, g)\) by 2.12. We obtain \(\mathcal{G}(C, h) = \mathcal{G}_I + (-n^* + \mathcal{G}(B, g)) = \mathcal{G}_I\) by 2.13.

Proof of (c'). \(\mathcal{G}_I = \sup \mathcal{G}(A_i, f_i) = \mathcal{G}(A, f)\) by 1.20. Further, \(B^0 \neq \emptyset\) because \(B - B^\infty \neq 0\) and we have \(S(C, h) (x) = \mathcal{G}(A, f)\) for each \(x \in B^0\) by (a) and 2.10 (g). It implies \(B^0 \subseteq C^{\mathcal{G}_I}\). We obtain \(n^* = 0\) by 4.1 (a) and \(\mathcal{G}(C, h) = \mathcal{G}_I + (-n^* + \mathcal{G}(B, g)) = \mathcal{G}(A, f) + \mathcal{G}(B, g)\) by 2.13.

Proof of (d). We put \(\varphi(E(A_i, f_i)) = \{e_i\}\) for each \(i \in I\); then \(B - B^\infty \subseteq \bigcup_{i \in I} [e_i]_{(C, h)}\) by (a). Let us have \(x < \mathcal{G}_I\). We put \(m^* = |C^x \cap (B - B^\infty)|\). Then \(C^x \cap (B - B^\infty) \subseteq \bigcup_{i \in I} (C^x \cap [e_i]_{(C, h)})\) which implies \(m^* \leq \sum_{i \in I} |C^x \cap [e_i]_{(C, h)}| \leq |I|\) because \(C^x \cap (d') // there is an oo-algebra \((A, f)\) such that \((A, f_i) \cong (A, f)\) for each \(i \in I\) then \(\mathcal{G}_I < \mathcal{G}(C, h)\) and, for each \(x\), \(\mathcal{G}_I \leq x < \mathcal{G}(C, h)\), \(C^x = B^{(\infty)}\) where \(n(x)\) is defined according to 4.1 (a).
\[ \cap [e_i]_{i \in I} \leq 1 \text{ by 1.19 (b)}. \] We have \( C^\omega \subseteq (B - B^\omega) \cup \bigcup_{i \in I(x)} (A_i - A_i^\omega) \) by 2.10 (d) which implies \( C^\omega = C^\omega \cap ((B - B^\omega) \cup \bigcup_{i \in I(x)} (A_i - A_i^\omega)) = (C^\omega \cap (B - B^\omega)) \cup \bigcup_{i \in I(x)} A_i \) with disjoint summands by 2.10 (a) and 1.13. It follows \( |C^\omega| = m^* + \sum_{i \in I(x)} |A_i^\omega| = = m^* + \sum_{i \in I(x)} |l(x)| \) because \( m_x \geq N_0 \), \( m_x \geq |l| \geq |l(x)| \), \( m_x \geq |l| \geq m^* \) and \( l(x) \neq 0 \).

\( C^\omega = B^\omega \) follows by 2.10 (b).

**Proof of (d').** \( \mathcal{H}(C, h) = \mathcal{H}(A, f) + \mathcal{H}(B, g) = \mathcal{H}_f + \mathcal{H}(B, g) \) by (c') and \( \mathcal{H}(B, g) > 0 \) because \( B - B^\omega \neq 0 \). It follows \( \mathcal{H}_f < \mathcal{H}(C, h) \). Further, we have proved \( 0 \neq B_0 \subseteq C^\omega \) in the proof of (c'). It implies \( C^\omega = B^{\omega(x)} \) for each \( \kappa \), \( \mathcal{H}_f \leq \kappa < \mathcal{H}(C, h) \), by 4.1 (b).

4.4. **Definition.** Let us have \( \alpha \in \text{Ord} \), let \( \{m_\alpha\}_{\alpha \in W_\alpha \cup \{\infty\}} \) be a suitable sequence of cardinals, \((A, f)\) an \( \omega \)-algebra. Then \((A, f)\) is said to have the property (\( \beta \)) with respect to the given sequence if \( \beta \in W_\alpha \), \( \mathcal{H}(A, f) = \beta \) and \( |A_{\alpha}| = m_\alpha \) for each \( \alpha \in W_\beta \).

