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# A QUANTITATIVE REFINEMENT OF THE CLOSED GRAPH THEOREM

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In a recent paper [1] the author has obtained a simple theorem of the closed graph type which may be applied in a variety of situations [2] where the classical closed graph theorem is not sufficient. It is the purpose of the present remark to clarify the relation between this theorem – the so called induction theorem – and the closed graph theorem.

We formulate first the induction theorem in its simplest form (for applications, a somewhat different formulation is more suitable). For the sake of completness we give a proof; the proof is simple and straightforward. The interest of the theorem lies in the fact that it represents an abstract form of an approximation process very useful in analysis. In this respect the theorem does not differ from the closed graph theorem which is nothing more than the abstract description of an approximation process. The approximation process of the closed graph theorem has been used before Banach; however, its abstract form, the closed graph theorem, – once it is formulated – saves us the inconvenience of going through this process in concrete situations.

The induction theorem is stronger than the closed graph theorem. To clarify their mutual relation we give, in the second section, a proof of a strengthening of the classical closed graph theorem formulated in such a manner that the theorem becomes an immediate consequence of the induction theorem. At the same time, the proof is arranged in such a manner that the redundancy available when the induction theorem is applied becomes obvious.

We prove the closed graph theorem in its "closed relation" form [3]; this form eliminates all inessential assumptions which obscure its substance.

**1. Metric spaces.** Given a metric space $(E, d)$ with distance function $d$, a point $x \in E$ and a positive number $r$, we denote by $U(x, r)$ the open spherical neighbourhood of $x$ with radius $r$, $U(x, r) = \{y \in E; d(y, x) < r\}$. Similarly, if $M \subset E$, we denote by $U(M, r)$ the set of all $y \in E$ for which $d(y, M) < r$. If we are given, for each sufficiently small positive $r$, a set $A(r) \subset E$, we define the limit $A(0)$ of the family $A(.)$ as follows

$$A(0) = \bigcap_{s>0} \left( \bigcup_{r \leqq s} A(r) \right)^{-} .$$

Let $T$ be an interval of the form $T = \{t; 0 < t < t_0\}$. A *small function* on $T$ will be a mapping of $T$ into itself such that the sum

$$\sigma(t) = t + w(t) + w(w(t)) + w(w(w(t))) + \dots$$

is finite for each $t \in T$. In the sequel we shall use the abbreviation $w^k$ for the $k$-th iterate of the function $w$. In particular, if $w$ is linear, $w(t) = \alpha t$, then $w$ is small if and only if $0 < \alpha < 1$.

Now we may state the induction theorem.

**(1,1) Theorem.** *Let $(E, d)$ be a complete metric space, let $T$ be an interval $\{t; 0 < < t < t_0\}$ and $w$ a small function on $T$. For each $t \in T$ let $Z(t)$ be a subset of $E$; denote by $Z(0)$ the limit of the family $Z(.)$. Suppose that*

$$Z(t) \subset U(Z(w(t)), t)$$

*for each $t \in T$. Then*

$$Z(t) \subset U(Z(0), \sigma(t))$$

*for each $t \in T$.*

Proof. Suppose that $x \in Z(t)$. Since $Z(t) \subset U(Z(w(t)), t)$, there exists an $x_1 \in \in U(x, t) \cap Z(w(t))$. Now $x_1 \in Z(w(t)) \subset U(Z(w^2(t)), w(t))$ so that there exists an $x_2 \in \in U(x_1, w(t)) \cap Z(w^2(t))$. Continuing this process we obtain a sequence $x_n$ such that

$$x_{n+1} \in U(x_n, w^n(t)) \cap Z(w^{n+1}(t)) ;$$

it follows that $d(x_n, x_{n+1}) < w^n(t)$ so that $x_n$ is a Cauchy sequence. Since $(E, d)$ is complete, this sequence converges to a limit $x_\infty$. Since $x_n \in Z(w^n(t))$ and $w^n(t) \to 0$, we have $x_\infty \in Z(0)$. Furthermore $d(x, x_\infty) \leqq d(x, x_1) + d(x_1, x_2) + \dots < t + w(t) + + w^2(t) + \dots = \sigma(t)$ so that $x \in U(x_\infty, \sigma(t)) \subset U(Z(0), \sigma(t))$. The proof is complete.

