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REMARKS ON REGULAR FACTORS OF REGULAR GRAPHS

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The notions of this paper are used in the sense of [5]. A *graph* G will be denoted by (V, H), where $V = V(G)$ and $H = H(G)$ are the sets of its *points* and *lines* respectively. If $A \subset V$ then $G(A)$ denotes the *induced subgraph* of G on the points of A. An *r-regular graph* is a regular graph of degree r. An *f-factor* is an f-regular factor. If $A, B \subset V(G)$ then we denote $H_{A,B} = \{xy \in H(G) \mid x \in A, y \in B\}$. $d_G(v)$ denotes the *degree* of a point v in G.

In this paper we shall give certain sufficient conditions for the existence of regular factors of regular graphs. These conditions are related to the line connectivity of graphs. Such a research is not new. For example, PETERSEN [7] has proved *that every bridgeless cubic graph has a* 1-*factor*. Some other such results will be mentioned below. Moreover, we can require a factor not to contain several arbitrarily prescribed lines. The following result is due to SCHÖNBERGER [9] (see also [6], p. 192): *Every bridgeless cubic graph has a* 1-*factor not containing two arbitrarily prescribed lines.* Our aim is to derive results of this nature. At first we give the following well-known assertions which will be used in the sequel.

Lemma 1 (Petersen [7]). *Every r-regular graph G with even $r > 0$ has a 2-factor. Consequently, G can be decomposed into 2-factors.*

Lemma 2 (BAEBLER [1]). *A $(2n + 1)$-regular 2m-line-connected graph has a 2m-factor.*

Lemma 3 (GALLAI [4]). *Let $G(V)$ be an r-regular k-line-connected graph with $k > 0$ and let f be an integer with even $f|V|$. If $r/k \leqq f \leqq r - r/k$ then G has an f-factor.*

In fact, Gallai in [4] has proved:

Let be given an r-regular, connected, nontrivial graph $G(V)$ and let c denote the minimum of $|H_{M,V-M}|$ taken over all proper subsets M of V with odd $|M|$. Then G has an f-factor if $f|V|$ is even and $r/c \leqq f \leqq r - r/c$.

Petersen, Baebler and Gallai obtained their results by using the method of alternating paths. This method provides also for a new proof of Tutte's theorem on 1-factors [10], see Gallai [4]. The following Lemma 4 was proved in [8] as a consequence of this Tutte's theorem. TUTTE in [11] has given a criterion for the existence of a factor with prescribed degree sequence and later (in [12]) proved this result as a straightforward consequence of his theorem on 1-factors. Lemma 5 is a special case of this general result and BELCK [2] has obtained it a time before.

Lemma 4 (PLESNÍK [8]). *Let $G = (V, H)$ be an r-regular $(r - 1)$-line-connected graph with $r > 0$ and even $|V|$. Let $X \subset H$ be an arbitrary set of $r - 1$ lines. Then the graph $G' = (V, H - X)$ has a 1-factor. Consequently, G has a 1-factor containing an arbitrarily prescribed line* (BERGE [3], *p.* 155).

Lemma 5 (Belck [2], Tutte [11, 12]). *A graph $G = (V, H)$ is without an f-factor if and only if there exist sets $S \subset V$ and $T \subset V - S$ such that $fs < q + ft - \sum_{v \in T} d_{G(V-S)}(v)$, where $s = |S|$, $t = |T|$ and q is the number of such components C of $G(V - S - T)$ that $|H_{T,V(C)}| + f|V(C)|$ is odd.*

The following theorem shows that Lemma 3 can be slightly strengthened.

Theorem 1. *Let $G = (V, H)$ be an r-regular k-line-connected graph with $k > 0$. Let x and f be nonnegative integers with $k > x$, even $f|V|$ and*

(a) $$\frac{r}{k - x} \leqq f \leqq \frac{r}{k}\left(k - 1 - \frac{2xk}{2x + 1}\right).$$

Then for any $X \subset H$ with $|X| = x$, the graph $G' = (V, H - X)$ has an f-factor.