4.5. **Lemma.** Let us have \( \alpha \in \text{Ord} \), \( \alpha \geq 2 \), let \( \{m_\alpha\}_{\alpha \in W_\alpha \cup \{\infty\}} \) be a suitable sequence of cardinals with the property \( m_\alpha \leq 1 \). If, for each \( \beta \in W_\alpha \), there is an \( \omega \)-algebra having the property (\( \beta \)) with respect to the given sequence then there exists a connected unary algebra \((A, f)\) such that \( \mathcal{H}(A, f) = \alpha \) and \( |A_{\alpha}| = m_\alpha \) for each \( \alpha \in \in W_\beta \).

**Proof.** (I) If \( \alpha \) is an isolated ordinal then \( m_\alpha = 1 \) because \( m_\alpha \neq 0 \) by 3.7. Thus, there exists \( \alpha - 1 \in \text{Ord} \) because \( \alpha \geq 2 \) and an \( \omega \)-algebra \((A, f)\) having the property \( \alpha - 1 \) with respect to the given sequence.

Two cases can occur:

(1) Suppose \( m_\alpha \geq N_0 \) for each \( \alpha \in W_{\alpha - 1} \).

Let \( \{(A_i, f_i) : i \in I\} \) be a set of mutually disjoint \( \omega \)-algebras such that \((A_i, f_i) \cong (A, f)\) for each \( i \in I \) and \( |l| = m_{\alpha - 1} \). Let \((B, g)\) be a cone disjoint with all algebras \((A_i, f_i)\) such that \( B = B^0 \cup B^\omega \), \( |B^0| = m_{\alpha - 1} \), \( |B^\omega| = 1 \).

Then \( \mathcal{H}(A_i, f_i) = \mathcal{H}(A, f) = \alpha - 1 > 0 \) for each \( i \in I \) by 1.20. Further, we have \( \mathcal{H}_f = \sup_{i \in I} \mathcal{H}(A_i, f_i) = \mathcal{H}(A, f) = \alpha - 1 \) and, for each \( \kappa < \alpha - 1 \), \( m_\kappa \geq m_{\alpha - 1} = = |l| \) which implies \( m_\kappa \geq \max \{ |l|, N_0 \} \) and \( |A_{\kappa}| = m_\kappa \) for each \( \kappa \in I = I(\alpha) \) (see 4.3). Finally, we have \( |B - B^\omega| = |B^0| = m_{\alpha - 1} = |l| \). Then there exists a surjection \( \varphi : \bigcup_{i \in I} E(A_i, f_i) \to B - B^\omega \) such that, for each \( \alpha \in B - B^\omega \), there is (precisely one) \( i \in I \) such that \( \varphi^{-1}(\alpha) = E(A_i, f_i) \) by 4.3 (a). We put \((C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g)\).

By 4.3 (b), \((C, h)\) is a connected unary algebra and \( \mathcal{H}(C, h) = \mathcal{H}(A, f) + \mathcal{H}(B, g) = = (\alpha - 1) + 1 = \alpha \) by 4.3 (c') because \( \mathcal{H}(B, g) = 1 \).
By 4.3 (d), we have $|C^x| = m_x$ for each $x < \alpha - 1$. By 4.3 (d'), we obtain $C^{\alpha - 1} = B^{\alpha}$ because $\vartheta(C, h) = \alpha$. It implies $|C^{\alpha - 1}| = |B^{\alpha}| = m_{\alpha - 1}$. By 4.3 (d), we obtain $|C^x| = |B^{\alpha}| = 1 = m_\infty$.

We have constructed a connected unary algebra $(C, h)$ with the following properties: $\vartheta(C, h) = \alpha$, $|C^x| = m_x$ for each $x \in W_{\alpha} \cup \{\infty\}$.

