Czechoslovak Mathematical Journal

Robert H. Johnson

A new class of enumeration problems

Czechoslovak Mathematical Journal, Vol. 24 (1974), No. 4, 573–583

Persistent URL: http://dml.cz/dmlcz/101276

Czechoslovak Mathematical Journal, 24 (99) 1974, Praha

A NEW CLASS OF ENUMERATION PROBLEMS

R. H. Johnson, Berkeley

(Received August 24, 1973)

INTRODUCTION

Because of the large number of results in graph theory which involve the degrees of the vertices one is led very naturally to attempt to articulate in some general way the relation between the degrees of the vertices and the structure of a graph. This gives rise to the concept of the degree sequence of a graph: If a graph G has vertices $x_1, \ldots, x_p$ where $d_1 = \deg x_1 \geqq \ldots \geqq d_p = \deg x_p$ then $(d_1, \ldots, d_p)$ is said to be the *degree sequence* of G. Conversely a sequence S is *graphical* if there is a graph G with degree sequence S in which case we say G *realizes* S, or G is a realization of S, or G *belongs* to S.

The study of the relation between graphs and their degree sequences leads, in turn to questions of the following sort. How many non-isomorphic graphs realize a given sequence? Given a positive integer n is there a graphical sequence with exactly n realizations? Given a positive integer n are there usefull bounds on the number of realizations on a sequence of length n. This paper develops methods for answering these questions.

In section A ordinary generating functions (o.g.f.'s) are used to enumerate the realizations of graphical sequences of elementary and common character. In section B we enumerate the realizations of two results are then used in section C where the problem of bounds on realizations of sequences is taken up. In this section the question "For a given positive integer n is there a sequence with n realizations" is also discussed. The latter, "Nat Turner's Problem", goes back to Senior ([5], 1951).

For convenience we introduce the following notation. If $S = (s_1, \ldots, s_p)$ is graphical sequence, then $|S| = |(s_1, \ldots, s_p)|$ denotes the number of nonisomorphic realizations of S. If S is not graphical, we set $|S| = 0$. Also if $r_i = |\{s_j \mid s_j = i, 1 \leqq j \leqq p\}|$ for $i = 0, \ldots, p - 1$, then S can also be written $0^{r_0}1^{r_1}2^{r_2} \ldots$ meaning that S has r_0 zero entries, r_1 one entries, $\ldots$. For example if S is a sequence of length p such that $S = (2, \ldots, 2)$ then $S = 2^p$. This latter notation is commonly used in connection with 'partitions'.

Throughout this paper the concept of 'ordinary generating function' (o.g.f.) is used. One o.g.f. that is used frequently in the sequel is $P(x) = 1/(1 - x)(1 - x^2)$. $(1 - x^3) \ldots$ where the coefficient of x^n is p_n, the number of partitions of n. The terminology and notation used in this paper follows that of [1].

A. COUNTING REALIZATIONS FOR ELEMENTARY GRAPHICAL SEQUENCES

We first find ordinary generating functions which enumerate the number of realizations of sequences of rather simple structure. These latter are then used to construct the ordinary generating functions for sequences that are somewhat more complicated.

Proposition 1. *The number of realizations of* $S = (2, \ldots, 2) = 2^p$ *is the coefficient* c_p *of* x^p *in*

$$(*) \qquad C(x) = 1/(1 - x^3)(1 - x^4) \ldots .$$

Proof. For p a natural number consider any partition $(p_1, \ldots, p_r)$ of p where $p_1 \geqq \ldots \geqq p_r \geqq 3$, $p_1 + \ldots + p_r = p$. It is clear that the graph $C_{p_1} \cup \ldots \cup C_{p_r}$ a realization of $2^p = (2, \ldots, 2)$.

Conversely consider any realization G of 2^p. Since each point of VG has degree two G is the union of disjoint cycles – say $C_{p_1}, \ldots, C_{p_r}$ where $p_i \geqq 3$ for $i = 1, \ldots, r$ is and $p_1 + \ldots + p_r = p$.

Thus there is a one-to-one correspondence between partitions of p, each part greater than or equal to three and the realizations of 2^p. Now by the same reasoning that $P(x) = 1/(1 - x)(1 - x^2)(1 - x^3) \ldots$, it follows that $1/(1 - x^3)(1 - x^4) \ldots$ has as coefficient of x^n, $n > 0$, equal to $(2, \ldots, 2) = 2^n$. For convenience we say that $c_0 = 1$ so that now

$$C(x) = \sum_{i=0}^{\infty} c_i x^i = 1/(1 - x^3)(1 - x^4) \ldots .$$

Proposition 2. *The number of realizations of the sequence* $(2, \ldots, 2, 1, 1)$ *of length p is the coefficient of* x^p *in*

