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SOME GENERATING FUNCTIONS FOR POLYNOMIALS

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\[
\sum_{n=0}^{\infty} L_n^{(a+mn)}(x) t^n = \frac{(1 + v)^{1+a}}{1 - mv} e^{-xv},
\]

where \( v = t(1 + v)^{1+m} \), \( m \) being an integer. Also, assuming

\[
\sum_{n=0}^{\infty} I_n^{(a+mn)}(x) t^n = A(t) e^{xB(t)},
\]

he proved that

\[
\sum_{n=0}^{\infty} I_n^{(-a-[1+m]n)}(x) t^n = \frac{A(-t)}{1 - B(-t)} \exp \left\{ \frac{-xB(-t)}{1 - B(-t)} \right\}.
\]

L. Carlitz [2], extended the results of Brown and proved that (1.1) and (1.3) hold for any \( m \).

A natural question arises as to whether we can find results of the forms (1.1) and (1.3) for other known polynomials, \( m \) being a constant.

The present paper is an answer to this question. We have found results of the forms (1.1) and (1.3) for generalized Laguerre, generalized Gould-Hopper, generalized Bessel and Jacobi polynomials. These results hold for all values of \( m \). The results of this paper also extend the results, which we found on another occasion [9]. There [9], we found explicit expressions of the form (1.1) for \( m = -1, 0, 1, 2 \), for generalized Laguerre, generalized Gould-Hopper, generalized Bessel and Jacobi polynomials. The treatment being formal, we shall obtain our results quite heuristically.

2. Operational formulae. Recently [8], we considered a class of polynomials \( F_n(x, a, m, p_c(x)) \), defined by the Rodrigues formula

\[
F_n(x, a, m, p_c(x)) = x^{-a} e^{p_c(x)} D^n \left[ x^{mn} e^{-p_c(x)} \right],
\]
\( a, m, \) being constants, \( p_r(x) \) being a polynomial in \( x \) of degree \( r \). It is immediate that the above polynomial reduces to the generalized Laguerre polynomial \([7]\) for \( m = 1 \) and to the generalized Gould-Hopper polynomial \([8]\) for \( m = 0 \). If, however, \( p_r(x) = b/x \), then \((2.1)\) reduces to the Bessel polynomial of KRAALL and FRINK \([6]\). If \( p_r(x) \) be a polynomial in \( 1/x \) of degree \( r \), then for \( m = 2 \), we get a generalization of the Bessel polynomials.

In \([7]\), we dealt with an operator \( T_k \), defined by \( T_k = x(k + xD) \), and hence such that

\[
T_k^n\{x^{k+r}\} = (b + r + k)_n x^{k+r+n},
\]

\( k \) being a constant. It is easily seen that

\[
T_k^n = x^n \prod_{j=0}^{n-1} (\delta + k + j), \quad \delta = xD.
\]

Since,

\[
D^n[x^{mn+a}e^{-p_r(x)}] = x^{-n} \prod_{j=0}^{n-1} (\delta - j) \{x^{mn+a}e^{-p_r(x)}\} =
\]

\[
= \prod_{j=1}^{n} (\delta + j) \{x^{mn+a-n}e^{-p_r(x)}\} = x^{k-n} \prod_{j=0}^{n-1} (\delta + j + k) \{x^{mn+a-n-k+1}e^{-p_r(x)}\} =
\]

\[
= x^{k-n-1} T_k^n\{x^{mn+a-n-k+1}e^{-p_r(x)}\}_{j},
\]

we have from \((2.1)\), the class of operational formulae

\[
F_n(x, a, m, p_r(x)) = x^{k-n-a-1}e^{p_r(x)} T_k^n\{x^{mn+a-n-k+1}e^{-p_r(x)}\}.
\]

Giving different values to \( k \) in \((2.4)\), we get different operational representations for the polynomial \( F_n(x, a, m, p_r(x)) \). If, however, we take \( k = mn + a - n + 1 \), we get the interesting result

\[
T_k^n\{e^{-p_r(x)}\} = x^{2n-mn}e^{-p_r(x)} F_n(x, a, m, p_r(x)),
\]

from which, by giving different values to \( m \), we get operational formulae for different polynomials.

