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ADDITIVE COMPOUND MATRICES AND AN INEQUALITY FOR EIGENVALUES OF SYMMETRIC STOCHASTIC MATRICES

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1. Introduction. In [2], the notion of measure of irreducibility $\mu(A)$ of a doubly stochastic matrix $A$ was introduced and the distance of the "nonstochastic" eigenvalues of $A$ from 1 was estimated from below by a function of $\mu(A)$.

F. SALZMANN [7] observed that if a nonnegative matrix has trace zero and multiple Perron root then a proper non-void subset of its eigenvalues has sum zero. In the present paper, we shall find a quantitative extension of this fact for symmetric stochastic matrices using the mentioned measure $\mu(A)$. In the proof, properties of so called additive compound matrices will play a substantial role. Since these matrices seem to be of interest for themselves, a brief sketch of their theory is included. Related questions have been studied in [4].

2. Generalized compound matrices. Let $X$ be an $n$-dimensional vector space over a field $K$. Let $k$ be a fixed integer, $1 \leq k \leq n$. Denote by $A^{(k)}X$, the $k$-th exterior power of $X$, the vector space of all $k$-vectors, i.e. of all linear combinations (over $K$) of exterior products of $k$ vectors in $X$:

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k, \quad x_j \in X, \quad j = 1, \ldots, k.$$

For the exterior products, the following rules are assumed: (cf. [1]):

R 1 If $P = (j_1, \ldots, j_k)$ is any permutation of indices $1, \ldots, k$, $\sigma(P)$ is the sign of $P$ and $x_i \in X$, $i = 1, \ldots, k$, then

$$x_{j_1} \wedge x_{j_2} \wedge \ldots \wedge x_{j_k} = \sigma(P) x_1 \wedge x_2 \wedge \ldots \wedge x_k;$$

R 2

$$(x_1 + y_1) \wedge x_2 \wedge \ldots \wedge x_k = x_1 \wedge x_2 \wedge \ldots \wedge x_k + y_1 \wedge x_2 \wedge \ldots \wedge x_k, \quad x_i \in X, \quad y_1 \in X;$$

R 3

$$(\lambda x_1) \wedge x_2 \wedge \ldots \wedge x_k = \lambda (x_1 \wedge x_2 \wedge \ldots \wedge x_k), \quad x_i \in X, \quad \lambda \in K;$$

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R 4 There exist linearly independent vectors \( x_1, x_2, \ldots, x_n \) in \( X \) such that the \( \binom{n}{k} \) vectors
\[
x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_k}, \quad 1 \leq i_1 < i_2 < \ldots < i_k \leq n,
\]
are linearly independent in \( \Lambda^{(k)}X \).
From these rules the following propositions follow easily:

P 1 \( \) The space \( \Lambda^{(k)}X \) has dimension \( \binom{n}{k} \) and, for any basis \( e_1, \ldots, e_n \) of \( X \) the \( k \)-vectors
\[
e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}, \quad 1 \leq i_1 < i_2 < \ldots < i_k \leq n,
\]
form a basis of \( \Lambda^{(k)}X \).

P 2 \( \) If \( x_i \in X, i = 1, \ldots, k, \) then \( x_1 \wedge x_2 \wedge \ldots \wedge x_k \) is zero in \( \Lambda^{(k)}X \) iff \( x_1, \ldots, x_k \) are linearly dependent in \( X \).

P 3 \( \) If \( X' \) is a dual space to \( X \) with respect to a bilinear form \( \langle x, x' \rangle \) then \( \Lambda^{(k)}X' \) is dual to \( \Lambda^{(k)}X \) with respect to the bilinear form determined by
\[
\langle x_1 \wedge \ldots \wedge x_k, x_1' \wedge \ldots \wedge x_k' \rangle = \det (\langle x_i, x_j' \rangle).
\]
Moreover, if \( e_1, \ldots, e_n \) and \( e_1', \ldots, e_n' \) are dual bases (i.e. \( \langle e_i, e'_j \rangle = \delta_{ij} \)) then \( e_{i_1} \wedge \ldots \wedge e_{i_k}, e'_{i_1} \wedge \ldots \wedge e'_{i_k}, 1 \leq i_1 < \ldots < i_k \leq n \), form also dual bases.

