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Štefan Znám

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# VEKTOR-COVERING SYSTEMS OF ARITHMETIC SEQUENCES

ŠTEFAN ZNÁM, Bratislava

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*To Prof. Š. Schwarz on the occasion of the 60-th anniversary of his birthday.*

A generalization of well-known disjoint covering systems of arithmetic sequences is given in this article (see [1]). It is shown here that the majority of results concerning disjoint covering systems can be extended to the case of the so called vector-covering systems of arithmetic sequences.

## I

Let $Z$ be the set of all integers, $a, n \in Z$ with $0 \leqq a < n$. Denote by $a(n)$ the set of all numbers of the form $a + sn$, where $s \in Z$. In the following such a set will be called arithmetic sequence with modulus $n$. Let $f$ be the characteristic function of the set $a(n)$ on $Z$, i.e. if $r \in Z$ then

$$f(r) = \begin{cases} 1 & \text{if} \quad r \in a(n) \\ 0 & \text{otherwise}. \end{cases}$$

Using this notion we can recall the definition of disjoint covering systems as follows:

**Definition 1** (see [1]). A system of arithmetic sequences

(1) $$a_j(n_j) \quad j = 1, 2, \dots, m, \quad 2 \leqq n_1 \leqq n_2 \leqq \dots \leqq n_m$$

is said to be *a disjoint covering* (DCS) if for any $r \in Z$ the equality

$$\sum_{j=1}^{m} f_j(r) = 1$$

holds.

Now we shall introduce a new kind of covering.

**Definition 2.** Let a vector $\varepsilon = (v_1, v_2, \ldots, v_m)$ with real $v_k$ be given. The system (1) will be called an *ε-covering* if for any $r \in Z$ we have

$$\sum_{j=1}^{m} v_j f_j(r) = 1 . \tag{2}$$

We say that (1) is a *vector-covering system* (VCS) if there exists such a vector ε that (1) is an ε-covering.

Obviously any DCS is a VCS for $\varepsilon = (1, 1, \ldots, 1)$. It is easy to show that to a given vector ε (with $m \geqq 2$) there exists an ε-covering system if and only if at least two components $v_j$ are positive.

**Corollary.** (1) *is an ε-covering if and only if the system*

$$b_j(n_j), \quad j = 1, 2, \ldots, m$$

*is an ε-covering with* $b_j = n_j - a_j - 1$.

Proof. Obviously, the function $g_j(r) = f_j(-r - 1)$ is the characteristic function of the set $b_j(n_j)$. We use simply (2).

The functions $f_j(r)$ are periodic with periods being divisors of $N = [n_1, n_2, \ldots, n_m]$ – the least common multiple of moduli $n_1, n_2, \ldots, n_m$. Thus we can easily prove the following

**Lemma.** *The system* (1) *is an ε-covering if and only if* (2) *holds for the numbers* $0, 1, \ldots, N - 1$.

Example. The system

$$1(3),\ 2(3),\ 3(4),\ 1(4),\ 0(6),\ 1(6),\ 5(6)$$

is a $(1, 1, 1, 1, 1, -1, -1)$-covering. This could be checked showing that for each number $0, 1, \ldots, 11$ the equality (2) holds (see the preceding Lemma).

In a vector-covering system, superfluous sequences can exist in the sense that deleting them we get a vector-covering system again. Some of our results hold only for VCS without superfluous sequences.

**Definition 3.** The system (1) is called a *reduced* $\varepsilon = (v_1, v_2, \ldots, v_m)$*-covering* if it is an ε-covering but no such non-empty subsystem $a_{j_i}(n_{j_i})$ $i = 1, 2, \ldots, k$ exists that for any $r \in Z$ the equality

$$\sum_{i=1}^{k} v_{j_i} f_{j_i}(r) = 0$$

holds. A system is said to be a *reduced* VCS if it is a reduced covering for a vector ε.

The system from our example could be shown to be reduced. Obviously, deleting the superfluous sequences in a VCS we get a reduced one.

## II

**Theorem 1.** (1) *is a* $(v_1, v_2, \ldots, v_m)$*-covering if and only if for any function g given on Z the equation*

$$\sum_{t=0}^{N-1} g(t) = \sum_{j=1}^{m} v_j\Big(\sum_{s=0}^{N/n_j-1} g(a_j + sn_j)\Big) \tag{3}$$

*holds.*

Proof. Suppose (1) is a $(v_1, v_2, \ldots, v_m)$-covering. Take some $t_0 \in \{0, 1, \ldots, N-1\}$. The term $g(t_0)$ occurs in the inner sum

$$\sum_{s=0}^{N/n_j-1} g(a_j + sn_j)$$

exactly if $t_0 \in a_j(n_j)$; therefore the coefficient of $g(t_0)$ on the right hand side of (3) is

$$\sum_{j=1}^{m} v_j f_j(t_0),$$

but this is equal to 1 since (1) is a $(v_1, v_2, \ldots, v_m)$-covering (see (2)) and hence (3) follows.