3. The generating functions. By making use of the Lagrange's expansion formula

\[
(1 + v)^{a+1} = 1 + (a + 1) \sum_{n=1}^{\infty} \frac{(a + (b + 1)n)}{n - 1} \frac{t^n}{n},
\]

where

\[
v = t(1 + v)^{b+1}, \quad v(0) = 0,
\]

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L. CARLITZ [2], proved that

\[(3.3) \quad \frac{(1 + v)^{a + 1}}{1 - bv} = \sum_{n=0}^{\infty} \binom{a + (b + 1)n}{n} t^n,\]

where \(v = t(1 + v)^{b+1}\). Let

\[(3.4) \quad f(x) = \sum_{r=0}^{\infty} a_r x^r.\]

Then,

\[\sum_{n=0}^{\infty} \frac{t^n}{n!} T_{(m-1)n+a+1}(x^n f(x)) = \sum_{n,r=0}^{\infty} \frac{t^n}{n!} a_r (b + r + (m-1)n + a + 1) x^{b+r+n} =\]

\[\sum_{n,r=0}^{\infty} a_r \binom{b + r + mn + a}{n} t^n x^{b+r+n}.\]

Hence,

\[(3.5) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} T_{(m-1)n+a+1}(x^n f(x)) = \sum_{n,r=0}^{\infty} a_r \binom{b + r + mn + n}{n} t^n x^{b+r+n}.\]

Making use of (3.3), we get from (3.5)

\[\sum_{n=0}^{\infty} \frac{t^n}{n!} T_{(m-1)n+a+1}(x^n f(x)) = \sum_{r=0}^{\infty} a_r x^{b+r} \frac{(1 + v)^{b+r+a+1}}{1 - (m-1)v},\]

where \(v = xt(1 + v)^m\). Hence, we have the general operational generating function

\[(3.6) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} T_{(m-1)n+a+1}(x^n f(x)) = \frac{x^b(1 + v)^{a+b+1}}{1 - (m-1)v} f[x(1 + v)],\]

where \(v = xt(1 + v)^m\).

Again, from (3.1), we have by putting \(b = m - 1\),

\[(3.7) \quad (1 + v)^{a+1} = 1 + (a + 1) \sum_{n=1}^{\infty} \binom{a + mn}{n-1} \frac{t^n}{n},\]

where \(v = t(1 + v)^m\). Putting \(a = 0\) in (3.7), we get

\[(3.8) \quad v = \sum_{n=1}^{\infty} \binom{mn}{n-1} \frac{t^n}{n}.\]

Similarly, for \(a = -2\), we get from (3.7)

\[(3.9) \quad \frac{v}{1 + v} = \sum_{m=1}^{\infty} \binom{mn - 2}{n-1} \frac{t^n}{n}.\]
Again, following Carlitz [2], we define

\begin{equation}
B(t, c) = -\sum_{n=1}^{\infty} \binom{c+1}{n-1} \frac{t^n}{n}
\end{equation}

and

\begin{equation}
A(t, a, c, d) = \frac{(1 - B(t, c))^{a+d+1}}{1 + c B(t, c)}.
\end{equation}

In view of (3.8), (3.10) and (3.11), the generating function (3.6) becomes

\begin{equation}
\sum_{n=0}^{\infty} \frac{t^n}{n!} T_{(m-1)n+a+1}^{n}\{x^b f(x)\} = \\
x^b A(xt, a, m-1, b) f[x(1 - B(xt, m-1))].
\end{equation}

Again, as is shown by Carlitz [2],

\begin{equation}
B(t, -c - 1) = \frac{-B(-t, c)}{1 - B(-t, c)},
\end{equation}

and

\begin{equation}
A(t, -a, -c - 1, -d) = \frac{A(-t, a, c, d)}{1 - B(-t, c)}.
\end{equation}

Therefore, by (3.12), (3.13) and (3.14), we have the operational generating function

\begin{equation}
\sum_{n=0}^{\infty} \frac{t^n}{n!} T_{(m-1)n-a+1}^{n}\{x^{-b} f(x)\} = \\
x^{-b} A(-xt, a, m-2, b) f\left[\frac{1 + B(-xt, m-2)}{1 - B(-xt, m-2)}\right].
\end{equation}

Now, let \( b = 0 \) and \( f(x) = e^{-\nu x} \) in (3.6), then

\begin{equation}
\sum_{n=0}^{\infty} \frac{t^n}{n!} T_{(m-1)n+a+1}^{n}\{e^{-\nu x}\} = \frac{(1 + \nu)^{a+1}}{1 - (m-1)\nu} \exp\left\{ -\nu x (1 + \nu) \right\},
\end{equation}

where \( \nu = xt(1 + \nu)^m \). Using (2.5) in (3.16), we get the general result

\begin{equation}
\sum_{n=0}^{\infty} \frac{t^n}{n!} F_n(x, a, m, \nu x) = \frac{(1 + \nu)^{a+1}}{1 - (m-1)\nu} e^{\nu x} e^{-\nu x (1 + \nu)},
\end{equation}

where \( \nu = x^{m-1} t(1 + \nu)^m \).
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4. Generalized Gould-Hopper Polynomial. Earlier [8], we considered a generalization of Gould-Hopper polynomial [5], defined by

\[(4.1)\quad H_n(x, a, p_r(x)) = (-1)^n x^{-a} e^{p_r(x)} D^n [x^a e^{-p_r(x)}] ,\]