$$C(x) \cdot \frac{x}{1 - x}$$

where $C(x)$ *is as above.*

Proof. Let d_p denote the number of realizations of the p-part sequence $(2, \ldots$ $\ldots, 2, 1, 1)$. Any realization of $(2, \ldots, 2, 1, 1)$ must have the following form: $c_{p_1}, \ldots$ $\ldots, c_{p_t}, c_{p_{t+1}}$ where $p + 1 = p_1 + \ldots + p_t + p_{t+1}$. If c_i is the coefficient of x^i, $i \geqq 0$, in $\sum_{i=0}^{\infty} c_i x^i$ then there are c_{p-2} realizations of $(2, \ldots, 2, 1, 1)$ that have a path of length one, c_{p-3} realizations that have a path of length two, and so on. It follows that

$$d_n = c_{n-2} + c_{n-3} + \dots + c_3 + 1 =$$

$$= c_{n-2} + c_{n-3} + \dots + c_3 + c_2 + c_1 + c_0 =$$

$$(\text{since } c_2 = c_1 = 0 \text{ and } c_0 = 1)$$

$$= \sum_{i=0}^{n} c_i - c_n - c_{n-1}$$

where $n \geqq 2$. Setting $d_1 = 0$, $d_0 = 0$ we have

$$d_n x^n = \sum_{i=0}^{n} c_i x^n - c_n x^n - c_{n-1} x^n , \quad n \geqq 1$$

and thus

$$\sum_{i=1}^{\infty} d_i x^i = \sum_{i=1}^{\infty} \sum_{j=0}^{i} c_j x^i - \{C(x) - 1\} - x \sum_{i=0}^{\infty} c_i x^i = \sum_{i=1}^{\infty} \sum_{j=0}^{i} c_j x^i - C(x)(1 + x) + 1 .$$

Thus since $d_0 = 0$ and $c_0 = 1$ we have

$$\sum_{i=0}^{\infty} d_i x^i = \sum_{i=0}^{\infty} (\sum_{j=0}^{i} c_j) x^i - C(x)(1 + x) = C(x)(1 - x)^{-1} - C(x)(1 + x) =$$

$$= C(x) \left[\frac{1}{1 - x} - (1 + x) \right] = C(x) \frac{x^2}{1 - x} .$$

Proposition 3. *The sequence* $(4, 2, \dots, 2)$ *of length* n, *has* p_{n-5} *realizations, where* p_i, $i = 1, 2, \dots$, *is the coefficient of* x^i *in* $P(x) = 1/(1 - x)(1 - x^2) \dots$. *Or, the ordinary generating function of* $(4, 2, \dots, 2) = 42^{n-1}$ *is* $x^5 P(x)$.

Proof. First it is clear that any realization consists of the union of a 'figure eight' (a connected graph with one point of degree four, the rest of degree two) with disjoint cycles. In the same spirit as Proposition 2 we count the number of realizations of $(4, 2, \dots, 2)$ by counting those with a figure eight with five points, six points, $\dots$. To this end we first find the ordinary generating function (o.g.f.) for $\langle f_n \rangle$ where f_n is number of figure eights on n points. If a figure eight has n points then it has $n - 1$ points distributed on two loops. Hence there is a one-to-one correspondence between figure eights on n points and partitions of $n - 1$ into two parts each part greater than or equal to two, $n \geqq 5$. It is easy to verify that the values of f_n for $n = 1, \dots, 12$ are $0, \dots, 0, 1, 1, 2, 2, 3, 4, 4$. It is easily shown by induction that the latter pattern continues. We first find the o.g.f. for $\langle 1, 1, 2, 2, 3, 3, \ldots \rangle$. The open form for the generating function is

$$\sum_{k=0}^{\infty} (k+1) x^{2k} + \sum_{k=0}^{\infty} (k+1) x^{2k+1} = \sum_{k=0}^{\infty} x^k + \sum_{k=1}^{\infty} kx^{2k+1} + \sum_{k=1}^{\infty} kx^{2k} =$$

$$= \sum_{k=0}^{\infty} x^k + x \sum_{k=0}^{\infty} ky^k + \sum_{k=0}^{\infty} ky^k = \frac{1}{1-x} + (1+x) \frac{y}{1-y^2} = \frac{1}{1-x} +$$

$$+ (1+x) \frac{x^2}{(1-x^2)^2} = \frac{1}{(1-x)(1-x^2)} \text{ (where } y = x^2\text{)}.$$

To obtain the ordinary generating function for the f_n we merely multiply the above by x^5:

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = x^5/(1-x)(1-x^2).$$

Back to computing $a_p = |(4, 2, \ldots, 2)|$, where $(4, 2, \ldots, 2)$ is of length p. Let $C(x) = \sum c_i x^i$ be as per Proposition 1. Then by the remark at the beginning of the proof, for $p \geqq 5$,

$$a_p = c_0 f_p + f_{p-1} c_1 + \ldots + f_0 c_p = \sum_{k=0}^{p} f_k c_{p-k}.$$

Defining $a_0 = \ldots = a_4 = 0$ we have

$$a_p = \sum_{k=0}^{p} f_k c_{p-k} \quad \text{for} \quad p = 0, 1, 2, \ldots.$$

Hence the o.g.f. for a_p must be

$$F(x)\, C(x) = [x^5/(1-x)(1-x^2)]\, [1/(1-x^3)(1-x^4)\ldots] =$$
$$= x^5/(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots = x^5 P(x).$$

We now give a sequence a lemmas which will yield a generalization of Proposition 3. The proofs depend on 'Ferrers' graphs (see [4, p. 114–117]) and I doubt that these lemmas are original.

Lemma 4. *There is a one-to-one correspondence between partitions of $n-1$ into m parts each part being greater than or equal to two and the partitions of $n-1-2m$ having no part greater than m.*

Proof. It will suffice to consider the Ferrers graphs and their conjugates. The conjugate of a partition of $n-1$ into m parts each part greater than or equal to two will have m dots in each of the first two rows. The remaining rows of the conjugate will have m or fewer dots, since the original partition had m or fewer dots in each column. Finally it is clear that the third, fourth, ..., etc, rows of the conjugate uniquely determine the original and vice versa. The result follows.

Lemma 5. *The number of partitions of $n - 1$ into m parts, each part being greater than or equal to two is the coefficient of x^n in the expansion of $x^{2m+1}(1 - x)^{-1} \dots \dots (1 - x^m)^{-1}$.*

Proof. The number of partitions of n into parts no greater than m is the coefficient of x^m in $(1 - x)^{-1} \dots (1 - x^m)^{-1}$. Hence the coefficient of x^n in $x^{2m+1}(1 - x)^{-1} \dots (1 - x^m)^{-1}$ is number of partitions of $n - 2m + 1 = (n - 1) - 2m$ with parts no greater than m. By Lemma 4 the result follows.

Lemma 6. *The number of realizations of the n-part sequence $(2m, 2, \dots, 2)$ which are connected is the coefficient of x^n in $x^{m+1}/(1 - x)^{-1} \dots (1 - x^m)^{-1}$.*

Proof. Any connected realization of $(2m, 2, \dots, 2)$ consists of m cycles 'joined' at a single point. If the cycles are $C_{r_1}, \dots, C_{r_m}$ then $r_i \geqq 3$ for $i = 1, \dots, m$ and $(r_1 - 1) + \dots + (r_m - 1) = n - 1$. Thus the number of non-isomorphic graphs of this type is clearly the number of partitions of $n - 1$ into $m - 1$ parts each part greater than or equal to two. The result now follows from the previous lemmas.

Remark. A connected realization of the n-part $(2m)\, 2^{n-1} = (2m, 2, \dots, 2)$ is called a *rose* on n points with m *petals*. Thus the o.g.f. $R^m(x)$ of m-petaled roses given by $\sum_{n=1}^{\infty} r_n^m x^n = x^{2m+1}/(1 - x) \dots (1 - x^m)$.

Proposition 7. *Let S_n be the n-part sequence $(2m, 2, \dots, 2)$. Then the o.g.f. of $\langle |S_n| \rangle$ is $C(x)\, R^m(x)$ when R^m and C are as above.*

Proof. Again, in the spirit of previous proofs, it is clear that any realization of $S_n = (2m, 2, \dots, 2)$ must be the union of a m-petaled rose and disjoint cycles. Reasoning as before we have for $n \geqq 2m + 1$, $|s_n| = r_{2m+1}^m c_{n-2m+1} + \dots + r_{n-3}^m c_3 + r_n^m$. Since $r_j^m = 0$ for $0 \leqq j \leqq 2m$ we have $|s_n| = \sum_{k=0}^{n} r_k^m c_{n-k}$. But this latter shows that the o.g.f. for $\langle |s_n| \rangle$ is the product of $R^m(x)$ and $C(x)$ and hence the result.

Remark. Taking $m = 2$ in Proposition 7 yields Proposition 3. The interest of Proposition 3 is the curious occurrence of $P(x)$.

Now consider realizations of the p-part partition $F_p = (2m, 2n, 2, \dots, 2)$ with $m > n > 1$. Let r_j^i and c_j denote the same quantities as in the previous propositions and let $p_k(j)$ denote the number of partitions of j into k parts. If G is a realization of F_p let $a, b \in V(G)$ be the points of degree $2m$ and $2n$ respectively.