Proof. Let us suppose G' has no f-factor. Then according to Lemma 5 there are disjoint sets $S, T \subset V$ with

(1) $$fs < ft + q - \sum_{v \in T} d_{G'(V-S)}(v).$$

Denote $H - X$ by H' and let P denote the union of point sets of all such components C of $G'(V - S - T)$ for which $|H'_{T,V(C)}| + f|V(C)|$ is odd. Denoting by R the set $V - S - T - P$, the set V is decomposed into four disjoint sets: S, T, P and R. We shall write: $X_{S,S} = H_{S,S} \cap X$, $x_{S,S} = |X_{S,S}|$, $X_{S,T} = H_{S,T} \cap X$, $x_{S,T} = |X_{S,T}|$ etc. Further, denote: $m = |H_{S,T}|$, $y_1 = 2x_{T,T} + x_{T,R} + x_{T,P}$, $y_2 = 2x_{P,P} + x_{P,R}$ and $y_3 = 2x_{S,S} + 2x_{T,T} + x_{R,S} + x_{R,T}$. Thus we have $\sum_{v \in T} d_{G'(V-S)}(v) = rt - m - y_1$ and (1) gives

(2) $$fs < ft + q - rt + m + y_1.$$

It can be observed that

$$(3)\qquad 2x + 1 \leqq k\,.$$

In the opposite case by the assumption (a) we should have

$$f \leqq (r/k)\,(k - 1 - 2xk/(2x + 1)) = (r/k)\,(-1 + k/(2x + 1)) < 0\,,$$

which is impossible.

Now we shall prove that

$$(4)\qquad s + t > 0\,.$$

Indeed, if $s + t = 0$ then $G'(V - S - T) = G'(V)$ and since $x < k$, it follows $q = 1$ by Lemma 5. Then $f|V|$ is odd which contradicts our assumption.

If we denote $h_{S,P} = |H_{S,P}|$, $h_{T,P} = |H_{T,P}|$, etc, then, since G is k-line-connected and $s + t > 0$, it follows

$$(5)\qquad h_{S,P} + h_{T,P} + h_{R,P} + 2x_{P,P} \geqq kq\,.$$

Obviously, $x_{S,P} \leqq h_{S,P}$, $x_{S,S} \leqq h_{S,S}$ etc, $x_{S,T} \leqq h_{S,T} = m$ and $x_{P,R} = h_{P,R}$. Further,

$$h_{S,P} + h_{T,P} = rt + rs - 2h_{S,T} - 2h_{S,S} - 2h_{T,T} - h_{S,R} - h_{T,R}\,.$$

Thus (5) gives

$$(6)\qquad rt + rs \geqq kq - x_{P,R} - 2x_{P,P} + 2m + 2x_{S,S} + 2x_{T,T} + x_{S,R} + x_{R,T}$$

or in our notation

$$(6')\qquad rt + rs \geqq kq + 2m - y_2 + y_3\,.$$

Let us show that

$$(7)\qquad t > s \quad \text{for} \quad q \leqq 1\,.$$

Suppose $t \leqq s$. If $q = 0$ then by (1) we have a contradiction. If $q = 1$ and $s \geqq t + 1$ then by (1) we have $f < 1 - \sum_{v\in T} d_{G'(V-S)}(v) \leqq 1$, which is a contradiction again. If $q = 1$ and $s = t$, then (1) gives

$$0 < 1 - \sum_{v\in T} d_{G'(V-S)}(v)\,,$$

i.e.

$$\sum_{v\in T} d_{G'(V-S)}(v) = 0 \quad \text{or} \quad rt - m - 2x_{T,T} - x_{T,R} - x_{T,P} = 0\,.$$

Since $s = t$, we can write

$$rt + rs = 2m + 4x_{T,T} + 2x_{T,R} + 2x_{T,P}$$

which being substituted into (6) gives:

$$2x_{T,T} + x_{T,R} + 2x_{T,P} + x_{P,R} + 2x_{P,P} \geqq kq + 2x_{S,S} + x_{R,S}$$

and then also $2x \geqq kq = k$, but this contradicts (3) and hence (7) is true.

