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SOME INEQUALITIES FOR CONVEX, STARLIKE, AND CLOSE-TO-CONVEX MAPPINGS

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Introduction. In this paper, we first apply a simple geometric lemma to obtain some inequalities for functions which are starlike and convex of order α where $\alpha \geqq \frac{1}{2}$. Let Δ denote the unit disk $\{z : |z| < 1\}$. Let S consists of the functions F that are analytic in Δ, univalent and satisfy $F(0) = 0$ and $F'(0) = 1$. The family of starlike functions of order α denoted by $St(\alpha)$ and the family of convex functions of order α denoted by $K(\alpha)$ were introduced by M. S. ROBERTSON in [5]. A function F analytic in Δ belongs to $St(\alpha)$ if $F(0) = 0$, $F'(0) = 1$ and $\operatorname{Re}(z\,F'(z)/F(z)) > \alpha$ for $0 \leqq \alpha < 1$. We remark that $St(0) = St$ the starlike univalent functions and $St(\alpha) \subset S$. A function F analytical in Δ belongs to $K(\alpha)$ if $F(0) = 0$, $F'(0) = 1$, and $\operatorname{Re}(1 + z\,F''(z) : : F'(z)) > \alpha$ for $0 \leqq \alpha < 1$. We note that $K(0) = K$ the set of convex univalent mappings. We recall the result of A. MARX in [4] that if $F \in K$, then $\operatorname{Re}(F(z)/z) > \frac{1}{2}$. In [1] L. BRICKMAN, T. H. MACGREGOR, D. R. WILKEN, and the present author prove that if $F \in K(\alpha)$, then $\operatorname{Re}(F(z)/z) > (2^{2\alpha-1} - 1)/(2\alpha - 1)$ for $0 \leqq \alpha < 1$. We use that result in this paper. Another consideration in this paper deals with the class $K(\alpha, \beta)$ which is a subclass of the close-to-convex function of order α and type β introduced by R. J. LIBERA in [3]. A function F analytic in Δ and normalized to satisfy $F(0) = 0$ and $F'(0) = 1$ belongs to $K(\alpha, \beta)$ if there exists a function g in $St(\beta)$ so that $\operatorname{Re}(z\,F'(z)/g(z)) > \alpha$ for z in Δ where $0 \leqq \alpha < 1$ and $0 \leqq \beta < 1$.

A final result deals with the class of convex mappings which are also K-fold symmetric denoted by K_k. A function $F(z)$ analytic in Δ is said to be K-fold symmetric if its power series has the form $F(z) = \sum_{M=0}^{\infty} a_{MK+1} z^{MK+1}$. We also recall the definition of the class of starlike K-fold symmetric functions. In [1] the authors determine that every function F in St_K has a representation

$$F(z) = \int_X \frac{z}{(1 - xz^K)^{2/K}}\, d\mu(x)$$

where μ is a probability measure on $X = \{x : |x| = 1\}$. We will use this result in the proof of theorem 5.

1. Geometric Lemma. Lemma 1. *If* $\operatorname{Re} z_K > \alpha_K$ *and* $\alpha_K > 0$ *for* $K = 1, 2, \ldots, n$ *then,*

$$\operatorname{Re} \sqrt[n]{(z_1 z_2 \ldots z_n)} > \sqrt[n]{(\alpha_1 \alpha_2 \ldots \alpha_n)}\,.$$

Proof. We note that $\operatorname{Re} z_K = r_K \cos \theta_K$ where $z_K = r_K e^{i\theta_K}$ for $K = 1, 2, \ldots, n$. We find that

$$\operatorname{Re} \sqrt[n]{(z_1 z_2 \ldots z_n)} = \sqrt[n]{(r_1 r_2 \ldots r_n)} \cos \frac{\theta_1 + \theta_2 + \ldots + \theta_n}{n}$$

by proper choice of the n-th root. We note that by hypothesis $\operatorname{Re} z_K = r_K \cos \theta_K > \alpha_K$ for $K = 1, 2, \ldots, n$. We conclude that

$$r_1 r_2 \ldots r_n \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n > \alpha_1 \alpha_2 \ldots \alpha_n$$

since $\alpha_K > 0$ for all K. Hence, we see that

$$\sqrt[n]{(r_1 r_2 \ldots r_n)} \sqrt[n]{(\cos \theta_1 \cos \theta_2 \ldots \cos \theta_n)} > \sqrt[n]{(\alpha_1 \alpha_2 \ldots \alpha_n)}\,.$$

We finish the proof by showing that

$$\cos \frac{\theta_1 + \theta_2 + \ldots + \theta_n}{n} \geqq \sqrt[n]{(\cos \theta_1 \cos \theta_2 \ldots \cos \theta_n)}$$

from which the lemma follows. The above inequality is an immediate consequence of the fact that the function $\phi(\theta) = \log \cos \theta$ is concave on the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

2. Inequalities for $St(\alpha)$ and $K(\alpha)$ where $\alpha \geqq \frac{1}{2}$. Theorem 1. *If* $F \in St(\alpha)$ *and* $\alpha \geqq \frac{1}{2}$, *then* $\operatorname{Re} \sqrt{(F'(z))} > \sqrt{(\alpha)}\,(\frac{1}{2})^{1-\alpha}$ *for* z *in* Δ.