(2) Suppose the existence of $x_0 \in W_{\alpha - 1}$ such that $m_{x_0} < \omega_0$.

Then $x_0 \geq 2$ implies $x_0 - 1 \geq 1$. Clearly, $x_0 - 1 \in \text{crit}(m_{x_0})$. If $x_0 - 1$ were a limit ordinal then we should have $m_{x_0} \geq |\text{cf} (x_0 - 1)|$ by 3.7 (3) which is a contradiction to the finiteness of $m_{x_0}$. Thus, $x_0 - 1$ is an isolated ordinal.

Let $(B, g)$ be a cone disjoint with $(A, f)$ such that $B = B^0 \cup B^\infty$ and $|B^0| = m_{x_0} - 1$, $|B^\infty| = 1$.

$\vartheta(A, f) = x_0 - 1 > 0$ is an isolated ordinal and we have $B^0 = 0$, $|B^0| = m_{x_0 - 1} \leq m_{x_0 - 2} = |A^{\alpha - 2}| = |A^{\vartheta(A, f) - 1}|$. Then there exists a surjection $\psi : A^{\vartheta(A, f) - 1} \rightarrow B^0$ which is a restriction of a surjection $\varphi : E(A, f) \rightarrow B^0$ by 4.2 (a). We put $I = \{1\}$, $A_1 = A, f_1 = f$, $(C, h) = \bigcup_i (A_i, f_i) \oplus (B, g)$.

By 4.2 (b), $(C, h)$ is a connected unary algebra. Clearly, $\vartheta(B, g) = 1$ which implies $\vartheta(C, h) = \vartheta(A, f) + \vartheta(B, g) = (x_0 - 1) + 1 = x_0$ by 4.2 (c).

Further, $C^x = A^x$ for each $x < x_0 - 1$, $C^{x_0 - 1} = B^0$, $C^\infty = B^\infty$ by 4.2 (d) because $\vartheta(C, h) = \alpha$. It follows $|C^x| = m_x$ for each $x \in W_{\alpha} \cup \{\infty\}$.

(II) Suppose that $\alpha$ is a limit ordinal. We put $I = W_{cf\alpha}$. Then there exists an increasing sequence of positive ordinals $(\beta_i)_{i \in I}$ such that $sup i = \alpha$. For each $i \in I$ there exists an $\infty$-algebra $(A_i, f_i)$ having the property $(\beta_i)$ with respect to the given sequence. We can suppose, without loss of generality, that the $\infty$-algebras $(A_i, f_i)$ are mutually disjoint.

The set $\{(A_i, f_i); i \in I\}$ of $\infty$-algebras has the following properties: $\vartheta(A_i, f_i) = \beta_i > 0$ for each $i \in I$; $\beta_i = sup \vartheta(A_i, f_i) = sup i = \alpha$; if we put $I(\alpha) = \{i \in I; \alpha \in W_{\vartheta(A_i, f_i)}\}$ for each $\alpha < \beta$ (see 4.3) then, for each $i \in I(\alpha)$, we have $|A_i^\alpha| = m_x$ because $(A_i, f_i)$ is an $\infty$-algebra having the property $(\beta_i)$ with respect to the given sequence.

Two cases can occur:

(i) Let us have $m_\infty = 1$. Since $(m_x)_{x \in W_{\alpha} \cup \{\infty\}}$ is a suitable sequence then $\text{crit}(m_x)_{x \in W_{\alpha} \cup \{\infty\}} = W_{\alpha} \cup \{\infty\}$ and, by 3.7 (3), we have $m_x \geq |\text{cf} x| = |I|$ for each $x \in W_{\alpha}$. Thus, for each $x < \alpha = \vartheta$, we have $m_x \geq max\{|I|, \omega_0\}$ because $|I| \geq \omega_0$.

We take a cone $(B, g)$ disjoint with all $\infty$-algebras $(A_i, f_i)$ such that $B = B^0 \cup B^\infty$ where $|B^0| = |I| = |\text{cf} \alpha|, |B^\infty| = 1$.