P 4 \( \) If \( X \) is unitary (over the field of complex numbers) with the inner product \( \langle x, y \rangle \) then \( \Lambda^{(k)}X \) is also unitary with respect to the inner product determined by
\[
\langle x_1 \wedge \ldots \wedge x_k, y_1 \wedge \ldots \wedge y_k \rangle = \det (\langle x_i, y_j \rangle).
\]
Let now \( Y \) be an \( m \)-dimensional vector space over the same field \( K \) and assume that the integer \( k \) also satisfies \( 1 \leq k \leq m \). Let \( \mathcal{A} \) be a linear operator from \( X \) into \( Y \); we shall write this \( \mathcal{A} \in L(X, Y) \). Then the \( k \)-th compound operator \( \mathcal{A}^{(k)} \) is the linear operator in \( L(\Lambda^{(k)}X, \Lambda^{(k)}Y) \) defined by
\[
\mathcal{A}^{(k)}(x_1 \wedge x_2 \wedge \ldots \wedge x_k) = \mathcal{A}x_1 \wedge \mathcal{A}x_2 \wedge \ldots \wedge \mathcal{A}x_k
\]
for any \( k \) vectors \( x_1, \ldots, x_k \) in \( X \).

The following propositions follow then easily:

P 5 \( \) If \( k \) does not exceed dimensions of any of the spaces \( X, Y, Z \) and if \( \mathcal{A} \in L(X, Y), \mathcal{B} \in L(Y, Z) \) then
\[
(\mathcal{B}\mathcal{A})^{(k)} = \mathcal{B}^{(k)}\mathcal{A}^{(k)}.
\]
If \( \mathcal{A} \in L(X, X) \) is the identity operator then \( \mathcal{A}^{(k)} \in L(\Lambda^{(k)}X, \Lambda^{(k)}X) \) is the identity operator.

If \( \mathcal{A} \in L(X, X) \) is nonsingular then so is \( \mathcal{A}^{(k)} \) and
\[
(\mathcal{A}^{(k)})^{-1} = (\mathcal{A}^{-1})^{(k)}.
\]

If \( \mathcal{A}^T \in L(Y, Y) \) is the transpose operator to \( \mathcal{A} \in L(X, Y) \), i.e. \( \langle \mathcal{A}x, y' \rangle = \langle x, \mathcal{A}^T y' \rangle \) for all \( x \in X, y' \in Y' \), then \( (\mathcal{A}^T)^{(k)} \in L(\Lambda^{(k)}Y', \Lambda^{(k)}X') \) is the transpose operator to \( \mathcal{A}^k \in L(\Lambda^{(k)}X, \Lambda^{(k)}Y) \).

If \( X \) is a unitary space and \( \mathcal{A} \in L(X, X) \) is symmetric (unitary, normal) then \( \mathcal{A}^{(k)} \in L(\Lambda^{(k)}X, \Lambda^{(k)}X) \) is symmetric (unitary, normal) as well.

Let now \( \mathcal{A} \in L(X, X) \) where \( X \) is a general n-dimensional vector space. If \( 1 \leq k \leq n \) and \( m = 0, 1, 2, \ldots, k \), we define the generalized k-th compound operators \( \mathcal{A}^{(k)}_m \in L(\Lambda^{(k)}X, \Lambda^{(k)}X) \) as linear operators determined by
\[
\mathcal{A}^{(k)}_m(x_1 \wedge x_2 \wedge \ldots \wedge x_k) = \sum_{(e_1, \ldots, e_k) \in \{0, 1\}^k} \mathcal{A}^{e_1}_{x_1} \wedge \mathcal{A}^{e_2}_{x_2} \wedge \ldots \wedge \mathcal{A}^{e_k}_{x_k},
\]
(where \( \mathcal{A}^0 = \mathcal{I} \), the identity).

In particular, \( \mathcal{A}^{(k)}_k \) is the k-th compound operator, \( \mathcal{A}^{(k)}_0 = \mathcal{A}^{(k)} \). The operator \( \mathcal{A}^{(k)}_1 \) will be called additive k-th compound operator and denoted by \( \mathcal{A}^{(k)} \).