Proof. We know that $\operatorname{Re}(z F'(z)/F(z)) > \alpha$. Furthermore, it is a simple consequence of known result of I. S. Jack [2, p. 472] that $\operatorname{Re}(F(z)/z) > (\frac{1}{2})^{2-2\alpha}$ where $\alpha \geqq \frac{1}{2}$. Hence, by previous lemma when $n = 2$, we have

$$\operatorname{Re} \sqrt{\left(\frac{z\,F'(z)}{F(z)} \frac{F(z)}{z}\right)} = \operatorname{Re} \sqrt{(F'(z))} > \sqrt{(\alpha)}\,(\tfrac{1}{2})^{1-\alpha}$$

where $\alpha \geqq \frac{1}{2}$.

Remarks. By examining the proof of the lemma and the known conditions for equality in $\operatorname{Re}(F(z)/z) > (\frac{1}{2})^{2-2\alpha}$, we may conclude that theorem 1 is sharp for each z in Δ. It is also possible to prove that $\sqrt{(\alpha)}\,(\frac{1}{2})^{1-\alpha} \geqq \alpha$ for $1 > \alpha \geqq \frac{1}{2}$ with equality only for $\alpha = \frac{1}{2}$. Finally, since $g(z) \in K(\alpha)$ if and only if $z\,g'(z) \in St(\alpha)$. Theorem 1 implies that when $F \in K(\alpha)$ and $\alpha \geqq \frac{1}{2}$, then $\operatorname{Re} \sqrt{(F'(z) + z\,F''(z))} > \sqrt{(\alpha)}\,(\frac{1}{2})^{1-\alpha}$.

Theorem 2. *If* $F \in St(\alpha)$ *and* $\alpha \geqq \frac{1}{2}$ *then*

$$\operatorname{Re} \sqrt{\sqrt{\left(\frac{F(z)\,F'(z)}{z}\right)}} > (\tfrac{1}{2})^{1-\alpha} \sqrt{\sqrt{\alpha}}$$

for z *in* Δ.

Proof. It follows from a known result [2, p. 472] that $\operatorname{Re}\sqrt{(F(z)/z)} > (\frac{1}{2})^{1-\alpha}$ when $\alpha \geqq \frac{1}{2}$. We proved in theorem 1 that $\operatorname{Re}\sqrt{F'(z)} > \sqrt{(\alpha)}\,(\frac{1}{2})^{1-\alpha}$ when $\alpha \geqq \frac{1}{2}$. Hence, by lemma 1 when $n = 2$ we conclude that

$$\operatorname{Re}\sqrt{\sqrt{\left(\frac{F(z)\,F'(z)}{z}\right)}} > (\tfrac{1}{2})^{1-\alpha}\sqrt{\sqrt{\alpha}}\,.$$

Remarks. By examining the proof of the lemma, the known conditions for equality in $\operatorname{Re}(F(z)/z) > (\frac{1}{2})^{2-2\alpha}$, and the proof of theorem 1, we conclude that the above result is sharp for each $\alpha \geqq \frac{1}{2}$ and $|z| < 1$.

Theorem 3. *If $F \in K(\alpha)$ and $\alpha \geqq \frac{1}{2}$, then*

$$\operatorname{Re}\sqrt{\left(\frac{F(z)\,F'(z)}{z}\right)} > \left(\frac{1}{2}\right)^{1-\alpha}\sqrt{\left(\frac{2^{2\alpha-1}-1}{2\alpha-1}\right)}$$

for z in Δ.

Proof. It follows from a known result [2, p. 473] that $\operatorname{Re}F'(z) > (\frac{1}{2})^{2-2\alpha}$ where $\alpha \geqq \frac{1}{2}$ for all z in Δ. In [1] L. Brickman, T. H. MacGregor, D. R. Wilken, and the present author proved that if $F \in K(\alpha)$, then

$$\operatorname{Re}\frac{F(z)}{z} > \frac{2^{2\alpha-1}-1}{2\alpha-1}$$

for $0 \leqq \alpha < 1$ so in particular for $\frac{1}{2} \leqq \alpha < 1$. Hence, the desired result follows by lemma 1 where $n = 2$.

Remarks. It can be shown that

$$\phi(\alpha) = \left(\frac{1}{2}\right)^{1-\alpha}\sqrt{\left(\frac{2^{2\alpha-1}-1}{2\alpha-1}\right)}$$

satisfies

$$\phi(\alpha) > \frac{1}{\sqrt{2}}\sqrt{\ln 2} > \frac{1}{2}\,.$$

We remark that this technique does not yield the result for $\operatorname{Re}\sqrt{(z\,F'(z)/F(z))}$ since we now have only $\operatorname{Re}(z/F(z)) > 0$ for all α satisfying $0 \leqq \alpha < 1$.