From (2) we have

$$y_1 > (r - f)\,t - (r - f)\,s - q + rs - m$$

and hence (since $rs \geqq m$ and $2x \geqq y_1$) we obtain

(8) $$2x > (r - f)(t - s) - q\,.$$

Now we shall show that

(9) $$q \geqq 2\,.$$

In the opposite case, by (7) used in (8) we should have $2x > r - f - q$. Then (3) and (a) would give:

$$f > r - 2x - q \geqq r - 1 - 2x \geqq r - 1 - 2x\frac{k}{2x + 1} =$$
$$= r - \left(1 + \frac{2xk}{2x + 1}\right) \geqq r - \frac{r}{k}\left(1 + \frac{2xk}{2x + 1}\right) \geqq f\,,$$

which is impossible.

From (6) it follows

(10) $$rt + rs - 2m + y_2 \geqq kq\,.$$

By (2) we have $kq > kfs - kft + krt - km - ky_1$ which combined with (10) gives:

(11) $$m(k - 2) + y_2 + ky_1 > s(kf - r) + t(kr - kf - r)\,.$$

Since $rt \geqq m + y_1$, we have either

(12) $$rt = m + y_1\,,$$

or

(13) $$rt \geqq m + y_1 + 1\,.$$

First, let us consider the case (12). By (2) it follows

(12.1) $$fs < ft + q$$

and by (6′) we then see that

(12.2) $$rs + rt - 2m + y_2 - y_3 \geqq kq > kfs - kft\,.$$

By (12) $t = (m + y_1)/r$ and this used in (12.2) gives

$$\left(1 + \frac{kf}{r}\right) y_1 + y_2 - y_3 > \frac{1}{r}(kf - r)(rs - m) . \tag{12.3}$$

From (6′) and (12) we have

$$rs - m \geqq kq - y_1 - y_2 + y_3 . \tag{12.4}$$

Since by (a) $kf - r \geqq 0$, we can use (12.4) in (12.3) to obtain:

$$(2y_1 + y_2 - y_3) f > (kf - r) q . \tag{12.5}$$

We see that

$$2y_1 - y_2 + y_3 = 4x_{T,T} + 2x_{T,R} + 2x_{T,P} + 2x_{P,P} + x_{P,R} - \\ - 2x_{S,S} - 2x_{T,T} - x_{R,S} - x_{R,T} \leqq 2x$$

and then by (9) (12.5) gives:

$$2xf > (kf - r) q \geqq 2(kf - r) ,$$

i.e. $r/(k - x) > f$, which contradicts (a). Hence the case (12) is not possible.

Now let (13) be true. Note that by (a) $kf - r \geqq 0$ and $kr - kf - r \geqq 0$. Since $s \geqq m/r$ and by (13) $t \geqq (m + y_1 + 1)/r$, we obtain from (11):

$$h(k - 2) + y_2 + ky_1 > (kf - r)\frac{k}{r} + (kr - kf - r)\frac{m + y_1 + 1}{r}$$

or equivalently,

$$f > \frac{r}{k}\left(k - 1 - \frac{y_2 + ky_1}{1 + y_1}\right). \tag{13.1}$$

Denoting $a = x_{T,T}$, $b = x_{T,R} + x_{T,P}$, $c = x_{P,P}$ and $d = x_{P,R}$, (3) implies

$$(2c + d)(2x + 1) \leqq (2c + d) k$$

which again implies (since $0 \leqq kb + kd$):

$$(2c + d + 4cx + 2dx) + 2k(x - b - c - d) \leqq 2kx - kb . \tag{13.2}$$

We see that $x \geqq a + b + c + d$, or equivalently, $a \leqq x - b - c - d$. Thus (13.2) gives

$$2c + d + 4cx + 2dx + 2ka \leqq 2kx - kb ,$$

or equivalently,

$$(2c + d - k)(1 + 2x) \leqq -k(1 + 2a + b) ,$$

i.e. $(2c + d - k)/(1 + 2a + b) \leqq -k/(1 + 2x)$, or by the definition of y_1 and y_2:

