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ON COMPACT $N$-SEMIGROUPS

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1. INTRODUCTION

A topological semigroup is a non-empty Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition $(x, y) \rightarrow xy$. When there is no possible ambiguity we shall simply refer to $S$ as a topological semigroup. If $S$ contains a zero, that is, an element $0$ such that $x0 = 0x = 0$ for all $x \in S$, $S$ is said to be a topological semigroup with zero. In this paper, we consider only topological semigroups with zero and hence we shall use the term "semigroup" to mean topological semigroup with zero.

If $S$ is a semigroup, an element $b$ of $S$ is called nilpotent if $b^n \rightarrow 0$, that is, if for every neighbourhood $U$ of $0$ there exists an integer $n_0$ such that $b^n \in U$ for all $n \geq n_0$. The set of all nilpotent elements of $S$ shall be denoted by $N$. If $N$ is an open subset of $S$, then $S$ is called an $N$-semigroup. In addition, if $S$ is a compact space, then $S$ will be called a compact $N$-semigroup.

In [2] we studied some properties of compact commutative $N$-semigroups with zero and local zeros. The following definition was introduced there. If $a \in S$, the set of all right topological zero divisors of $a$ is the set Tod$_r a = \{ x \in S \mid ax \in N \}$. The set Tod$_l a$ of all left topological zero divisors of $a$ is similarly defined. If $S$ is commutative we shall denote them both by Tod $a$. We observe that Tod $a$ is always non-empty since $0 \in$ Tod $a$. In this paper we shall study the properties of $N$ in terms of Tod $e$ where $e$ is a non-zero idempotent of $S$. We shall prove that in fact $N$ is the intersection of all such Tod $e$. We shall also show that if $e$ is a non-zero primitive idempotent of a compact $N$-semigroup $S$, then Tod $e$ is an open prime ideal of $S$. Finally, we show that in a compact $N$-semigroup, under some conditions, a nil ideal is nilpotent, thus transporting the well known Hopkins-Levitzki theorem from ring theory to compact $N$-semigroups, with the chain conditions being replaced by compactness.

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2. PRELIMINARIES

We shall use the following notation. Let $A$ be any subset of a semigroup $S$, and let $a \in S$. Then

- $\bar{A}$ = topological closure of $A$ in $S$
- $A^\prime$ = complement of $A$ in $S$
- $|A|$ = cardinal number of the set $A$
- $J(A) = A \cup AS \cup SA \cup SAS$, that is, the smallest ideal of $S$ containing $A$
- $J_0(A) = \text{the union of all ideals contained in } A$, that is, the largest ideal contained in $A$ if $J_0(A) \neq \emptyset$
- $R(A) = \{ x \in S \mid x^n \in A \text{ for some integer } n \geq 1 \}$
- $\Gamma(a) = \lim_{n \to \infty} a^n$, that is, the set of cluster points of the sequence $\{a^n\}_{n=1}^{\infty}$.

It is well-known that if $\Gamma(a)$ is compact, it contains a unique idempotent. Moreover, $K(a)$ is a group and $K(a) = e \Gamma(a) = \Gamma(a) e$ where $e \in \Gamma(a)$ is the unique idempotent (see [6], pages 22–25).

We recall some definitions and results that we shall need.

**Lemma 2.1** (Numakura [4]). The set $E$ of idempotents of $S$ is a closed subspace of $S$ which is partially ordered under the relation $e \leq f$ if $ef = fe = e$, and this partial order is closed, that is, it has a closed graph. If $ef = fe$ for all $e, f \in E$, then $E$ is a semigroup and $ef$ is the greatest lower bound of $\{e, f\}$ relative to $\leq$.

**Definition 2.2.** An idempotent $e$ is called primitive if $f^2 = f \in eSe$ implies that $f = 0$ or $f = e$. It is obvious that the non-zero primitive idempotents are the atoms of the partially ordered set $(E^*, \leq)$, where $E^* = E - \{0\}$.