Now suppose (3) holds for any $g$. We choose $r \in Z, 0 \leqq r \leqq N-1$. Putting $g(r) = 1$ and $g(n) = 0$ otherwise we get from (3)

$$1 = \sum_{j=1}^{m} v_j\Big(\sum_{s=0}^{N/n_j-1} g(a_j + sn_j)\Big) = \sum_{j=1}^{m} v_j f_j(r)$$

and according to Lemma (1) is a $(v_1, v_2, \ldots, v_m)$-covering.

If we consider a $(v_1, v_2, \ldots, v_m)$-covering ,where $v_j$ are integers, one can prove the following (in a sense stronger).

**Theorem 2.** *Let* $v_1, v_2, \ldots, v_m$ *be integers. Then the system* (1) *is a* $(v_1, v_2, \ldots, v_m)$*-covering if and only if the equality*

$$\frac{v_1 e^{a_1}}{e^{n_1} - 1} + \ldots + \frac{v_m e^{a_m}}{e^{n_m} - 1} = \frac{1}{e - 1} \tag{4}$$

*holds.*

Proof. Putting $g(t) = e^t$ we obtain from (3) (after some modifications) the relation (4). Now suppose (4) holds. Multiplying by $e^N - 1$ we can rewrite this relation in the form

$$(5) \qquad \sum_{t=0}^{N-1} e^t - \sum_{j=1}^{m} v_j \Big( \sum_{s=0}^{N/n_j - 1} e^{a_j + s n_j} \Big) = 0 \, .$$

Thus we have a vanishing polynomial in $e$ with integral coefficients and therefore all coefficients must be zero ($e$ is a transcendental number). But the coefficient by $e^r$ is equal to

$$1 - \sum_{j=1}^{m} v_j f_j(r)$$

$r = 0, 1, \ldots, N - 1$. According to Lemma, (1) is a $(v_1, v_2, \ldots, v_m)$-covering.

**Corollary 1.** *Putting $g(t) = 1$ in (3) we get*

$$\sum_{j=1}^{m} \frac{v_j}{n_j} = 1 \, .$$

**Corollary 2.** *Putting $g(t) = t$ in (3) we have*

$$\sum_{j=1}^{m} v_j \left( \frac{a_j}{n_j} - \frac{1}{2} \right) = - \frac{1}{2} \, .$$

A. S. Fraenkel proved in [2] the following interesting result:

(1) *is a DCS if and only if*

$$\sum_{j=1}^{m} n_j^{t-1} B_t \left( \frac{a_j}{n_j} \right) = B_t$$

*holds for $t = 0, 1, 2, \ldots$, where $B_t(x)$ is the t-th Bernoulli polynomial and $B_t$ the t-th Bernoulli number.*

In [8] another proof of Fraenkel's result is given. This one can be applied (with some modifications) to prove the following theorem (generalizing Fraenkel's result for vector-covering systems):

**Theorem 3.** *The system (1) is $(v_1, v_2, \ldots, v_m)$-covering if and only if*

$$\sum_{j=1}^{m} v_j n_j^{t-1} B_t \left( \frac{a_j}{n_j} \right) = B_t$$

*holds for $t = 0, 1, 2, \ldots$*

Using the properties of Bernoulli polynomials some coherences could be found between Theorems 2 and 3 (see [8]).

## III

Let (1) be a $(v_1, v_2, \ldots, v_m)$-covering system. Let $z$ be any complex number with $z \neq (2\pi i/N)\, u$, $u$ integer. Then putting $g(t) = z^t$ in (3) we get

$$\sum_{j=1}^{m} \frac{v_j z^{a_j}}{z^{n_j} - 1} = \frac{1}{z-1}. \tag{6}$$

Comparing the residues on both sides of (4) we have for all $j = 1, 2, \ldots, m$ (see [3]):

$$\sum_{\substack{t=1 \\ n_j | sn_t}}^{m} \frac{v_t}{n_t} e^{2\pi i s a_t/n_j} = \begin{cases} 0 & \text{if} \quad s = 1, 2, \ldots, n_j - 1 \\ 1 & \text{if} \quad s = n_j \end{cases}. \tag{7}$$

Remark. Similarly as in [3] it can be proved that (7) is a necessary and sufficient condition for (1) to be a $(v_1, \ldots, v_m)$-covering*). We showed here only that (7) is a necessary condition.