The following propositions are immediate:

For \( \mathcal{A} \in L(X, X), \mathcal{B} \in L(X, X) \), we have
\[
(\mathcal{A} + \mathcal{B})^{(k)} = \mathcal{A}^{(k)} + \mathcal{B}^{(k)}.
\]

For \( k \) fixed, the operators \( \mathcal{A}^{(k)}_m \) commute with each other.

If \( \mathcal{A} \in L(X, X), \mathcal{B} \in L(X, X) \) are commutative, \( \mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A} \), then all \( \mathcal{A}^{(k)}_m, \mathcal{B}^{(k)}_m \) commute with each other.

If \( X \) is unitary and \( \mathcal{A} \in L(X, X) \) is symmetric (normal) then \( \mathcal{A}^{(k)}_m \) are all symmetric (normal).

Let us proceed to square matrices. Since any square matrix can be considered as matrix of a linear operator with respect to a basis, assume that \( X \) is an n-dimensional vector space, \( X' \) its dual with respect to a bilinear \( \langle x, x' \rangle \) and let \( e_1, \ldots, e_n, e'_1, \ldots, e'_n \) be dual bases. If \( \mathcal{A} \in L(X, X) \), its matrix \( A = (a_{ik}) \) where, as usually, \( a_{ij} = \langle \mathcal{A}e_j, e'_i \rangle \). We can then define analogously the k-th compound matrix \( A^{(k)} \)
of a matrix $A$ as matrix of the operator $\mathcal{A}^{(k)}$ with respect to the basis $\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_n}, 1 \leq i_1 < i_2 < \ldots < i_n \leq n\}$. Similarly, the generalized $k$-th compound matrices of $A$, denoted by $A^{(k)}_m$, are matrices of $\mathcal{A}^{(k)}_m$ with respect to the same basis as before.

Let us show that $A^{(k)}$ coincides with the usual $k$-th compound matrix of $A$ (see e.g. [5]). We shall use abbreviations $(i) = \{i_1, \ldots, i_k\}$, $1 \leq i_1 < i_2 < \ldots < i_k \leq n$; $(j) = \{j_1, \ldots, j_k\}$, $1 \leq j_1 < j_2 < \ldots < j_k \leq n$; $A^{(k)}_{(i), (j)}$ will then denote the entry of $A^{(k)}$ with row “index” $(i)$ and column “index” $(j)$.

P 14 We have

$$A^{(k)}_{(i), (j)} = \det A((i); (j))$$

where $A((i); (j))$ is the submatrix of $A$ consisting of rows with indices $i_1, \ldots, i_k$ and columns with indices $j_1, \ldots, j_k$.

Proof. By P 3,

$$A^{(k)}_{(i), (j)} = \langle \mathcal{A}^{(k)}(e_{j_1} \wedge \ldots \wedge e_{j_k}), e'_{i_1} \wedge \ldots \wedge e'_{i_k} \rangle =$$

$$= \langle \mathcal{A}e_{j_1} \wedge \ldots \wedge \mathcal{A}e_{j_k}, e'_{i_1} \wedge \ldots \wedge e'_{i_k} \rangle =$$

$$= \det (\langle \mathcal{A}e_{j_1}, e'_{i_1} \rangle) = \det (a_{i_1j_1}) = \det A((i); (j)).$$

From general properties of matrices of linear operators and from P 5, P 6 and P 7 the well known properties follow:

P 15 If $A$ and $B$ are $n \times n$ matrices then

$$(AB)^{(k)} = A^{(k)}B^{(k)},$$

if $I$ is the $n$-rowed identity matrix then $I^{(k)}$ is the $\binom{n}{k}$-rowed identity matrix

and if $A$ is nonsingular then $A^{(k)}$ is nonsingular and

$$(A^{(k)})^{-1} = (A^{-1})^{(k)}.$$

Let us order now the vectors $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$ according to the lexicographical ordering of indices. Then the following proposition holds:

P 16 If the matrix $A = (a_{ij})$ of $\mathcal{A} \in L(X, X)$ is triangular (i.e. $a_{ij} = 0$ for $i > j$) then all generalized compound matrices $A^{(k)}_m$ are triangular as well. The diagonal entry of $A^{(k)}_m$ corresponding to the index $(i) = (i_1, \ldots, i_k)$, $1 \leq i_1 < \ldots < i_k \leq n$, is equal to $E_m(a_{i_1i_1}, \ldots, a_{i_ki_k})$ where $E_m(\xi_1, \ldots, \xi_k)$ denotes the $m$-th elementary symmetric function of $\xi_1, \ldots, \xi_k$.