**Definition 2.3.** Two non-zero idempotents $e$ and $f$ of $S$ are said to be orthogonal if $ef = fe = 0$. We shall denote this by $e \perp f$.

**Definition 2.4.** An ideal $P$ of $S$ is said to be prime if $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$ where $A$ and $B$ are ideals of $S$. An ideal $Q$ of $S$ is said to be completely prime if $ab \in Q$ implies that $a \in Q$ or $b \in Q$, where $a$ and $b$ are elements of $S$.

**Remark.** An ideal which is completely prime is prime, but the converse need not be true. (For a counter example, see [6], page 51.) However, these concepts coincide in the case of commutative semigroups.

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**Theorem 2.5** (Numakura [5]). If $S$ is a compact semigroup with zero, then each open prime ideal $P \neq S$ has the form $J_0(S - e)$ where $e$ is a non-zero idempotent of $S$. Conversely, if $e$ is a non-zero idempotent, then $J_0(S - e)$ is an open prime ideal.

**Lemma 2.6** (Numakura [4]). Let $S$ be a semigroup with zero and let $a \in S$. If $a^n$ is nilpotent for some integer $n \geq 1$, then a itself is a nilpotent element.

**Lemma 2.7** (Hoo-Shum [2]). If $S$ is a compact commutative semigroup, then the set $N$ is an ideal of $S$.

### 3. NILPOTENT ELEMENTS AND TOPOLOGICAL ZERO DIVISORS

In this section we shall study the set $N$ of nilpotent elements in a compact commutative semigroup $S$, and give a characterization of this set in terms of the sets $\text{Tod } e_i$ where the $e_i$ are in $E^*$. Some of the results are closely related to those obtained in our previous paper [2]. Throughout this section, $S$ will denote a commutative semigroup.

**Lemma 3.1.** If $S$ is compact but not nil, then $N$ is the intersection of all the sets $\text{Tod } e$ where $e \in E$.

**Proof.** Since $S$ is a commutative semigroup, it follows from Lemma 2.7 that $N \subseteq \bigcap_{e \in E} \text{Tod } e$. We now show that $\bigcap_{e \in E} \text{Tod } e \subseteq N$. Let $x \in \bigcap_{e \in E} \text{Tod } e$. Then $e x \in N$ for all $e \in E$. Since $S$ is compact, it follows that $\Gamma(x)$ is compact, and hence there exists an idempotent $e_1 \in \Gamma(x)$. Since $K(x) = e_1 \Gamma(x)$ is a group, it follows that $e_1 x \in K(x)$ has an inverse $y \in K(x)$. Hence applying Lemma 2.7 once more, since $N$ is an ideal of $S$, we have $e_1 = (e_1 x) y \in N S \subseteq N$. This implies that $e_1 = 0$, that is, $K(x) = \{0\}$. But $K(x)$ is the set of all cluster points of the sequence $\{x^n\}_{n=1}^\infty$. Hence $x^n \to 0$, that is, $x \in N$. Therefore $N = \bigcap_{e \in E} \text{Tod } e$.

**Theorem 3.2.** Let $S$ be compact and let $E^*$ be the set of all non-minimal idempotents of $S$. Then $N = \bigcap_{e \in E^*} \text{Tod } e$.

**Proof.** Since $\text{Tod } 0 = S$, by Lemma 3.1, we immediately have $N = \bigcap_{e \in E^*} \text{Tod } e$. Now let $e_1, e_2$ be idempotents of $S$ and let us suppose that $e_1 \leq e_2$, that is, $e_1 e_2 = e_2 e_1 = e_1$. Then if $x \in \text{Tod } e_2$ we have $e_2 x \in N$. Thus $(e_1 e_2) x = e_1 (e_2 x) \in e_1 N \subseteq N$ by Lemma 2.7; that is, $e_1 x \in N$, or $x \in \text{Tod } e_1$. Thus, if $e_1 \leq e_2$ we have $\text{Tod } e_2 \subseteq \text{Tod } e_1$. This proves the theorem.