**Theorem 4.** *Let $n_u$ be a modulus of a $(v_1, \ldots, v_m)$-covering system. If $v_u \neq 0$ then there exists a modulus $n_t$ $(u \neq t)$ so that $n_u \mid n_t$.*

Proof (see [3]). If no $n_t$ $(t \neq u)$ is divisible by $n_u$, then we get (putting $j = u$, $s = 1$ in (7))

$$\frac{v_u}{n_u} e^{2\pi i a_u/n_u} = 0$$

which is impossible.

**Corollary.** *Due to Theorem 4 the modulus $n_m$ is also a divisor of some $n_u$, $u \neq m$, provided $v_m \neq 0$. Owing to (1) this is possible only if $n_m = n_{m-1}$. For DCS this is a well-known fact* (see [1]).

However, we can prove a little more:

**Theorem 5.** *Let (1) be a $(v_1, \ldots, v_m)$-covering with $v_m \neq 0$ and let $q$ be the smallest prime divisor of $n_m$. Then (1) contains at least $q$ equal moduli.*

Proof. Suppose $n_1 \leqq n_2 \leqq \ldots \leqq n_{m-t} < n_{m-t+1} = n_{m-t+2} = \ldots = n_m$ (from Corollary of Theorem 4 the inequality $t \geqq 2$ follows). It is sufficient to prove that $t \geqq q$. Putting $j = m$, $s = 1, 2, \ldots, q - 1$ in (7) we get the system of equalities

$$\sum_{z=0}^{t-1} v_{m-z} e^{2\pi i s a_{m-z}/n_m} = 0.$$

Hence the system of equations

$$\sum_{z=0}^{t-1} x_z e^{2\pi i s a_{m-z}/n_m} = 0, \quad s = 1, 2, \ldots, t$$

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*) The equation (6) in [3] contains some misprints.

has a solution $x_0 = v_m, \ldots, x_{t-1} = v_{m-t+1}$, but this is impossible if $t < q$ (because then the determinant of this system is not 0). The proof is complete.

Remark. The analogous result for DCS was conjectured in [7]; later it was proved in [4] and [3]. Our proof is a slight modification of that from [3].

S. K. Stein proved in [6] the following interesting theorem:

*If in a DCS* (1) *there exist exactly two equal moduli* (*and the remaining ones are distinct*) *then*

$$n_j = 2^j \quad for \quad j = 1, 2, \ldots, m-2\,, \quad n_{m-1} = n_m = 2^{m-1}\,. \tag{8}$$

**Theorem 6.** *If* (1) *is a* $(v_1, \ldots, v_m)$*-covering with* $v_k \neq 0$ *in which there exist exactly two equal moduli then* (1) *is a DCS and* (8) *holds.*

Proof. We shall proceed by induction concerning the number of sequences $m$. For $m = 2$ the assertion obviously holds. Suppose the assertion holds for all systems with less than $m$ sequences. From the conditions of our theorem and from the Corollary of Theorem 4 we have

$$n_1 < n_2 < \ldots < n_{m-2} < n_{m-1} = n_m\,.$$

Thus from (7) putting $j = m$, $s = 1$ we get

$$v_{m-1}e^{2\pi i a_{m-1}/n_m} + v_m e^{2\pi i a_m/n_m} = 0 \tag{9}$$

and hence $|v_m| = |v_{m-1}|$. Let us distinguish two cases:

a) $v_{m-1} = -v_m$. Then we get from (9) $a_m = a_{m-1}$. This is a contradiction because deleting the equal sequences $a_m(n_m)$ $a_{m-1}(n_{m-1})$ we should get a VCS with distinct moduli (see Theorem 5).

b) $v_m = v_{m-1}$. Then it can be shown by elementary considerations that (9) implies (supposing $a_{m-1} < a_m$)

$$a_m = a_{m-1} + \frac{n_m}{2}\,.$$

Hence the sequences $a_{m-1}(n_{m-1})$ and $a_m(n_m)$ can be replaced by a single sequence $a_{m-1}(n_m/2)$. In such a way we obtain a VCS having $m - 1$ sequences and exactly two equal moduli (see Theorem 5). Now use the inductive assumption and (8) follows. From Corollary 1 of Theorem 1 we have $v_{m-1} = v_m = 1$ and hence (1) is a DCS, too.

Remark. For DCS similar results were proved in the cases that there exist exactly 3, 4, 5, 7 equal moduli (see 5 and [7]).

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*Author's address:* 816 31 Bratislava, Mlynská dolina, ČSSR (Katedra algebry PFUK).