Proof. Let $(i) = \{i_1, \ldots, i_k\}$, $(j) = \{j_1, \ldots, j_k\}$, $1 \leq i_1 < \ldots < i_k \leq n$, $1 \leq j_1 < \ldots < j_k \leq n$. Then the entry of $A^{(k)}_m$ in the position $(i), (j)$ is

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\[
(A_m^{(k)})_{(i), (j)} = \langle \mathcal{A}_m^{(k)}(e_{j_1} \wedge \ldots \wedge e_{j_k}, e_{i_1} \wedge \ldots \wedge e_{i_k}) = \\
= \sum_{(e_1, \ldots, e_n), e_i \in (0, 1)} \langle \mathcal{A}^e_{j_1} e_{j_1} \wedge \ldots \wedge \mathcal{A}^e_{j_k} e_{j_k}, e_{i_1} \wedge \ldots \wedge e_{i_k} \rangle = \\
= \sum_{(e_1, \ldots, e_n), e_i \in (0, 1), \sum_i = m} \det (\langle \mathcal{A}^e_{j_q} e_{j_q}, e'_{p} \rangle)
\]

where \(\langle \mathcal{A}^e_{j_q} e_{j_q}, e'_{p} \rangle\) are \(k \times k\) matrices, \(p, q = 1, \ldots, k\).

Let now \((i) > (j)\) in the lexicographical ordering. Then there exists an index \(s, 1 \leq s < n\), such that \(i_1 \geq j_1, \ldots, i_s \geq j_s, i_s > j_s\). Denote, for a moment, by \(\alpha_{pq}, p, q = 1, \ldots, k\), the entries of one such matrix \(\langle \mathcal{A}^e_{j_q} e_{j_q}, e'_{p} \rangle\). We shall show that \(\alpha_{pq} = 0\) whenever \(p \geq s \geq q\) which will imply, by the Laplace expansion theorem, singularity of this matrix. Thus if \(p \geq s \geq q\) then

\[i_p \geq i_s > j_s \geq j_q.\]

If \(e_q = 0\) then \(\alpha_{pq} = \langle e_{j_q}, e'_{p} \rangle = 0\). If \(e_q = 1\) then \(\alpha_{pq} = \langle \mathcal{A} e_{j_q}, e'_{p} \rangle = a_{ipjq} = 0\) since \(i_p > j_q\).

Consequently, \((A_m^{(k)})_{(i), (j)} = 0\) and \(A_m^{(k)}\) is triangular.

To prove the last assertion, let \((i) = (j)\). Then, by triangularity of \(A\) and \(A_m^{(k)}\),

\[
(A_m^{(k)})_{(i), (i)} = \sum_{(e_1, \ldots, e_n), e_i \in (0, 1), \sum_i = m} \det (\langle \mathcal{A}^e_{i_q} e_{i_q}, e'_{p} \rangle) = \\
= \sum_{(e_1, \ldots, e_n), e_i \in (0, 1), \sum_i = m} \prod_{p=1}^k a_{ipij} = E_m(a_{i_1i_1}, \ldots, a_{i_1i_k})
\]

(by definition, \(a_{kk}^0 = 1\)). The proof is complete.

We are able now to prove

**Theorem 2.1.** Let \(A\) be an \(n \times n\) matrix over a field \(K\) with eigenvalues \(\alpha_1, \ldots, \alpha_n\). Let \(k, m\) be integers, \(1 \leq k \leq n, 0 \leq m \leq k\). Then the generalized \(k\)-th compound matrix \(A_m^{(k)}\) has eigenvalues \(E_m(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}), 1 \leq i_1 < i_2 < \ldots < i_k \leq n\).