**Corollary 1.** If $S$ is compact, then $N$ is a closed ideal of $S$ if and only if for each $e \in E^*$, $\text{Tod } e$ is a closed ideal of $S$.  

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Proof. If for each \( e \in E^* \), Tod \( e \) is a closed ideal of \( S \), then by Theorem 3.2 it follows immediately that \( N \) is a closed ideal of \( S \). The converse was proved by us in [2] and also by A. D. Wallace in [11].

Remark. We point out that K. Numakura said in [4] that the structure of semigroups in which \( N \) is not open was not known to him. The Corollary above suggests that it may be worthwhile for us to consider the sets Tod \( e \) when \( N \) is not open.

Corollary 2. (Another characterization of compact \( N \)-semigroups.) A compact semigroup \( S \) is an \( N \)-semigroup if and only if \( S \) contains only a finite number of open ideals Tod \( e \) with \( e \in E^* \).

Proof. Since the intersection of finitely many open sets is open, in one direction, this results is obvious. The converse was proved by us in [2] and by A. D. Wallace in [11].

In [2] we proved that \( S \) is a compact \( N \)-semigroup if and only if \( E^* = E - \{0\} \) is compact. The characterization above is an improvement of our previous result. Also, in [2] we called a semigroup an \( A \)-semigroup if Tod \( a \) are all open for every \( a \in S \), and we asked (Colloquium Mathematicum problem P796): if \( S \) is an \( A \)-semigroup, is \( S \) an \( N \)-semigroup? If \( S \) is compact and \( E^* \) is finite, this Corollary gives an affirmative answer to this problem.

Corollary 3. If \( e \in E^* \), then Tod \( e = R(\text{Tod } e) \).

Proof. Clearly Tod \( e \subseteq R(\text{Tod } e) \). Take \( y \in R(\text{Tod } e) \). Then there is an integer \( k \geq 1 \) such that \( y^k \in \text{Tod } e \), and hence \( ey^k \in N \). Since \( e \) is an idempotent and \( S \) is commutative, we have \((ey)^k \in N \). By Lemma 2.6, it follows that \( ey \in N \), that is, \( y \in \text{Tod } e \). Hence Tod \( e = R(\text{Tod } e) \).

Remark 1. In general, \( N \) is properly contained in Tod \( e \) if \( e \) is a non-zero primitive idempotent. However, Tod \( e \) need not be the minimal non-nil ideal of \( S \). The next example due to S. Schwarz ([8], page 226) shows this.

Example 3.3. Let \( S \) be the discrete semigroup consisting of four elements \( \{0, a, e, f\} \) with the following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>e</td>
<td>0</td>
<td>0</td>
<td>e</td>
<td>0</td>
</tr>
<tr>
<td>f</td>
<td>0</td>
<td>a</td>
<td>0</td>
<td>f</td>
</tr>
</tbody>
</table>
Clearly, $e$ and $f$ are non-zero primitive idempotents of $S$. The lattice of ideals of $S$ is given by Figure 1. Obviously, Tod $f$ is not the minimal non-nil ideal of $S$.

\[ \text{Tod } e = \{0, a, f\} \]
\[ \text{Tod } f = \{0, a, e\} \]
\[ N = \{0, a\} \]
\[ \{0\} \]

Figure 1.

**Remark.** If $S$ is not compact, Theorem 3.2 need not hold. This can be seen from the following example.

**Example 3.4.** Let $S_1$ be the set of all non-negative real numbers with the ordinary multiplication. Let $S_2$ be the set of all integers $\leq -2$, the multiplication being the ordinary multiplication of numbers with a negative sign affixed. Define in $S_1 \cup S_2 = S$ a commutative multiplication $\ast$ by $x \ast y = 0$ if $x \in S_1$, $y \in S_2$, while the products in $S_1$ and $S_2$ are as above. Then $S$ is a semigroup. Clearly $N = [0, 1)$ and Tod $1 = [0, 1) \cup S_2$. Thus $N \uplus \bigcap_{e \in E} $ Tod $e$.

