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APPLICATION OF ROTHE'S METHOD TO ABSTRACT PARABOLIC EQUATIONS

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Introduction. The method of Rothe, introduced by Rothe in his paper [1], for the investigation of parabolic equations, was used by many authors, see for example O. A. LADYŽENSKAJA [2], A. M. ILJIN, A. S. KALAŠNIKOV, O. A. OLEJNIK [3]. Recently, a paper of K. REKTORYS [4] and papers of J. KAČUR [5] led the author to apply this method on the initial value problem for abstract parabolic equations $\mathrm{d}u/\mathrm{d}t + A(u(t)) = f(t)$ with A a nonlinear operator. The author followed the paper of F. E. BROWDER [6], where the theory of semigroups is used and generalised his result for nonvanishing $f(t)$; see also H. BRÉZIS [7]. The method of Rothe consists in replacing the equation in fixed points $t_j = jh$, by $(u(t_j) - u(t_{j-1}))/h + A(u(t_j)) = f(t_j)$. The nonlinear operator A is supposed monotone in its domain $D(A) \subset H$, where H is a real Hilbert space, and such that Range $(A + I) = H$. If $u_0 \in D(A)$ and if f is a continuous function from the interval $\langle 0, T \rangle$ to H and of bounded variation, then a unique solution of the problem exists in the sense precised below and the piecewise linear functions or piecewise constant functions inset in points t_j tend uniformly in the interval $\langle 0, T \rangle$ for $h \to 0$ to the solution.

Assertions and main theorem. Let the operator A be monotone in $D(A)$:

$$(Au - Av, u - v) \geqq 0 \;\forall\, u, v \in D(A) \tag{1}$$

and let us suppose that

$$(A + I)(D(A)) = H\,. \tag{2}$$

Let u_0 be in $D(A)$ and f in $C(\langle 0, T \rangle, H)$ with bounded variation in the interval $\langle 0, T \rangle$. By a solution of the problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) = f(t)\,, \quad 0 \leqq t \leqq T\,, \quad u(0) = u_0\,, \tag{3}$$

we mean a strongly continuous function $u : \langle 0, T \rangle \to H$, weakly differentiable to H for $t \in \langle 0, T \rangle$, such that $u(t) \in D(A)$ for $t \in \langle 0, T \rangle$, $A(u(t))$ is weakly continuous,

$u(t)$ satisfies the equation (3) for $t \in \langle 0, T \rangle$ and the initial condition in the obvious sens. (See [6].)

Put for $\lambda > 0 : A_\lambda = A + \lambda I$. We obtain as in [6]:

Assertion 1. *For $\lambda > 0$, Range $(A_\lambda) = H$ and $\|A_\lambda^{-1}x - A_\lambda^{-1}y\| \leqq \lambda^{-1}\|x - y\|$.*

Proof. Let $u, v \in D(A)$. Then $(A_\lambda u - A_\lambda v, u - v) \geqq \lambda\|u - v\|^2$, hence $\|u - v\| \leqq \leqq \lambda^{-1}\|A_\lambda u - A_\lambda v\|$. If for some $\lambda_0 > 0$ the Range $(A_{\lambda_0}) = H$, then $A_{\lambda_0}^{-1}$ is defined and is Lipschitzian with Lipschitz constant λ_0^{-1}. To solve the equation $A_\lambda u = y$, put $u = A_{\lambda_0}^{-1}x$. We obtain the equation

(4) $$x + (\lambda - \lambda_0) A_{\lambda_0}^{-1}x = y\,,$$

whose solution gives a solution of the original equation. If $|\lambda - \lambda_0|\,\lambda_0^{-1} < 1$, then by the theorem on contractive mappings, the equation (4) has a unique solution, q.e.d.

Let $h = T/n$, $t_j = jh$, $j = 0, 1, \ldots, n$ and put $z_0 = u_0$, $Az_1 + (z_1 - z_0)/h = f(t_1)$, $A(z_j) + (z_j - z_{j-1})/h = f(t_j)$, $j = 1, 2, \ldots, n$. It follows from the Assertion 1 that z_j are uniquely determined. We have

$$(Az_1 - Az_0, z_1 - z_0) + \frac{\|z_1 - z_0\|^2}{h} = (f(t_1), z_1 - z_0) - (Az_0, z_1 - z_0)\,,$$

hence

(5) $$h^{-1}\|z_1 - z_0\| \leqq \|f(t_1)\| + \|Az_0\|$$

and similarly for $j \geqq 2$: $(Az_j - Az_{j-1}, z_j - z_{j-1}) + \|z_j - z_{j-1}\|^2/h = (f(t_j), z_j - z_{j-1}) - (f(t_{j-1}), z_j - z_{j-1}) + h^{-1}(z_{j-1} - z_{j-2}, z_j - z_{j-1})$, so