**Proof.** In a suitable extension field \(K'\), \(A\) can be expressed in the form

\[A = STS^{-1}\]

where \(S\) is nonsingular and \(T\) triangular with diagonal entries \(\alpha_{i_1}, \ldots, \alpha_n\). Hence in an \(n\)-dimensional vector space \(X\) over \(K'\), there exist two bases \(e_1, \ldots, e_n\) and \(\tilde{e}_1, \ldots, \tilde{e}_n\),
and in $L(X, X)$ a linear operator $\mathcal{A}$ such that $A$ is the matrix of $\mathcal{A}$ with respect to the basis $e_1, \ldots, e_n$ and $T$ is the matrix of $\mathcal{A}$ with respect to $\tilde{e}_1, \ldots, \tilde{e}_n$. The generalized $k$-th compound operator $\mathcal{A}^{(k)}_m \in L(A^{(k)}X, A^{(k)}X)$ has then matrices $A^{(k)}_m$ with respect to the basis $e_{i_1} \wedge \ldots \wedge e_{i_k}$ and $T^{(k)}_m$ with respect to the basis $\tilde{e}_{i_1} \wedge \ldots \wedge \tilde{e}_{i_k}$. By P 16, $T^{(k)}_m$ is triangular and its eigenvalues, equal to diagonal entries, are $E_m(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}), 1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Thus $\mathcal{A}^{(k)}_m$ as well as $A^{(k)}_m$ has these eigenvalues. The proof is complete.

This Theorem can be generalized as follows:

**Theorem 2.2.** Let $A$ be an $n \times n$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$, $1 \leq k \leq n$. Let $S(\xi_1, \ldots, \xi_k)$ be a polynomial symmetric function of $\xi_1, \ldots, \xi_k$. If $S$ is expressed (which is, according to a well known theorem, always possible) as a polynomial in the elementary symmetric functions $E_1, \ldots, E_k$:

$$S = \Phi(E_1, E_2, \ldots, E_k),$$

then the matrix polynomial

$$\Phi(A^{(k)}_1, A^{(k)}_2, \ldots, A^{(k)}_k)$$

is a matrix the eigenvalues of which are exactly all numbers $S(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}), 1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

**Proof.** If $A$ is triangular, the assertion follows immediately from P 16 and from the fact that the diagonal entries of $\Phi(A^{(k)}_1, \ldots, A^{(k)}_k)$ are $\Phi(E_1(\alpha_{i_1}, \ldots, \alpha_{i_k}), \ldots, E_k(\alpha_{i_1}, \ldots, \alpha_{i_k})) = S(\alpha_{i_1}, \ldots, \alpha_{i_k})$. The general case follows from the decomposition (in a suitable extension field) $A = STS^{-1}$ where $S$ is nonsingular and $T$ triangular. The proof is complete.

**Corollary 2.3.** Let $A$ be an $n \times n$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$. Then the matrix $(A^{(2)})^2 - 4A^{(2)}$ (or $(A^{(1)})^2 - 4A^{(2)}$) has eigenvalues $(\alpha_i - \alpha_j)^2, 1 \leq i < j \leq n$.

**Remark.** This matrix can be called discriminant of $A$ since its nonsingularity is equivalent to simplicity of all eigenvalues of $A$.

**Proof** is immediate.

In the sequel, we shall be interested in the $k$-th additive compound matrix $A^{[k]}$ only. The following Theorem presents the complete description of $A^{[k]}$ if $A$ is given.

**Theorem 2.4.** Let $A = (a_{ij})$ be an $n \times n$ matrix. Let $1 \leq k \leq n$. Then $A^{[k]}$ has the following entry $A^{[k]}_{(i,j)}$ in the row corresponding to the set $(i) = \{i_1, \ldots, i_k\}$ and column corresponding to $(j) = \{j_1, \ldots, j_k\}$, $1 \leq i_1 < \ldots < i_k \leq n$, $1 \leq j_1 < \ldots < j_k \leq n$: 397
\[ A_{(i), (j)}^{(k)} = \sum_{s=1}^{k} a_{i,s} \quad \text{if card } (i) \cap (j) = k \text{ (i.e. if } (i) = (j)); \]
\[ A_{(i), (j)}^{(k)} = (-1)^{\sigma} a_{pq} \quad \text{if card } (i) \cap (j) = k - 1, \text{ where } p = (i) \setminus (i) \cap (j), \quad q = (j) \setminus (i) \cap (j) \text{ and } \sigma = \text{ the number of elements in } (i) \cap (j) \text{ between } p \text{ and } q; \]
\[ A_{(i), (j)}^{(k)} = 0 \quad \text{if card } (i) \cap (j) \leq k - 2. \]