**Proposition 3.5.** Let $S$ be a compact $N$-semigroup and let $e$ be a non-zero idempotent of $S$. Then

(i) Tod $e$ is a nil ideal of $S$ if there does not exist any non-zero idempotent of $S$ which is orthogonal to $e$.

(ii) If $N$ is itself a prime ideal of $S$, then $N = \text{Tod } e$ for all non-zero idempotents $e$.

(iii) If Tod $e$ is not a minimal non-nil ideal of $S$, then Tod $e$ contains a non-zero primitive idempotent $f$ such that $fS \cap N$. Conversely, if $f$ is a non-zero primitive idempotent in Tod $e$ such that $N - fS \cap N$, then Tod $e$ is not a minimal non-nil ideal of $S$.

**Proof.** The proofs of (i) and (ii) are trivial, and the proof of (iii) is similar to the arguments of Numakura in [4]. We omit the details.

4. OPEN PRIME IDEALS IN $N$-SEMIGROUPS

Throughout this section all semigroups under consideration are commutative compact $N$-semigroups. Unless otherwise specified, $S$ will be such a semigroup.
Theorem 4.1. If e is a non-zero primitive idempotent of S, then Tod e is an open prime ideal of S.

We need the following lemma for the proof.

Lemma 4.2. Let e be a non-zero idempotent of S. If I is an ideal of S which is not contained in Tod e, then there is a non-zero idempotent f such that f ∈ I − Tod e.

Proof. Let x ∈ I − Tod e and consider the principal ideal J(x) generated by x. Clearly J(x) ⊂ I(x) ⊂ I. Since S is compact, I(x) is a compact semigroup. Thus there is an idempotent f ∈ I(x) ⊂ I. Suppose, for an indirect proof, that f ∈ Tod e. Then we have fe ∈ N, which implies that fe = 0. Thus, by continuity of multiplication, we have (xe)n → fe = 0. That is xe ∈ N. But this implies that x ∈ Tod e, which is a contradiction. Hence we conclude that f ∉ Tod e.

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Since e ∉ Tod e, we have Tod e ⊂ J₀(S − e). If Tod e ⊃ + J₀(S − e), then by Lemma 4.2, there is an idempotent f ∈ J₀(S − e) − Tod e. Hence ef ∉ 0. Since (ef) e = ef, we have 0 ⊃ ef ⊂ e. But e is a non-zero primitive idempotent of S. Hence ef = e. Thus e ∈ J(e) J(f) ⊂ J(f) ⊂ J₀(S − e) which is a contradiction. Hence Tod e = J₀(S − e). Now, applying the well-known theorem of K. Numakura (Theorem 2.5), we obtain immediately that Tod e is an open prime ideal of S.

Corollary 1. If E² consists of non-zero primitive idempotents, then N can be expressed as the intersection of a family of open prime ideals properly containing N.

Proof. Immediate from Theorem 3.2 and Theorem 4.1.

Remark. In [5] K. Numakura proved that the set N is the intersection of all open prime ideals of S. His result is clearly strengthened here by considering the ideals Tod e in place of all open prime ideals.

Corollary 2. Let Bᵦ = {x ∈ S ∣ eᵦ ∈ I(x)} and let e be a non-zero primitive idempotent. Then Bᵦ is a subsemigroup of S and Tod e is a union of Bᵦ, that is, Tod e = ∪ᵦ Bᵦ.

Proof. This follows from Schwarz’s results on compact commutative semigroups [7].