\(p_r(x)\) being a polynomial in \(x\) of degree \(r\), \(a\) being arbitrary. \(H_n(x, a, p_r(x))\) reduces to the Gould-Hopper polynomial for \(p_r(x) = px^r\). In terms of the polynomial \(F_n(x, a, m, p_r(x))\), we have

\[(4.2)\quad F_n(x, a, 0, p_r(x)) = (-1)^n H_n(x, a, p_r(x)) .\]

The operational formula for the relevant polynomial is

\[(4.3)\quad T_{a+1-n}^n [e^{-px(x)}] = x^{2a} e^{-p(x)} F_n(x, a, 0, p_r(x)) .\]

Putting \(m = 0\) in (3.17) and using (4.2), we get immediately the generating function; for the Gould-Hopper polynomial, to be

\[(4.4)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, a, p_r(x)) = x^{-a} (x - t)^a e^{p_r(x)} e^{-p_r(x-x)} .\]

Again, making use of (4.3), we have from (3.16), the general result

\[(4.5)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, a + mn, p_r(x)) = \frac{(1 + v)^{a+1}}{1 - (m - 1) v} \frac{e^{p_r(x)} e^{-p_r(x(1+v))}}{1} ,\]

where \(v = -(t/x) (1 + v)^m\), \(m\) being a constant. For \(m = 0\), (4.5) reduces to (4.4).

From (4.5), we get generating functions for the generalized Gould-Hopper polynomials of different orders.

Again, putting \(b = 0\) and \(f(x) = e^{-p_r(x)}\) in (3.15) and using (4.3), we get the general result

\[(4.6)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, -a - (m - 2) n, p_r(x)) = \frac{A(t/x, a - m - 2, 0)}{1 - B(t/x, m - 2)} e^{p_r(x)} \exp \left\{ -p_r \left[ x \left\{ 1 + \frac{B(t/x, m - 2)}{1 - B(t/x, m - 2)} \right\} \right] \right\} ,\]

\(m\) being a constant.

Again, using (2.5) and (4.3), we get the relation

\[(4.7)\quad F_n(x, a, 0, p_r(x)) = x^{-mn} F_n(x, a - mn, m, p_r(x)) ,\]

where \(m\) is a constant. From (4.7), we get readily the relation between the generalized Gould-Hopper polynomial and other general polynomials.
5. Generalized Laguerre Polynomials. Elsewhere [7], we considered a generalization of the Laguerre polynomial, defined by

\[ T_{rn}(a)(x) = \frac{1}{n!} x^{-a} e^{p_r(x)} D_r^{a} [x^n e^{-p_r(x)}] , \]

where \( a \) being a constant, \( p_r(x) \) being a polynomial in \( x \) of degree \( r \). \( T_{rn}(a)(x) \) reduces to the generalized Laguerre polynomial of CHATTERJEA [3] for \( p_r(x) = px^r \) and to the Laguerre polynomial for \( p_r(x) = x \). In terms of the polynomial \( F_n(x, a, m, p_r(x)) \), we have

\[ F_n(x, a, 1, p_r(x)) = n! \ T_{rn}(a)(x) . \]

The operational formula for the generalized Laguerre polynomial is

\[ T_{a+1}^n \{e^{-p_r(x)}\} = n! x^n e^{-p_r(x)} T_{rn}(a)(x) , \]

\( p_r(x) \) being a polynomial in \( x \) of degree \( r \), \( a \) being a constant.

Putting \( m = 1 \) in (3.17), and using (5.2), we get immediately the generating function for the generalized Laguerre polynomial

\[ \sum_{n=0}^{\infty} t^n T_{rn}(a)(x) = (1 - t)^{-a-1} e^{p_r(x)} x e^{-p_r[x(1 - t)^{-1}]} . \]

Again, making use of (5.3), we have from (3.16), the general generating function

\[ \sum_{n=0}^{\infty} t^n T_{rn}^{(a + (m - 1)n)}(x) = \frac{(1 + v)^{a+1}}{1 - (m - 1)\ v} e^{p_r(x)} x e^{-p_r[x(1 + v)^m]} , \]

where \( v = t(1 + v)^m \), \( m \) being a constant. For \( m = 1 \), (5.5) reduces to (5.4). From (5.5) we get the generating functions for the generalized Laguerre polynomials of different orders.