**2. Closed relations.** A relation $T$ from a set $E$ into a set $F$ is a subset of $E \times F$. We write $y \in Tx$ and $x \in T^{-1}y$ if $[x, y] \in T$. Similarly, if $B \subset F$ then $T^{-1}B$ is the set of all $x \in E$ such that $[x, b] \in T$ for some $b \in B$, in other words

$$T^{-1}B = \cup \{T^{-1}b; b \in B\} = \{x \in E; Tx \cap B \neq 0\} .$$

For $A \subset E$

$$TA = \cup \{Ta; a \in A\} = \{y \in F; T^{-1}y \cap A \neq 0\} .$$

It will be useful to restate inclusions concerning $T$ in terms of $T^{-1}$ and vice versa. In particular, the following three inclusions are equivalent:

$$x \in U(T^{-1}y, r), \quad y \in TU(x, r), \quad T^{-1}y \cap U(x, r) \neq 0.$$

Restating these for $T^{-1}$, we obtain the equivalence of the following three conditions

$$y \in U(Tx, r), \quad x \in T^{-1}U(y, r), \quad Tx \cap U(y, r) \neq 0.$$

Now we can give a proof of the closed relation theorem based on the induction theorem.

**(2,1) Theorem.** *Let $E$ be a complete metric space, $F$ a metric space. Let $T$ be a closed subset of $E \times F$. If the relation $T$ is uniformly almost open then it is uniformly open.*

*More precisely: suppose that, for each $r > 0$, there exists a positive number $q(r)$ such that*

$$(1) \qquad (TU(x, r))^{-} \supset U(Tx, q(r))$$

*for each $x \in D(T)$. Then, for each $r' > r$ and each $x \in D(T)$,*

$$(2) \qquad TU(x, r') \supset U(Tx, q(r)).$$

Proof. Let $r > 0$, $r' > r$ and $x \in D(T)$ be fixed and consider an arbitrary $y_0 \in U(Tx, q(r))$. For each $t > 0$, set

$$(3) \qquad W(t) = T^{-1}\, U(y_0, t).$$

It will be useful to note that $W(t) = \{x \in E;\ Tx \cap U(y_0, t) \neq 0\} = \{x \in E;\ y_0 \in U(Tx, t)\}$.

It is easy to see, $T$ being closed, that

$$(4) \qquad W(0) = T^{-1}y_0.$$

The theorem will be established if we prove the inclusion

$$(5) \qquad W(q(r)) \subset U(W(0), r').$$

Indeed, since $x \in W(q(r))$, the inclusion (5) implies $x \in U(W(0), r') = U(T^{-1}y_0, r')$ whence $y_0 \in TU(x, r')$. Since $y_0$ was an arbitrary point in $U(Tx, q(r))$, this yields the inclusion $U(Tx, q(r)) \subset TU(x, r')$.

The theorem thus reduces to the proof of the inclusion (5). We begin by showing that our assumption about $T$ implies the following property of the family $W(.)$.

If $t > 0$ is given then

$$W(q(t)) \subset U(W(s), t) \tag{6}$$

for arbitrarily small positive $s$.

To see that, take arbitrary $t > 0$ and $s > 0$. If $z \in W(q(t))$, then $y_0 \in U(Tz, q(t)) \subset \subset (TU(z, t))^-$ so that $U(y_0, s)$ intersects $TU(z, t)$. If $v \in U(z, t)$ is such that $U(y_0, s)$ intersects $Tv$ then $v \in W(s)$ and $z \in U(v, t) \subset U(W(s), t)$.

Introduce now a new function $q^*(t) = \min(q(t), r^{-1} q(r) t)$ so that $q^*(r) = q(r)$ and $q^*(t) \to 0$. Since $q^*(t) \leqq q(t)$, we have

$$W(q^*(t)) \subset U(W(s), t) \tag{7}$$

for each positive $t$ and $s$.

Now let $\varepsilon > 0$ be chosen so as to have $r/(1 - \varepsilon) = r'$. Setting $s = q^*(\varepsilon t)$ in the inclusion (7) we obtain

$$W(q^*(t)) \subset U(W(q^*(\varepsilon t)), t)$$

for each $t > 0$. It follows from the induction theorem that

$$W(q^*(t)) \subset U\left(W(0), \frac{1}{1 - \varepsilon} t\right) = U\left(T^{-1} y_0, \frac{1}{1 - \varepsilon} t\right)$$

for each $t > 0$. In particular, for $t = r$, we obtain

$$W(q(r)) \subset U(T^{-1} y_0, r')$$

which is the desired inclusion. The proof is complete.

## *References*


[1] *V. Pták:* Deux théorèmes de factorisation, Comptes Rendus Ac. Sci. Paris. *278* (1974), 1091—1094.

[2] *V. Pták:* A theorem of the closed graph type, Manuscripta Mathematica, in print.

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