Corollary 3. Let S be a compact connected N-semigroup. If e ∈ E² then there exists a compact group lying in the boundary of the set Tod e.
Proof. Since $S$ is a compact $N$-semigroup, $\text{Tod } e$ is an open ideal of $S$. Then $\overline{\text{Tod } e} = \overline{\text{Tod } e}$. By Lemma 4.2, we can find an idempotent $f \in \overline{\text{Tod } e}$. Clearly $f$ lies on the boundary $\overline{\text{Bd } (\text{Tod } e)}$ for $\overline{\text{Bd } (\text{Tod } e)} = \overline{\text{Tod } e} \cap (S - \overline{\text{Tod } e}) = \overline{\text{Tod } e} \cap (S - \overline{\text{Tod } e})$. Now let $H(e)$ be the maximal group containing the idempotent $e$. As both $\overline{\text{Tod } e}$ and $\overline{\text{Tod } e}$ are ideals of $S$, we have $H(e) \subset \overline{\text{Bd } (\text{Tod } e)}$, completing the proof.

Remark. We observe that the converse of Theorem 4.1 need not be true, that is, an open prime ideal of $S$ need not correspond to an ideal $\text{Tod } e$ for some $e \in E^*$. The following example illustrates this.

Example 4.3. Let $S$ be the teeth of the comb-space with zero adjoined, that is, $S = (\{0\} \cup \{1/n \mid n = 1, 2, \ldots\}) \times [0,1)$. The multiplication $*$ defined on $S$ is given by

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2, \min \{y_1, y_2\}).$$

We easily check that $S$ is a topological semigroup with zero, and that all points lying on the lines $\{0\} \times [0,1)$ and $\{1\} \times [0,1)$ are idempotents of $S$. The non-zero primitive idempotent is the point $(1, 0) = e$. Clearly $\text{Tod } e = J_0(S - e) = S - S - \{1\} \times [0,1))$ which is an open prime ideal of $S$. If we consider the idempotent $e_1 = (1, \frac{1}{2})$, then for all $e \in E^*$, $\text{Tod } e$ is not equal to $J_0(S - e_1)$.

In general, $\text{Tod } e$ need not be a prime ideal. We have the following remark on finite semigroups.

Proposition 4.4. Let $S$ be a finite semigroup such that $N$ is not equal to $\text{Tod } e$ for all $e \in E^*$, then all $\text{Tod } e$ must be prime ideals of $S$ if $|S| \leq 4$.

Proof. If we want to construct a non-prime ideal $\text{Tod } e$ in $S$, according to Theorem 4.1, we must require that $e_1$ to be non-primitive, that is, there is some non-zero idempotent $g$ in $S$ such that $g < e_1$. Moreover, we also observe that for any non-nil ideal $\text{Tod } e$, there exists always an idempotent $f \in \text{Tod } e$ such that $f \perp e$. Combining these two facts, one can easily derive that in order to construct a non-prime ideal $\text{Tod } e$ in $S$, we must require that $S$ contains at least one non-primitive idempotent and at least three other non-zero idempotents, or require that $S$ contains at least one non-primitive idempotent, two non-zero primitive idempotents plus at least one other element. Thus, a non-prime ideal $\text{Tod } e$ cannot exist unless $|S| \geq 5$. We omit the details.

The following example shows how a non-prime ideal $\text{Tod } e$ can be constructed in a semigroup $S$. 

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Example 4.5. Consider the semigroup with the following multiplication table.

\[
\begin{array}{cccccc}
  & 0 & e & f & g & a & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & e & e & 0 & 0 & 0 \\
f & 0 & e & f & g & 0 & 0 \\
g & 0 & 0 & g & g & 0 & a \\
a & 0 & 0 & 0 & 0 & a & c \\
c & 0 & 0 & 0 & a & c \\
\end{array}
\]

Then \( N = \{0, a\} \), \( \text{Tod } f = \{0, a, c\} \). Clearly \( \text{Tod } f \) is not prime since \( e \notin \text{Tod } f \) and \( g \notin \text{Tod } f \), but \( eg = 0 \in \text{Tod } f \).

Moreover \( \text{Tod } e = \{0, g, a\} \), \( \text{Tod } g = \{0, e, a, c\} \), \( \text{Tod } c = \{0, e, f, g, a\} \). Thus \( N = \text{Tod } f \cap \text{Tod } g \cap \text{Tod } e \cap \text{Tod } c \).