Thus, $B - B^\infty = B^0$ and $|B - B^\infty| = |I|$. 241
By 4.3 (a), there exists a surjection \( \varphi : \bigcup_{i \in I} E(A_i, f_i) \to B - B^\infty \) such that, for each \( x \in B - B^\infty \), there is (precisely one) \( i \in I \) such that \( \varphi^{-1}(x) = E(A_i, f_i) \). We put \( (C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g) \).

Then \( (C, h) \) is a connected unary algebra by 4.3 (b). Further, \( \vartheta(C, h) = \vartheta_I = \alpha \) by 4.3 (c).

Finally, \( |C^x| = m_x \) for each \( x < \vartheta_I = \alpha \), \( |C^\infty| = |B^\infty| = 1 = m_\infty \) by 4.3 (d).

Thus, we have constructed a connected unary algebra \( (C, h) \) such that \( \vartheta(C, h) = \alpha \) and \( |C^x| = m_x \) for each \( x \in W_s \cup \{\infty\} \).

(ii) Let us have \( m_\infty = 0 \). Since \( (m_\infty)_{x \in W_s \cup \{\infty\}} \) is a suitable sequence we have \( \text{cf} \, \alpha = \omega_0 \) by 3.7.

Two cases are possible:

1) Suppose \( m_x \geq \omega_0 \) for each \( x \in W_s \).

Then, for each \( x < \alpha = \vartheta_I \), we have \( m_x \geq \max \{ |I|, \omega_0 \} \) because \( |I| = \text{cf} \, x = \omega_0 \).

Let \( (B, g) \) be the cone (constructed in 2.5, 2) such that \( |B^n| = 1 \) for each \( n \in \mathbb{N} \).

Suppose that \( (B, g) \) is disjoint with all \( \omega \)-algebras \( (A_i, f_i) \).

Thus, \( B^\infty = 0 \) and \( |B - B^\infty| = |B| = \omega_0 = |I| \).

We take, by 4.3 (a), a surjection \( \varphi : \bigcup_{i \in I} E(A_i, f_i) \to B \) such that \( \varphi(E(A_i, f_i)) = B^i \) for each \( i \in I = W_\omega = \mathbb{N} \). We put \( (C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g) \).

Then \( (C, h) \) is a connected unary algebra by 4.3 (b) and \( \vartheta(C, h) = \vartheta_I = \alpha \) by 4.3 (c).

Further, \( |C^x| = m_x \) for each \( x < \alpha \), \( |C^\infty| = |B^\infty| = 0 = m_\infty \) by 4.3 (d).

Thus, we have constructed a connected unary algebra \( (C, h) \) such that \( \vartheta(C, h) = \alpha \) and \( |C^x| = m_x \) for each \( x \in W_s \cup \{\infty\} \).

2) Suppose the existence of \( x_0 \in W_s \) such that \( m_{x_0} \neq \omega_0 \).

Clearly, \( x \in W_s \) implies \( x \in \text{crit} \, (m_\infty)_{x \in W_s \cup \{\infty\}} \). If there is a limit ordinal \( x \), \( x_0 < x < \alpha \), then \( m_{x_0} \geq \text{cf} \, x \) by 3.7 (3) which is a contradiction to the finiteness of \( m_{x_0} \).

Thus, each \( x \) with the property \( x_0 < \alpha < x \) is isolated.

We take an arbitrary \( \lambda, x_0 < \lambda < \alpha \). Thus, there is an \( \omega \)-algebra \( (A, f) \) having the property \( (\lambda) \). Thus \( \vartheta(A, f) = \lambda > 0 \) is an isolated ordinal.

By 3.7 (2) (b), there is \( \mu \in W_s \) such that \( m_{\mu} = 1 \). It follows the existence of a cone \( (B, g) \) such that \( |B^n| = m_{\lambda + n} \) for each \( n \in \mathbb{N} \) by 2.5.