Theorem 4. *If $F(z) \in K(\alpha, \beta)$ and $\beta \geqq \frac{1}{2}$, then $\operatorname{Re}\sqrt{F'(z)} > \sqrt{(\alpha)}\,(\frac{1}{2})^{1-\beta}$ for z in Δ.*

Proof. We have $\operatorname{Re}(z\,F'(z)/g(z)) > \alpha$ for $|z| < 1$ where $g(z) \in St(\beta)$. Since $\beta \geqq \frac{1}{2}$, it is known [2, p. 472] that $\operatorname{Re}(g(z)/z) > (\frac{1}{2})^{2-2\beta}$. Hence, by lemma 1 we conclude that

$$\operatorname{Re}\sqrt{[F'(z)]} = \operatorname{Re}\sqrt{\left(\frac{z\,F'(z)}{g(z)}\,\frac{g(z)}{z}\right)} > \sqrt{(\alpha)}\left(\frac{1}{2}\right)^{1-\beta}.$$

Remarks. We notice that since $St(\alpha) \subset K(\alpha, \alpha)$ the above result includes theorem 1 when $\alpha \geqq \frac{1}{2}$.

We next prove a simple lemma which is necessary in the proof of theorem 5. We remark that theorem 5 generalizes the result of STRÖHACKER [6, p. 373] that $\operatorname{Re}(F(z)/z) > \frac{1}{4}\pi$ for a convex odd mapping F. Our result also generalizes the result of Marx [4] that $\operatorname{Re}(F(z)/z) > \frac{1}{2}$ for a convex mapping F. It is clear from known results that $F \in K_K$ if and only if $zF' \in St_K$. Hence, the representation theorem 3 [1] for St_K mentioned in the introduction yields the following representation for F' whenever $F \in K_K$:

$$F'(z) = \int_X \frac{1}{(1 - xz^K)^{2/K}} \, d\mu(x)$$

where μ is a probability measure on X the unit circle.

Lemma 2. *If*

$$F(z) = \frac{1}{(1 - xz^K)^{2/K}}$$

where $|x| = 1$, $K = 2, 3, \ldots$, and $|z| < 1$ *then*

$$\operatorname{Re} F(z) \geqq \frac{1}{(1 + |z|^K)^{2/K}}$$

for z *in* Δ.

Proof. We see that $F(z) = (1 - \phi(z))^{-2/K}$ where ϕ is analytic in $|z| < 1$ and satisfies $\phi(0) = 0$, $|\phi(z)| < 1$, and $|\phi(z)| \leqq |z|^K$. It follows by arguments based on subordination, conformality, and the fact that $1/(1 - z)^\varrho$ is convex for $|z| < 1$ and $0 < \varrho \leqq 1$ that $\operatorname{Re} F(z) \geqq (1 + |z|^K)^{-2/K}$ for $|z| < 1$.

Theorem 5. *If* $F \in K_K$, *then*

$$\operatorname{Re} \frac{F(z)}{z} > \Phi(K) \equiv \int_0^1 \frac{dt}{(1 + t^K)^{2/K}}$$

for z *in* Δ *and* $K = 2, 3, \ldots$.

Proof. As remarked in the previous paragraph, it follows from theorem 3 [1] that for $F \in K_K$ we have

$$F'(z) = \int_X \frac{1}{(1 - xz^K)^{2/K}} \, d\mu(x)$$

where μ is a probability measure on X the unit circle. Hence, by lemma 2 we have

$$\operatorname{Re} F'(z) = \int_X \operatorname{Re} \frac{1}{(1 - xz^K)^{2/K}} \, d\mu(x) \geqq \frac{1}{(1 + |z|^K)^{2/K}}$$

for $|z| < 1$. In general, we have

$$\frac{F(z)}{z} = \frac{1}{z} \int_0^z F'(\omega)\, \mathrm{d}\omega \, .$$

We choose the path of integration to be the straight line $\omega = tz$ for $0 \leqq t \leqq 1$. Then, it follows that

$$\frac{F(z)}{z} = \int_0^1 F'(tz)\, \mathrm{d}t$$

and consequently,

$$\operatorname{Re} \frac{F(z)}{z} = \int_0^1 \operatorname{Re} F'(tz)\, \mathrm{d}t \geqq \int_0^1 \frac{\mathrm{d}t}{(1 + |z|^K t^K)^{2/K}}$$

by direct application of lemma 2. So, we have

$$\operatorname{Re} \frac{F(z)}{z} \geqq \int_0^1 \frac{\mathrm{d}t}{(1 + |z|^K t^K)^{2/K}} > \int_0^1 \frac{\mathrm{d}t}{(1 + t^K)^{2/K}} = \Phi(K)$$

since $0 \leqq |z| < 1$.

Remarks. We note that $\Phi(2) = \frac{1}{4}\pi$ is the result of Ströhacker [6, p. 373]. The numbers $\Phi(K)$ increase monotonically to 1. We remark that this theorem can be proved by first obtaining the lower bound on the derivative through known results on subordination avoiding reference to the representation theorem for F' and then proceeding as above through integration.

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