We would like to thank Dr. P. N. Stewart here for his comments which lead to the following:

**Theorem 4.6.** Let \( E_2 \) be the set of non-zero primitive idempotent \( s \) of \( S \). Then \( N = \bigcap_{e \in E_2} \text{Tod } e \), where each \( \text{Tod } e \) is a minimal open prime ideal containing \( N \). Conversely if \( P \) is a minimal open prime ideal containing \( N \), then \( P = \text{Tod } e \) for some \( e \in E_2 \).

**Proof.** We first prove that if \( P \) is a minimal open prime ideal containing \( N \), then \( P = \text{Tod } e \) for some non-zero primitive idempotent \( e \). Let \( P \) be an ideal with this property, then by Theorem 2.5 we can write \( P = J_0(S - e) \) for some non-zero idempotent \( e \). If \( e \) is not a non-zero primitive idempotent, then there exists a non-zero idempotent \( e_1 < e \) such that \( J_0(S - e_1) \supseteq J_0(S - e) \). (See [6], page 119). But then \( J_0(S - e_1) \) is an open prime ideal of \( S \), which contradicts to the minimality of \( P \). Hence \( e \) is a non-zero primitive idempotent. Also \( \text{Tod } e \subset J_0(S - e) = P \) and \( \text{Tod } e \) is an open prime ideal. Thus \( \text{Tod } e = J_0(S - e) = P \). Now \( N = \bigcap_{e \in E_2} \text{Tod } e \). Our proof is completed.

**Remark.** If \( N \) itself is non-prime, then the set of all minimal open prime ideals of \( S \) properly containing \( N \) can be identified by the set of all non-zero primitive idempotents of \( S \).

We now give a new version of the theorem of Faucett, Koch and Numakura [1].

**Theorem 4.7.** Let \( e \) be a non-zero primitive idempotent of \( S \). If the intersection of maximal ideals of \( S \) is nil, then the following conditions are equivalent.
(1) $S$ — Tod $e$ is a disjoint union of groups.

(2) For each element of $S$ — Tod $e$ there exists a unit element.

(3) $a \in S$ — Tod $e$ implies that $a^2 \in S$ — Tod $e$.

(4) $S$ — Tod $e$ contains an idempotent and the product of any two idempotents of $S$ — Tod $e$ lies in $S$ — Tod $e$.

Proof. The proof uses a result of Schwarz [9]. It is proved there that a prime ideal of $S$ is a maximal ideal if and only if it contains the intersection of all maximal ideals of $S$. Now let $M$ be the intersection of all maximal ideals of $S$. By our hypothesis, $M$ is nil. Hence $M \subseteq N$. Since $N \subseteq$ Tod $e$ we have $M \subseteq$ Tod $e$. By Theorem 4.1, Tod $e$ is an open prime ideal; in fact, it is completely prime since $S$ is commutative. Then, by Schwarz’s result, Tod $e$ is a maximal ideal of $S$. Hence by the theorem of Faucett, Koch and Numakura [1], the theorem follows.

Remark. If $e$ is a non-zero idempotent of $S$, then (3) is always true by Corollary 3 of Theorem 3.2.

5. NIL IMPLIES NILPOTENT

The well-known theorem of Hopkins-Levitzki in ring theory states that if a ring $R$ satisfies the descending chain condition (ascending chain condition) on its one-sided ideals, then any nil ideal of $R$ is a nilpotent ideal of $R$. We show here that under some conditions, this theorem in ring theory can be transferred to compact $N$-semigroups without assuming the d.c.c. or a.c.c. on its ideals. In this section, the commutativity of $S$ is not assumed.

Remark. In a compact $N$-semigroup, a nil ideal need not be nilpotent as the following example shows.

Example 5.1. Let $S$ be the unit interval with the usual multiplication. Then $I = [0, 1)$ is a nil ideal (nil in the topological sense). However, $I$ is not nilpotent since $I^n = I$ for all integers $n \geq 1$.