Then \( |A^{\vartheta(A, f) - 1}| = |A^{\lambda - 1}| = m_{\lambda - 1} \geq m_{\lambda} = |B^0| \).

By 4.2 (a), there exists a surjection \( \psi : A^{\vartheta(A, f) - 1} \to B^0 \) which is a restriction of a surjection \( \varphi : E(A, f) \to B^0 \).

We put \( J = \{1\}, A_1 = A, f_1 = f \) and \( (C, h) = \bigcup_{i \in J} (A_i, f_i) \oplus (B, g) \).
By 4.2 (b), $(C, h)$ is a connected unary algebra and $\mathcal{G}(C, h) = \mathcal{G}(A, f) + \mathcal{G}(B, g) = \lambda + \omega_0 = \alpha$ by 4.2 (c) because $\mathcal{G}(B, g) = \omega_0$ and $\lambda + \omega_0, \alpha$ are both equal to the least limit ordinal greater than $\lambda$.

Further, $C^x = A^x$ for each $x < \lambda$ and $C^x = B^{n(x)}$ for each $x, \lambda \leq x < \alpha$ where $n(x)$ is the only element of $N$ such that $x = \lambda + n(x)$ (see 4.1 (a)), $C^\infty = B^\infty = 0$ by 4.2 (d). It follows $|C^x| = |A^x| = m_x$ for each $x < \lambda$, $|C^\infty| = |B^{n(x)}| = m_{\lambda+n(x)} = m_\alpha$ for each $x, \lambda \leq x < \alpha$ and $|C^\infty| = 0 = m_\alpha$. Thus, $|C^x| = m_x$ for each $x \in W_\alpha \cup \{\infty\}$.

4.6. Corollary. Let $\alpha \in \text{Ord}$, let $(m_x)_{x \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals such that $m_\infty = 1$. Then there is a connected unary algebra $(A, f)$ such that $\mathcal{G}(A, f) = \alpha$ and $|A^x| = m_x$ for each $x \in W_\alpha \cup \{\infty\}$.

Proof. For each ordinal, we denote by $V(\alpha)$ the following assertion: If $(m_x)_{x \in W_\alpha \cup \{\infty\}}$ is an arbitrary suitable sequence of cardinals such that $m_\infty = 1$ then there is a connected unary algebra $(A, f)$ such that $\mathcal{G}(A, f) = \alpha$ and $|A^x| = m_x$ for each $x \in W_\alpha \cup \{\infty\}$.

If we put $A = A^\infty$ where $|A^\infty| = 1 = m_\alpha$ then we see that $V(0)$ holds. Similarly, if we define the cone $A = A^0 \cup A^\infty$ where $|A^0| = m_0$, $|A^\infty| = 1 = m_\infty$ then we see that $V(1)$ holds.

Let us have $\beta \geq 2$ and suppose that $V(\gamma)$ holds for each $\gamma < \beta$. Let $(m_x)_{x \in W_\beta \cup \{\infty\}}$ be a suitable sequence of cardinals such that $m_\infty = 1$. If $\gamma \in W_\beta$ then the sequence $(m_x)_{x \in W_\gamma \cup \{\infty\}}$ is a suitable sequence of cardinals such that $m_\infty = 1$ by 3.9. Thus, by the induction hypothesis, there is a connected unary algebra $(A_\gamma, f_\gamma)$ such that $\mathcal{G}(A_\gamma, f_\gamma) = \gamma$ and $|A^x_\gamma| = m_x$ for each $x \in W_\gamma \cup \{\infty\}$. Thus, for each $\gamma \in W_\beta$, $(A_\gamma, f_\gamma)$ is an $\infty$-algebra having the property $(\gamma)$ with respect to the sequence $(m_x)_{x \in W_\gamma \cup \{\infty\}}$ (cf. 4.4). By 4.5, there is a connected unary algebra $(A, f)$ such that $\mathcal{G}(A, f) = \beta$ and $|A^x| = m_x$ for each $x \in W_\beta \cup \{\infty\}$. Thus, $V(\beta)$ holds.