Again, putting \( b = 0 \) and \( f(x) = e^{-p_r(x)} \) in (3.15) and using (5.3), we get the result

\[ \sum_{n=0}^{\infty} t^n T_{rn}^{(-a -(m - 1)n)}(x) = \]

\[ = A(-t, a, m-2, 0) e^{p_r(x)} \exp \left[ -p_r \left( x \left\{ 1 + \frac{B(-t, m-2)}{1 - B(-t, m-2)} \right\} \right] , \]

\( m \) being a constant. The results in (5.5) and (5.6) are the generalizations of the results of BROWN [1] and CARLITZ [2].

Again, using (2.5) and (5.3), we get the relation

\[ F_n(x, a, 1, p_r(x)) = x^{(1 - m)n} F_n(x, a - (m - 1)n, m, p_r(x)) , \]

where \( m \) is a constant. From (5.7), we get readily the relation between the generalized Laguerre polynomial and other general polynomials.
6. The Bessel Polynomials. The operational representation for the generalized Bessel polynomials [6], is easily seen to be

\[ T_{a+1}^n \{ e^{-b/x} \} = b^n e^{-b/x} y_n(x, a + 2, b). \]

Again, from (2.5), we have for \( p_i(x) = -b/x \) and \( m = 2 \)

\[ T_{a+1}^n \{ e^{-b/x} \} = e^{-b/x} F_n(x^{-1}, a, 2, -b/x), \]

and hence

\[ F_n(x^{-1}, a, 2, -b/x) = b^n y_n(x, a + 2, b). \]

Again, making use of (6.1), we get immediately from (3.16), the following generating function for the Bessel polynomials:

\[ \sum_{n=0}^{\infty} \frac{t^n b^n}{n!} y_n(x, a + (m - 2)n + 1, b) = \frac{(1 + v)^{m+1}}{1 - (m - 1)v} \exp \left\{ \frac{bv}{x(1 + v)} \right\}, \]

where we have put \( p_i(x) = -b/x \) in (3.16) and where \( v = xt(1 + v)^{m} \), \( m \) being a constant. The relation in (6.4) gives us immediately, generating functions for Bessel polynomials of different orders for different values of \( m \).

Again, putting \( b = 0 \) and \( f(x) = e^{-b/x} \) in (3.15), and using (6.1), we get the result

\[ \sum_{n=0}^{\infty} \frac{t^n b^n}{n!} y_n(x, -a - mn + 1, b) = \frac{A(-xt, a, m-2, b)}{1 - B(-xt, m-2)} \exp \left\{ \frac{b}{x} B(-xt, m-2) \right\}, \]

\( m \) being a constant.

7. The Jacobi Polynomials. It is easily seen that the operational representation for the Jacobi polynomial is given by

\[ T_{a+1}^n \{(1 - x)^n \} = n! (1 - x)^b x^n P_n^{(a,b)}(1 - 2x). \]

Putting \( b = 0, f(x) = (1 - x)^b \), \( m = 1 \) in (3.6), we get the result of Feldheim [4]

\[ \sum_{n=0}^{\infty} t^n P_n^{(a,b)}(x) = (1 - t)^b \left[ 1 - \frac{t}{2} (1 + x) \right]^{-a+b-1}. \]

Also, from (3.6), we have for \( b = 0, f(x) = (1 - x)^b \), the generating function for the Jacobi polynomials

\[ \sum_{n=0}^{\infty} t^n P_n^{(a+m-1),b-n}(x) = \frac{(1 + v)^{m+1}}{1 - (m - 1)v} \left[ 1 - \frac{(1 - x)(1 + v)}{2} \right]^{b} \left( \frac{1 + x}{2} \right)^{-b}, \]

where \( 2v = (1 + x)(1 + v)^{m} t, m \) being a constant.

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Again, if we put \( b = 0, f(x) = (1 - x)^b \) in (3.15), we get the result

\[
\sum_{n=0}^{\infty} t^n P_n^{(a - (m-1)n, b-n)}(x) = 
\]

\[
\left( \frac{1 + x}{2} \right)^{-b} \frac{A \left( \frac{-t}{2}, (1 + x), a, m - 2, 0 \right)}{1 - B \left( \frac{-t}{2}, (1 + x), m - 2 \right)} \left[ 1 - x \left( \frac{1}{1 - B \left( \frac{-t}{2}, (1 + x), m - 2 \right)} \right) \right]^{b}.
\]

It is interesting to note that similar results for other polynomials can be found by the method outlined above.

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References


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