$$\frac{y_2 - k}{1 + y_1} \leqq \frac{-k}{1 + 2x}\,. \tag{13.3}$$

Using (13.3), we can write

$$\frac{y_2 + ky_1}{1 + y_1} = k + \frac{y_2 - k}{1 + y_1} \leqq k + \frac{-k}{1 + 2x} = \frac{2kx}{2x + 1}$$

which being combined with (13.1) gives

$$f > \frac{r}{k}\left(k - 1 - \frac{2kx}{2x + 1}\right).$$

However, the last inequality contradicts (a). Hence also the case (13) is impossible and the theorem is proved.

Further, we shall show that for even f a stronger result can be proved.

Theorem 2. *Let $G = (V, H)$ be an r-regular k-line-connected graph with $k > 0$. Let $x < k$ and f be nonnegative integers, where f is even and*

$$f \leqq \frac{k - 1}{k}(r - 2x)\,. \tag{A}$$

Then for any $X \subset H$ with $|X| = x$, the graph $G' = (V, H - X)$ has an f-factor.

Proof. Let G' have no f-factor ($f \geqq 2$). Then using the same notation as in the proof of Theorem 1, we have again (1), (2), (4), (5), (6) and (6').

Since f is even, Lemma 5 implies $|H'_{T,V(C)}| \geqq 1$ for any component C, which means that

$$\sum_{v \in T} d_{G'(V-S)}(v) \geqq q\,. \tag{14}$$

From (14) and (1) it follows $fs < ft$, i.e.

$$t \geqq s + 1\,. \tag{15}$$

If q is eliminated by comparison of (6') and (2) then we obtain:

$$(t - s)(kr - r - fk) + (k - 2)(rs - m) \leqq ky_1 + y_2 - y_3\,. \tag{16}$$

By the assumption (A) $kr - r - fk \geqq 0$ and $k \geqq 2$ (since $f \geqq 2$). Obviously, $rs - m \geqq 0$ and by (15), $1 \leqq t - s$. Thus we obtain from (16):

$$kr - r - fk < ky_1 + y_2 - y_3\,. \tag{17}$$

Since $k \geqq 2$, it is $2k - 2 \geqq k$ and $2k - 2 \geqq 2$ and we can write:

$$\begin{aligned} ky_1 + y_2 - y_3 = &\ 2kx_{T,T} + kx_{T,R} + kx_{T,P} + 2x_{P,P} + x_{P,R} - \\ &- 2x_{S,S} - 2x_{T,T} - 2x_{R,S} - x_{R,T} \leqq \\ &\leqq 2(k-1)\, x_{T,T} + (k-1)\, x_{T,R} + kx_{T,P} + 2x_{P,P} + x_{P,R} \leqq \\ &\leqq 2(k-1)\,(x_{T,T} + x_{T,R} + x_{T,P} + x_{P,P} + x_{P,R}) \leqq \\ &\leqq 2(k-1)\, x\,. \end{aligned}$$

Hence by (17) we have:

$$kr - r - fk < 2(k-1)\, x$$

or equivalently:

$$f > \frac{k-1}{k}(r - 2x)\,,$$

which contradicts (A). This completes the proof.

Remark 1. Lemma 2 (Baebler) in the case $2m < 2n$ is a corollary of Theorem 2 (since $2m \leqq ((2m-1)/2m)(2n+1)$). If $2m = 2n$ then by Lemma 4 there is a 1-factor and thus $2n$-factor, too. Since Lemma 4 can be derived [8] from Tutte's theorem on 1-factors [10], we see that the mentioned result of Baebler can be derived also from the results [10, 11] of Tutte.

Remark 2. Also Lemma 1 (Petersen) can be derived from Lemma 5 through Theorem 2. Indeed, it is sufficient to prove that every component of an r-regular graph with an even $r \geqq 4$ has a 2-factor. However, this is true since such a graph has no bridge and so we can put $k = 2$, which means by Theorem 2 that such a graph has a 2-factor (because of $2 \leqq \frac{1}{2}r$).