It follows by transfinite induction that $V(\alpha)$ holds for each ordinal $\alpha$ which is our assertion.

4.7. Corollary. Let $\alpha \in \text{Ord}$, let $(m_x)_{x \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals such that $m_\infty = 0$. Then there is a connected unary algebra $(A, f)$ such that $\mathcal{G}(A, f) = \alpha$ and $|A^x| = m_x$ for each $x \in W_\alpha \cup \{\infty\}$.

Proof. Since $m_\infty = 0$ the ordinal $\alpha$ is a limit ordinal by 3.7 which implies $\alpha \geq 2$. We put $m_x = m_\alpha$ for each $x \in W_\alpha$, $m_\infty = 1$. If $\beta \in W_\alpha$ then $(m_x)_{x \in W_\beta \cup \{\infty\}}$ is a suitable sequence with the property $m_\infty = 1$ by 3.10. By 4.6, there is an $\infty$-algebra $(A_\beta, f_\beta)$ such that $\mathcal{G}(A_\beta, f_\beta) = \beta$ and $|A^x_\beta| = m_x = m_\alpha$ for each $x \in W_\beta$. Thus, for each $\beta \in W_\alpha$, $(A_\beta, f_\beta)$ has the property $(\beta)$ with respect to the sequence $(m_x)_{x \in W_\alpha \cup \{\infty\}}$ (cf. 4.4). The assertion follows by 4.5.
4.8. Lemma. Let $m > 0$ be a cardinal, $R \in \mathbb{N}$ an ordinal such that $m < \aleph_0$ implies $R = m$. Then there is a connected unary algebra $(A, f)$ such that $A = A^\infty$, $|A| = |A^\infty| = m$ and $R(A, f) = R$.

Proof. Let $A$ be an arbitrary set such that $|A| = m$. We take an arbitrary subset $B \subseteq A$ such that $|B| = R$. We have the following possibilities:

(I) $m < \aleph_0$.

Then $R = m$ and $B = A$. We put $A = \{a_1, a_2, \ldots, a_m\}$. We put $f(a_i) = a_{i+1}$ for each $i$, $1 \leq i \leq m - 1$, $f(a_m) = a_1$. Then $(A, f)$ is a connected unary algebra such that $A = A^\infty = Z(A, f)$ which implies $|A| = |A^\infty| = m = R = |Z(A, f)| = R(A, f)$.

(II) $m \geq \aleph_0$.

Then $|A - B| = m = \aleph_0 m$. We take an arbitrary set $K$ such that $|K| = m$ and, for each $x \in K$, we define a subset $B_x \subseteq A - B$ such that $|B_x| = \aleph_0$, $A - B = \bigcup_{x \in K} B_x$ with disjoint summands. We have $A = B \cup \bigcup_{x \in K} B_x$. Two cases can occur:

(1) $R = 0$.

Then we put $B = \{a_1, a_2, \ldots, a_R\}$, $B_x = \{a_x^i; i \in N\}$ for each $x \in K$. We define $f(a_i) = a_{i+1}$ for each $i$, $1 \leq i \leq R - 1$, $f(a_R) = a_1$, $f(a_x^i) = a_x^{i+1}$ for each $x \in K$, $i \in N - \{0\}$, $f(a_x^0) = a_1$ for each $x \in K$. Then $(A, f)$ is a connected unary algebra, $R(A, f) = |Z(A, f)| = |B| = R$, $A^\infty = A$ which implies $|A| = |A^\infty| = m$.

(2) $R > 0$.

Then we have $B = 0$. We put $B_x = \{a_x^i; i \in N\}$ for each $x \in K$, we take an arbitrary $x_0 \in K$ and we define $f(a_x^0) = a_{x_0}^{x+1}$ for each $i \in N$, $f(a_x^i) = a_x^{i+1}$ for each $x \in K$, $i \in N - \{0\}$, $f(a_x^0) = a_1$ for each $x \in K$ and each $x \in K - \{x_0\}$.