Theorem 5.2. Let $S$ be a compact $N$-semigroup. If a non-nilpotent ideal $I$ of $S$ contains at least one closed non-nilpotent left (right) ideal of $S$, then $I$ is non-nil. (This is the Hopkins-Levitzki theorem on compact semigroups.)

The proof requires the use of the following result

Lemma 5.3. Let $S$ be a compact space and let $F = \{B_\lambda \mid \lambda \in \Lambda\}$ be a family of closed subspaces of $S$ indexed by $\Lambda$. If $A$ is an open subspace of $S$ such that $\bigcap_{\lambda \in \Lambda} B_\lambda \subseteq A$, then there is a finite number of $B_\lambda$ whose intersection is also contained in $A$. 560
Proof. Since $\bigcap B_\lambda \subset A$ we have $A' \subset \bigcup_{\lambda \in A} B_\lambda'$. Each $B_\lambda'$ is an open subspace of $S$ and $A'$ is compact in $S$. Thus $\{B_\lambda'\}_{\lambda \in A}$ is an open covering of $A'$. By the compactness of $A'$, there is a finite subcovering of $A'$, say $\{B_\lambda'\}_{\lambda = 1}^m$. Hence $A' \subset \bigcup_{\lambda = 1}^m B_\lambda'$ and hence $\bigcap_{\lambda = 1}^m B_\lambda \subset A$.

Proof of Theorem 5.2. Let $I$ be a non-nilpotent ideal of $S$. Let $T$ be the collection of all closed non-nilpotent left ideals of $S$ contained in $I$. Now $T$ is partially ordered by inclusion and is non-empty by our hypothesis on $I$. Suppose $\{T_\alpha\}_{\alpha}$ is a linearly ordered subcollection of $T$. Then $\bigcap T_\alpha$ is non-empty since $S$ is compact. Hence $\bigcap T_\alpha$ is a closed non-empty ideal of $S$. We claim that $\bigcap T_\alpha$ is a non-nilpotent ideal. For if not, then $\bigcap T_\alpha$ is nilpotent and hence is nil. Hence $\bigcap T_\alpha \subset N$ where $N$ is the set of all nilpotent elements of $S$. Since $T_\alpha$ is closed for all $\alpha$ and $N$ is open, by Lemma 5.3, we can find finitely many $T_\alpha$ whose intersection is contained in $N$. Since $\{T_\alpha\}_{\alpha}$ is an inclusion tower, we have $T_\alpha \subset N$ for some $\alpha$. But since $T_\alpha$ is a closed left ideal of $S$, by a result of K. Numakura ([5], page 675), $T_\alpha$ is nilpotent. This contradiction establishes our claim. Thus $\{T_\alpha\}_{\alpha}$ has a lower bound, and Zorn's lemma assures us of the existence of a minimal closed non-nilpotent left ideal, say $L_1$ in $I$. We have $L_1^2 \subset L_1$, but since $L_1$ is non-nilpotent, we must have $L_1^2 = L_1$ by the minimality of $L_1$. Let $\mathcal{M}$ be the family of all left ideals $J$ in $S$ such that $L_1J \neq 0$ and $J \subset L_1$. Then $\mathcal{M}$ is non-empty since $L_1 \in \mathcal{M}$. Since $S$ is compact, applying the above arguments and Zorn's lemma, we see that $\mathcal{M}$ has a minimal closed left ideal of $S$, say $J_1$ such that $L_1J_1 \neq 0$. Let $0 \neq x \in J_1$ be such that $L_1x \neq 0$. Then $L_1x$ is a closed left ideal of $S$, and $L_1(L_1x) = L_1^2x = L_1x \neq 0$ and $L_1x \subset L_1J \subset L_1$. Hence $L_1x \in \mathcal{M}$. Moreover, $L_1^2x = L_1x \neq 0$. Now let $a \in L_1$ be such that $ax = x$. Then for any integer $n \geq 1$ we have $a^n x = x$, which implies that $a^n \to 0$. Since $a \in L_1 \subset I$, $I$ is therefore non-nil. This completes our proof.

References


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