(6) $$h^{-1}\|z_j - z_{j-1}\| \leqq \|f(t_j) - f(t_{j-1})\| + h^{-1}\|z_{j-1} - z_{j-2}\|\,.$$

We obtain from (5) and (6)

(7) $$h^{-1}\|z_j - z_{j-1}\| \leqq \operatorname*{Var}_{\langle 0,T\rangle} f + \operatorname*{Max}_{\langle 0,T\rangle} \|f(t)\| + \|Az_0\|\,.$$

Put in $[0, T]$, $z^n(t) = z_{j-1} + h^{-1}(t - t_{j-1})(z_j - z_{j-1}) = h^{-1}(h - (t - t_{j-1}))\,.\,z_{j-1} + h^{-1}(t - t_{j-1})\,z_j$ for $t_{j-1} \leqq t \leqq t_j$. It follows from (5) and (7)

Assertion 2.

(8) $$\|z^n(\tau) - z^n(\mu)\| \leqq |\tau - \mu|\,(\operatorname*{Var}_{\langle 0,T\rangle} f + \operatorname*{Max}_{\langle 0,T\rangle} \|f(t)\| + \|Az_0\|)\,.$$

Let us define $x^n(t)$ by $x^n(0) = u_0$, $x^n(t) = z_j$ for $(j - 1)\,h < t \leqq jh$. In the same manner we define $f^n(t)$. It follows from the construction above that

(9) $$\int_0^t A(x^n)(\tau))\,d\tau + z^n(t) = \int_0^t f^n(\tau)\,d\tau + u_0$$

and for the derivatives from the left:

$$A(x^n(t)) + \frac{\mathrm{d}^- z^n}{\mathrm{d}t} = f(t)\,. \tag{10}$$

It follows from the definition of the functions $x^n(t)$ and $f^n(t)$ that for all $t \in \langle 0, T\rangle$, $f^n(t) \to f(t)$ (strong convergence), uniformly in $\langle 0, T\rangle$, and

$$x^n(t) - z^n(t) \to 0 \quad \text{uniformly in} \quad \langle 0, T\rangle\,. \tag{11}$$

Assertion 3. *$x^n(t) \to u(t)$ uniformly in $\langle 0, T\rangle$.*

Proof. We have

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\|z^m(t) - z^n(t)\|^2 = 2\left(\frac{\mathrm{d}^- z^m}{\mathrm{d}t}(t) - \frac{\mathrm{d}^- z^n}{\mathrm{d}t}(t),\right. \tag{12}$$

$$\left. z^m(t) - z^n(t)\right) = 2(f^m(t) - A(x^m(t)) - f^n(t) + A(x^n(t)), z^m(t) - z^n(t)\Big)\,.$$

It follows from (8) and (10) that

$$\|A(x^n(t))\| \leqq 2 \operatorname*{Max}_{\langle 0,T\rangle} \|f(t)\| + \operatorname*{Var}_{\langle 0,T\rangle} f + \|Az_0\|\,, \tag{13}$$

hence (11), (12) and (13) give

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\|z^m(t) - z^n(t)\|^2 \leqq \varepsilon_{m,n} + 2(f^m(t) - A(x^m(t)) - f^n(t) +$$

$$+ A(x^n(t)), x^m(t) - x^n(t)) \leqq \varepsilon_{m,n} + 2\|f^m(t) - f^n(t)\|\,\|x^m(t) - x^n(t)\|\,,$$

where $\varepsilon_{m,n} \to 0$. Because $\|z^m(t) - z^n(t)\| \geqq 0$ and $\|z^m(0) - z^n(0)\| = 0$, it follows that with some $\bar{\varepsilon}_{m,n} \to 0 : \|z^m(t) - z^n(t)\| \leqq \bar{\varepsilon}_{m,n}t$, q.e.d.

We prove as in the paper [6] the following Assertions 4 and 5:

Assertion 4. *A is maximal monotone, i.e. if $(Av - w, v - u) \geqq 0$ for all v in $D(A)$, then $u \in D(A)$ and $Au = w$.*

Proof. We have $(Av - w + v - u, v - u) \geqq 0$. For fixed $z \in H$, $t > 0$, put $v_t = A_1^{-1}(u + w + tz)$. It follows that $Av_t + v_t - u - w = tz$, hence $t(z, v_t - u) \geqq \geqq 0$ so $(z, v_t - u) \geqq 0$. Letting $t \to 0+ : v_t \to A_1^{-1}(u + w)$, hence $(z, A_1^{-1}(u + w) - - u) \geqq 0$ so $u = A_1^{-1}(u + w)$, $u \in D(A)$, and $Au = w$, q.e.d.