**Proof.** Let \( X \) be an \( n \)-dimensional vector space over the field containing all elements of \( A, e_1, \ldots, e_n \) a basis of \( X, \mathcal{A} \) the operator which has \( A \) as its matrix with respect to this basis. Then \( A^{(k)} \) is the matrix, with respect to the basis \( e_{i_1} \wedge \ldots \wedge e_{i_k}, \]
\[ 1 \leq i_1 < \ldots < i_k \leq n, \]
of the operator \( \mathcal{A}^{(k)} \) defined above. Then
\[ \mathcal{A}^{(k)}(e_{j_1} \wedge \ldots \wedge e_{j_k}) = \mathcal{A}e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_k} + \]
\[ e_{j_1} \wedge \mathcal{A}e_{j_2} \wedge \ldots \wedge e_{j_k} + \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge \mathcal{A}e_{j_k} \]
and we obtain the entry \( A_{(i), (j)}^{(k)} \) as the coefficient at the term \( e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \) in the expansion of the right hand side as a linear combination of the vectors \( e_{p_1} \wedge \ldots \wedge e_{p_k}, \quad 1 \leq p_1 < \ldots < p_k \leq n. \) Since \( \mathcal{A}e_{j_s} = \sum_{i=1}^{n} a_{i,j_s}e_i, \quad s = 1, \ldots, k, \) the cases \( (i) = (j) \) and card \( (i) \cap (j) \) \( \leq k - 2 \) are clear. Let now card \( (i) \cap (j) = k - 1, \)
\[ (i) \cap (j) = \{s_1, \ldots, s_{k-1}\}, \quad 1 \leq s_1 < \ldots < s_{k-1} \leq n, \quad (i) = \{s_1, \ldots, s_{\ell}, p, s_{\ell+1}, \ldots, s_{k-1}\}, \]
\[ (j) = \{s_1, \ldots, s_w, q, s_{w+1}, \ldots, s_{k-1}\}. \]
We obtain the coefficient \( A_{(i), (j)}^{(k)} \) from the term
\[ e_{s_1} \wedge \ldots \wedge e_{s_w} \wedge \mathcal{A}e_q \wedge e_{s_{w+1}} \wedge \ldots \wedge e_{s_{k-1}} \]
only, in particular from the term \( a_{pq}e_p \) of \( \mathcal{A}e_q \). We obtain thus
\[ a_{pq}(e_{s_1} \wedge \ldots \wedge e_{s_w} \wedge e_p \wedge e_{s_{w+1}} \wedge \ldots \wedge e_{s_{k-1}}) = \]
\[ = (-1)^{\sigma} a_{pq}(e_{s_1} \wedge \ldots \wedge e_{s_w} \wedge e_p \wedge e_{s_{w+1}} \wedge \ldots \wedge e_{s_{k-1}}) \]
where \( \sigma \) is the number of indices \( s_i \) between \( p \) and \( q \). The proof is complete.

**Corollary 2.5.** Let \( A = (a_{ik}) \) be a real \( n \times n \) matrix, \( 1 \leq k \leq n. \) Let \( M = \{i_1, \ldots, i_k\}, \quad 1 \leq i_1 < \ldots < i_k \leq n. \) Then the sum of squares of all the off-diagonal entries of the matrix \( A^{(k)} \) in the row corresponding to indices in \( M \) is equal to
\[ \sum_{i \in M, j \notin M} a_{ij}^2. \]

**Proof.** It follows from Theorem 2.4 that the non-zero off-diagonal entries of the row of \( A^{(k)} \) corresponding to \( M \) are of the form \( \pm a_{pq} \) where \( p \in M \) and \( q \notin M \), each pair \( (p, q) \) occurring exactly once.
3. An application to symmetric stochastic matrices. In this section, all vectors and matrices will be real. We shall be using vector and matrix norms (see e.g. [3]). By the norm $\|x\|$ of a column vector $x = (x_1, \ldots, x_n)^T$ we shall mean the euclidean norm, i.e. $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. If $C$ is an $n \times n$ matrix then its norm $\|C\|$ is, as usual, defined as $\sup_{\|x\| \leq 1} \|Cx\|$ which is well known to be equal to the nonnegative square root of the maximum eigenvalue of $CCT$.