Then, clearly, $(A, f)$ is a connected unary algebra such that $Z(A, f) = 0$ which implies $R(A, f) = 0 = R$. Further, $A^\infty = A$ and $|A| = |A^\infty| = m$.

4.9. Theorem. Let $\alpha \in \text{Ord}$, let $(m_x)_{x \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals, let $R \in \mathbb{N}$ be such that $m_\infty < \aleph_0$ implies $R = m_\infty$. Then there is a connected unary algebra $(A, f)$ such that $R(A, f) = R$, $\theta(A, f) = \alpha$ and $|A^\infty| = m_x$ for each $x \in W_\alpha \cup \{\infty\}$.

Proof. (I) If $m_\infty = 0$ then there is a connected unary algebra $(A, f)$ such that $\theta(A, f) = \alpha$ and $|A^\infty| = m_x$ for each $x \in W_\alpha \cup \{\infty\}$ by 4.7. Further, $Z(A, f) \subseteq A^\infty$ by 1.15 which implies $R(A, f) = |Z(A, f)| \leq |A^\infty| = m_\infty = 0 = R$; thus, $R(A, f) = R$.

(II) If $m_\infty \neq 0$ then we put $m'_x = m_x$ for each $x \in W_\alpha$ and $m'_\infty = 1$. By 3.11, $(m'_x)_{x \in W_\alpha \cup \{\infty\}}$ is a suitable sequence with the property $m'_\infty = 1$. By 4.6, there is a connected unary algebra $(A, f)$ such that $\theta(A, f) = \alpha$ and $|A^\infty| = m'_x = m_x$ for each $x \in W_\alpha$. By 4.8, there is a cone $(B, g)$ such that $B = B^\infty$, $|B| = |B^\infty| = m_\infty$ and $R(B, g) = R$. We can suppose, without loss of generality, that $(A, f)$, $(B, g)$ are mutually disjoint.

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Two cases can occur:

(1) If \( a = 0 \) then \( W_\alpha \cup \{\infty\} = \{\infty\} \) and \((B, g)\) has the properties \( R(B, g) = R, \ |

(2) If \( a > 0 \) then \( \emptyset \neq \mathcal{A}^0 \subseteq A - A^\infty \) which implies \( E(\mathcal{A}, f) \neq \emptyset \) by 2.3 (a). Let \( \varphi : E(\mathcal{A}, f) \to B \) be an arbitrary map. We put \( I = \{1\}, A_1 = A, f_1 = f, (C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus (B, g). \)

Then \((C, h)\) is connected unary algebra by 2.9. If \( \mathcal{A}_f \) and \( n^* \) are defined by 2.12 then \( \mathcal{A}_f = \mathcal{A}(\mathcal{A}, f) \). Clearly, \( \mathcal{A}(B, g) = 0 \) which implies \( n^* = 0 \). By 2.13, \( \mathcal{A}(C, h) = \mathcal{A}_f + (-n^* + \mathcal{A}(B, g)) = \mathcal{A}(\mathcal{A}, f) = \alpha. \) If \( x \in W_\alpha \) then \( x < \mathcal{A}(\mathcal{A}, f) \) by 2.10 (d). It follows \( C^x = A^x \) for each \( x < \mathcal{A}(\mathcal{A}, f) \) by 2.10 (h). Thus, \( |C^x| = m_x \) for each \( x \in W_\alpha \). By 2.10 (b), (c), we have \( C^x = B^\infty \) and \( Z(C, h) = Z(B, g) \) which implies \( |C^x| = |B^\infty| = m_x, R(C, h) = |Z(C, h)| = |Z(B, g)| = R(B, g) = R. \) Thus, we have constructed a connected unary algebra \((C, h)\) such that \( R(C, h) = R, \mathcal{A}(C, h) = \alpha, |C^x| = m_x \) for each \( x \in W_\alpha \cup \{\infty\}. \)

4.10. Theorem. Let \( A \) be a set, \( S : A \to \operatorname{Ord} \cup \{\infty\} \) a map, \( R \in \mathbb{N}. \) Let the following conditions be satisfied:

(a) If \( |S^{-1}(\infty)| < \aleph_0 \) then \( R = |S^{-1}(\infty)|. \)

(b) The sequence \( (|S^{-1}(\infty)|)_{\alpha \in S(A)} \) is suitable.