Assertion 5. *If $u_j \to u$ and $Au_j \rightharpoonup w$* (weak convergence) *then $u \in D(A)$ and $Au = w$.*

Proof. For $v \in D(A) : (Av - Au_j, v - u_j) \geqq 0$. From this follows taking the limit as $j \to \infty$ that $(Av - w, v - u) \geqq 0$, hence by the Assertion 4, $u \in D(A)$ and $Au = w$, q.e.d.

Assertion 6.

$$\|u(t) - u(\tau)\| \leqq |t - \tau| \left(\operatorname*{Var}_{\langle 0,T\rangle} f + \operatorname*{Max}_{\langle 0,T\rangle} \|f(t)\| + \|Au_0\|\right). \tag{14}$$

Proof. (14) follows from (8), (11) and the Assertion 3.

Assertion 7. *$A(x^n(t)) \rightharpoonup A(u(t))$ and $A(u(t))$ is weakly continuous.*

Proof. We use estimations (13), (14) and Assertion 3 and 5.

Theorem. *Let A be a nonlinear operator from the domain $D(A) \subset H$ to H. Let (1) and (2) be satisfied. Let $u_0 \in D(A)$, $f(t)$ be continuous on $\langle 0, T\rangle$ with values in H and of bounded variation on $\langle 0, T\rangle$. Then there exists a unique solution $u(t)$ of the initial problem (3) and $u(t)$ is a Lipschitz continuous function satisfying (14). For the functions x^n, z^n constructed by the Rothe's method (8), (9), (10), (13) are valid. Also*

$$z^n(t) \to u(t) \quad \textit{uniformly in} \quad \langle 0, T\rangle, \tag{15}$$

$$A\, x^n(t) \rightharpoonup A(u(t)). \tag{16}$$

If u, v are two solutions corresponding to the f, u_0, g, v_0 respectively, then

$$\operatorname*{Max}_{\langle 0,T\rangle} \|u(t) - v(t)\| \leqq 2\int_0^T \|f(t) - g(t)\|\, dt + \|u(0) - v(0)\|. \tag{17}$$

If $f(t) \equiv 0$, then $\|A(u(t))\|$ is non-increasing.

Proof. First we prove (17) from which follows also uniqueness. We have

$$\frac{d}{dt}\|u(t) - v(t)\|^2 = 2(f(t) - g(t) - A(u(t)) + A(v(t)),$$

$$u(t) - v(t)) \leqq 2\|f(t) - g(t)\|\, \|u(t) - v(t)\|,$$

so

$$\frac{d}{dt}\|u(t) - v(t)\|^2 \leqq 2(\operatorname*{Max}_{\langle 0,T\rangle}\|u(\tau) - v(\tau)\|)\, \|f(t) - g(t)\|,$$

hence

$$\|u(t) - v(t)\|^2 \leqq \|u(0) - v(0)\|^2 + 2\operatorname*{Max}_{\langle 0,T\rangle}\|u(\tau) - v(\tau)\| \cdot \int_0^T \|f(\tau) - g(\tau)\|\, d\tau,$$

from which (17) follows. Let us consider first sequences $x^n(t)$, $z^n(t)$ introduced above.

It follows from (9) that for every $v \in H$

$$\int_0^t (A(x^n(\tau)), v)\,d\tau + (z^n(t), v) = \int_0^t (f^n(\tau), v)\,d\tau + (u_0, v)\,. \tag{18}$$

Hence from the Assertions 3,7 and from (11), it follows for $n \to \infty$:

$$\int_0^t (A(u(\tau)), v)\,d\tau + (u(t), v) = \int_0^t (f(\tau), v)\,d\tau + (u_0, v)\,, \tag{19}$$

hence it follows from (19) that $u(t)$ is weakly differentiable, because of the Assertions 7, and that the equation (3) is satisfied. (15) and (16) are evident.

From (13) and the Assertion 7 we obtain for $f(t) \equiv 0$

$$\|A(u(t))\| \leqq \|Au_0\|\,, \tag{20}$$

hence from the uniqueness

$$\|A(u(t))\| \leqq \|A(u(s))\|\,, \quad t \geqq s\,, \tag{21}$$

q.e.d.

Remark. If $f(t) \equiv 0$, then clearly for two solutions corresponding to u_0, v_0:

$$\frac{d}{dt}\|u(t) - v(t)\|^2 = 2(u'(t) - v'(t), u(t) - v(t)) =$$
$$= 2(A(u(t)) - A(v(t)), u(t) - v(t)) \leqq 0\,,$$

hence the transition operator $U(t)$ defined by $U(t)\,u(0) = u(t)$ is nonexpansive.

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