Let us also recall that a stochastic matrix is a nonnegative square matrix with all row-sums equal to one, a doubly stochastic matrix is a stochastic matrix with all column-sums equal to one. For such a doubly stochastic $n \times n$ matrix $A = (a_{ij})$, the measure of irreducibility $\mu(A)$ was defined in [2] as

$$\mu(A) = \min_{\theta + M = N, M + \theta = N} \sum_{i = j \in M} a_{ij}$$

where $N = \{1, 2, \ldots, n\}$. It was proved in [2] that every nonstochastic eingevalue $\lambda$ of $A$ satisfies the inequality

$$|1 - \lambda| \geq 2 \left(1 - \cos \frac{\pi}{n}\right) \mu(A).$$

(1)

We shall need the following particular case of a theorem by L. Mirsky [6]:

Let $A, B$ be symmetric $n \times n$ matrices with eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \ldots \geq \beta_n$ respectively. Then,

$$\max_{i=1,\ldots,n} |\alpha_i - \beta_i| \leq \|A - B\|.$$

From this we shall derive easily

**Corollary 3.1.** Let $A = \begin{pmatrix} a_{11} & a_1^T \\ a_1 & A_{22} \end{pmatrix}$ be a symmetric $n \times n$ matrix in a partitioned form, $A_{22}$ being $(n-1) \times (n-1)$. Then at least one eigenvalue of $A$ is contained in the interval

$$|a_{11} - x| \leq \|a_1\|.$$

**Proof.** We put

$$B = \begin{pmatrix} a_{11} & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Then $A - B = C$ where

$$C = \begin{pmatrix} 0 & a_1^T \\ a_1 & 0 \end{pmatrix}.$$
Since
\[ CC^T = \begin{pmatrix} a_1^T a_1 & 0 \\ 0 & a_1^T a_1 \end{pmatrix} \]
has eigenvalues \( \|a_1\|^2, \|a_1\|^2, 0, \ldots, 0, \|A - B\| = \|a_1\| \). The matrix \( B \) has an eigenvalue \( \beta = a_{11} \). It follows from the preceding theorem that for some eigenvalue \( \alpha \) of \( A \)
\[ |\alpha - \beta| \leq \|A - B\|, \]
i.e.
\[ |a_{11} - \alpha| \leq \|a_1\|. \]
The proof is complete.

We shall prove now the following generalization of this Corollary:

**Theorem 3.2.** Let \( A = (a_{ij}) \) be a symmetric \( n \times n \) matrix. Let \( 0 = M \subseteq N = \{1, 2, \ldots, n\}, \) card \( M = m \). Then there exist \( m \) eigenvalues \( \lambda_1, \ldots, \lambda_m \) of \( A \) such that
\[ \left| \sum_{i \in M} a_{ii} - \sum_{i = 1}^{m} \lambda_i \right| \leq \left( \sum_{i \in M, j \notin M} a_{ij}^2 \right)^{1/2}. \]

**Proof.** It suffices to prove this for \( M = \{1, \ldots, m\}, 1 \leq m \leq n - 1 \). Then the additive \( m \)-th compound matrix \( A^{[m]} \) has, according to Theorem 2.4 and Corollary 2.5, \( \sum_{i \in M} a_{ii} \) as its upper left corner entry and \( \sum_{i \in M, j \notin M} a_{ij}^2 \) as the sum of squares of all the off-diagonal entries in the first row. Since \( A^{[m]} \) is symmetric by P 13, it follows from Corollary 3.1 that there exists an eigenvalue \( \omega \) of \( A^{[m]} \) in the interval
\[ \left| \sum_{i \in M} a_{ii} - \omega \right| \leq \left( \sum_{i \in M, j \notin M} a_{ij}^2 \right)^{1/2}. \]
However, all eigenvalues of \( A^{[m]} \) are sums of \( m \) eigenvalues of \( A \) by Theorem 2.1. The proof is complete.