Then there is a unary operation \( f \) on \( A \) such that \((A, f)\) is a non empty connected unary algebra and \( S(A, f) = S, R(A, f) = R. \)

Proof. By 3.7 (1), there is \( \alpha \in \operatorname{Ord} \) such that \( S(A) = W_\alpha \cup \{\infty\} \). By 4.9, there is a connected unary algebra \((A_*, f_*)\) such that \( \mathcal{A}(A_*, f_*) = \alpha, R(A_*, f_*) = R \) and \( |A_*^x| = |S^{-1}(\infty)| \) for each \( x \in W_\alpha \cup \{\infty\} \). We have \( |A_*^x| = \sum_{x \in W_\alpha \cup \{\infty\}} |S^{-1}(\infty)| = |A|. \) Thus, there is a bijection \( \varphi : A_* \to A \) such that \( \varphi | A_*^x : A_*^x \to S^{-1}(\infty) \) is a bijection for each \( x \in W_\alpha \cup \{\infty\} \). We put \( f(x) = \varphi(f_*(\varphi^{-1}(x))) \) for each \( x \in A \). Then \( f \) is a unary operation on \( A \) such that \( \varphi^{-1}(f(x)) = f_*(\varphi^{-1}(x)) \) for each \( x \in A \). It follows that \( \varphi^{-1} \) is a bijective homorphism of \((A, f)\) onto \((A_*, f_*)\). Thus, \((A, f), (A_*, f_*)\) are isomorphic, \( \varphi \) is an isomorphism of \((A_*, f_*)\) onto \((A, f)\).

By 1.20, we have \( \mathcal{A}(A, f) = \mathcal{A}(A_*, f_*) \) for each \( x \in W_{\mathcal{A}(A, f)} \cup \{\infty\} \) and \( \mathcal{A}(Z(A_*, f_*)) = Z(A, f) \). Thus, \( R(A, f) = |Z(A, f)| = |\mathcal{A}(Z(A_*, f_*))| = |Z(A_*, f_*)| = R(A_*, f_*) = R. \) Further, for each \( x \in W_\alpha \cup \{\infty\} \) we have \( S^{-1}(A, f)(x) = A^x = \varphi(A_*^x) = \varphi(\varphi^{-1}(S^{-1}(\infty))) = S^{-1}(\infty) \) which implies \( S(A, f) = S. \)

If \( |S^{-1}(\infty)| \neq 0 \) then \( \emptyset \neq A^0 \subseteq A \); if \( |S^{-1}(\infty)| = 0 \) then \( \alpha \) is infinite by 3.7 (2) which implies \( |S^{-1}(0)| \neq 0 \) by 3.7 (1) which implies \( 0 \neq A^0 \subseteq A. \) Thus, \((A, f)\) is non-empty.

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5. SOLUTION OF THE PROBLEM

5.1. Main Theorem. Let $A$ be a set, $S : A \rightarrow \text{Ord} \cup \{\infty\}$ a map, $R \in \mathbb{N}$ a finite ordinal. Then the following conditions are equivalent:

(A) There is a unary operation $f$ on $A$ such that $(A, f)$ is a non empty connected unary algebra, $S(A, f) = S$, $R(A, f) = R$.

(B) The following conditions are satisfied:

(a) If $|S^{-1}(\infty)| < \aleph_0$ then $R = |S^{-1}(\infty)|$.

(b) The sequence $(|S^{-1}(x)|)_{x \in S(A)}$ is suitable.

It is a consequence of 3.8 and 4.10.

Bibliography


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