Now we add some simple results which together with Theorems 1 and 2 can be formulated in the following form.

Theorem 3. *Let $G = (V, H)$ be an r-regular k-line-connected graph and let x and f be nonnegative integers with even $|V|\, f$. Further, let at least one of the following conditions be satisfied:*

(1) *$k > x$ and $r/(k-x) \leqq f \leqq (r/k)(k/(2x+1) - 1)$;*
(2) *$k > 0$ and f is even with $f \leqq (r - 2x)(k-1)/k$;*
(3) *$k \geqq 0$, f and r are even with $f \leqq r - 2x$;*
(4) *$k \geqq r - 1$, f and r are odd with $f \leqq r - 2x$;*
(5) *$k \geqq r - 1$, f is even and r is odd with: $f \leqq r - 1$ if $x = 0$ and $f \leqq r + 1 - 2x$ if $x > 0$.*

Then for any $X \subset H$ with $|X| = x$, the graph $G' = (V, H - X)$ has an f-factor.

Proof. If (1) or (2) are valid then G' has an f-factor according to Theorem 1 or 2 respectively.

If (3) is true then by Lemma 1 G can be decomposed into $r/2$ line-disjoint 2-factors. If we delete from G any x lines, then we break no more than x of these 2-factors, i.e. $G' = (V, H - X)$ has at least $r/2 - x$ pairwise line-disjoint 2-factors and the required assertion follows.

If (4) is true then by Lemma 4 the graph $G' = (V, H - X)$ has a 1-factor $F_1 = (V, H_1)$, since $x \leqq r - 1$. The graph $G'' = (V, (H - H_1) - X)$ has an $(f - 1)$-factor F_2 whenever $(f - 1) \leqq (r - 1) - 2x$ since the condition (3) of Theorem 3 is satisfied. Thus the f-factor $F_1 \cup F_2$ has the required properties.

Finally, if the condition (5) is satisfied then for $x = 0$ the proof follows by means of Lemma 4 and then Lemma 1. Therefore let $x > 0$. Consider a line $h \in X$, then by Lemma 4 the graph $G = (V, H)$ has a 1-factor $F_1 = (V, H_1)$ containing the line h. If we denote $Y = X - H_1$, then the subgraph $G_1 = (V, (H - H_1) - Y)$ of the graph $G' = (V, H - X)$ has (by (3) of this theorem) an f-factor if f is even with $0 \leqq f \leqq r - 1 - 2|Y|$. Since $|Y| \leqq |X| - 1 = x - 1$, thus $r - 1 - 2|Y| \geqq r - 1 - 2(x - 1) = r + 1 - 2x$. This completes the proof.

Remark 3. Although we have formulated our results for graphs they are true for multigraphs and pseudographs, too. Indeed, our proofs are based on the above mentioned (see Remark 1 and 2) two results of Tutte [10, 11] which are true also for pseudographs, and we have used no properties valid only for graphs. (Note that ir an r-regular pseudograph has at least one loop then it cannot be $(r - 1)$-line-connected.)

Remark 4. In general, no weaker line-connectivity in Lemma 4 would guarantee the existence of a 1-factor (see [8]).

The following example shows an analogous situation for Lemma 3 if fkr is odd and $f < k/r$: Let G be the pseudograph formed from a complete bipartite graph $K_{r,k}$ on point sets V_1 and V_2 by adding $(r - k)/2$ loops to every its point of degree k (i.e. point of V_1). Thus G is r-regular k-line-connected pseudograph. If we put in Lemma 5: $S = V_2$, $T = 0$, then obviously $q = |V_1| = r$, $s = k$ and the assumption $fk < r$ implies $fs < q$, which means that G has no f-factor.

The next example shows that in general, it cannot be $f > r - 2x$ in part (3) of Theorem 3: Let r be even and let G be the pseudograph consisting of a single point and $r/2$ loops. If we delete x lines then the remaining degree is $r - 2x$ and obviously no f-factor with $f > r - 2x$ can exist.

Nevertheless, we have no such examples in the case of Theorem 1.

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