In the following two theorems, the matrix \( A \) will be a stochastic symmetric matrix (and therefore doubly stochastic).

**Theorem 3.3.** Let \( A = (a_{ij}) \) be a symmetric stochastic \( n \times n \) matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Then there exist two non-void proper subsets \( M_1, M_2 \) of \( N = \{1, \ldots, n\} \) with the same number of elements such that
\[ 0 \leq \sum_{i \in M_1} a_{ii} - \sum_{i \in M_2} \lambda_i \leq \mu(A). \]

**Proof.** According to the definition of \( \mu(A) \) there exists a non-void proper subset \( M_0 \) of \( N \) such that
\[ \mu(A) = \sum_{i \in M_0, j \notin M_0} a_{ij}. \]
By Theorem 3.2, there exists a subset $M$ of $N$, with the same number of elements as $M_0$, such that

$$\left| \sum_{i \in M_0} a_{ii} - \sum_{i \in M} \lambda_i \right| \leq \left( \sum_{i \in M_0, j \notin M_0} a_{ij}^2 \right)^{1/2} \leq \sum_{i \in M_0, j \notin M_0} a_{ij} = \mu(A).$$

If $\sum_{i \in M_0} a_{ii} - \sum_{i \in M} \lambda_i \geq 0$, we put $M_1 = M_0$, $M_2 = M$; in the other case, we put $M_1 = N \setminus M_0$, $M_2 = N \setminus M$. The proof is complete.

**Theorem 3.4.** Let $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of a symmetric $n \times n$ stochastic matrix. Then

$$\lambda_1 - \lambda_2 + 2 \left( 1 - \cos \frac{\pi}{n} \right) \left( \sum_{i \in N} \lambda_i - \min_{M + N, \emptyset + M = N, \sum_{i \in M} \lambda_i \geq 0} \sum_{i \in M} \lambda_i \right) \geq 0.$$

**Proof.** Let $A = (a_{ik})$ be that symmetric stochastic matrix. By Theorem 3.3, there exist two non-void proper subsets $M_1, M_2$ of $N$ with the same number of elements such that

$$0 \leq \sum_{i \in M_1} a_{ii} - \sum_{i \in M_2} \lambda_i \leq \mu(A).$$

This implies

$$\sum_{i \in M_2} \lambda_i \leq \sum_{i \in M_1} a_{ii} \leq \sum_{i \in N} a_{ii} = \sum_{i \in N} \lambda_i$$

and

$$\sum_{i \in M_2} \lambda_i + \mu(A) \geq \sum_{i \in M_1} a_{ii} \geq 0.$$

Thus, there exists a non-void proper subset of $N$, namely $M_2$, such that

$$\sum_{i \in N} \lambda_i \geq \sum_{i \in M_2} \lambda_i \geq -\mu(A),$$

which is equivalent to

$$0 \leq \sum_{i \in N, M_2} \lambda_i \leq \sum_{i \in N} \lambda_i + \mu(A).$$

Hence also

$$\min_{M + N, \emptyset + M = N, \sum_{i \in M} \lambda_i \geq 0} \sum_{i \in M} \lambda_i \leq \sum_{i \in N} \lambda_i + \mu(A).$$

By (1),

$$2 \left( 1 - \cos \frac{\pi}{n} \right) \mu(A) \leq \lambda_1 - \lambda_2.$$

Combining these two inequalities, we obtain the result of the Theorem.

**Remark.** The modified Salzman's example [7] stating that no stochastic $5 \times 5$ matrix can have eigenvalues 1, 1, $-\frac{3}{2}$, $-\frac{3}{2}$, $-\frac{3}{2}$ can be extended by Theorem 3.4 for symmetric stochastic matrices as follows:
None of the 5-tuples $1 - \alpha_1, -\frac{2}{3} - \alpha_2, -\frac{4}{3} - \alpha_3, -\frac{8}{3} - \alpha_4$ with $|\alpha_1| + |\alpha_2| + + |\alpha_3| + |\gamma| < \frac{2}{15}(3 - 2 \cos \frac{1}{2}\pi) \approx 0.184$ consists of eigenvalues of a symmetric stochastic $5 \times 5$ matrix.

References


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