Any realization G of F_p may be classified according to the number of such paths is of the form $2l$, $0 \leqq 2l \leqq 2n$. There are two cases: (a) one of these paths consists of an edge and (b) each of the paths contain at least two edges.

Case (a). If there are 2 edge disjoint paths of G joining a and b then there are $m - l$ cycles at a not containing b and $n - l$ cycles at be not containing a. Suppose that $i + 1$ points of G lie on cycles adjacent to a and $j + 1$ points of G lie on cycles containing b. Let $p_{t,2}(j)$ denote a number of partitions of j into t parts, each part greater than or equal to two. For the purpose that there are k points (other than a and b) distributed on the $2l - 1$ disjoint paths (one path consists simply of the edge ab). We then have

$$p_{n-l,2}(i)\, p_{m-l,2}(j)\, p_{2l-1}(k)\, c_t\,, \quad i + k + j + t = p - 2$$

such graphs G which realize F_p. Hence the total number is

$$(*) \qquad \sum_{i+j+k+t=p-2} p_{n-l,2}(i)\, p_{m-l,2}(j)\, p_{2l-1}(k)\, c_t$$

where $0 \leqq l \leqq n$ and for the sake of consistency we have set $p_{-1}(k) = p_{0,2}(k) = 1$ for all k.

Case (b). Arguing the same way the number of realizations is

$$(\substack{*\\*}) \qquad \sum_{i+j+k+t=p-2} p_{n-l,2}(i)\, p_{m-l,2}(j)\, p_{2l}(k)\, c_t\,.$$

Next we set

$$P_{t,2} = P_{t,2}(x) = \sum_{i=0}^{\infty} p_{t,2}(i)\, x^i\,,$$

$$P_t = P_t(x) = \sum_{i=0}^{\infty} P_t(i)\, x^i\,.$$

Then letting g_p^l be the sum of $(*)$ and $(\substack{*\\*})$, multiplying both sides by x^p and summing for $p \geqq 2$ we have

$$\sum_{p=2}^{\infty} g_p^l\, x^p = \sum_{p=2}^{\infty} \sum_{i+j+k+t=p-2} p_{n-l,t}^{(i)}\, p_{m-l,t}^{(j)}\, c_t(P_{2_l}(k) + P_{2l-1}(k))\, x^p\,.$$

Setting $g_0^s = g_1^s = 0$ we have

$$(\substack{*\\*\\*}) \qquad \sum_{p=0}^{\infty} g_p^l x^p = x^2 P_{n-l,2} P_{m-l,2} C[P_{2l} + P_{2l-1}]\,.$$

Proposition 8. *Let* $F_p = (2m, 2n, 2, \ldots, 2)$, $m > n \geqq 2$. *If* $|F_p| = g_p$ *the o.g.f. for* $\langle g_{p_n} \rangle$ *is*

$$x^2 \sum_{l=0}^{n} P_{n-l,2} P_{m-l,2} C[P_{2l} + P_{2l-1}]\,.$$

Proof. The formula follows from $\binom{*}{*}$ upon summing for $l = 0, \ldots, n$ where l denoted the fact that there were 2 edge disjoint paths joining a and b.

Remark. It is of interest to note that

$$P_{2s} = x^{2s}/(1 - x) \ldots (1 - x^{2s})$$

$P_{m-l,2} = x^{2(m-s)}/(1 - x) \ldots (1 - x^{m-s})$ and that P_{n-s} has a similar expression. In spite of this Proposition 8 does not admit of an obvious simplification. However it does follow that the o.g.f's in the previous proposition have form

$$P(x)\,[r(x)/s(x)]$$

where P is the o.g.f. for partitions, and r and s are polynomials.

Other sequences whose realizations can be enumerated without great difficulty are $(2, \ldots, 2, 1, \ldots, 1)$ and $(2m + 1, 2n + 1, 2, \ldots, 2)$. Lastly one should note the relative complexity of the o.g.f. for relatively simple sequences.

B. IMPORTANT SPECIAL CASES

In this section we give the o.g.f. for realizations of sequences which have a much different structure than those of Section A. The results will then be applied to the problem of finding bounds on the number of realizations of a graphical sequence.

Proposition 9. *Let S_p be a sequence of length p^2 such that $d_1 = \ldots = d_p = p - 1$ and $d_{p+1} = \ldots = d_{p^2} = 1$. Then the o.g.f. of $\langle |S_p| \rangle$ is the o.g.f. for the number of graphs on p points.*

Proof. $K_{1,p-1} \cup \ldots \cup K_{1,p-1}$ is a realization of S_p. Let $v_1, \ldots, v_p$ be the centers of $pK_{1,p-1}$. Let G be any graph with $\{v'_1, \ldots, v'_p\} = V(G)$. We construct an isomorphic copy of G on the centers $v_1, \ldots, v_p$ in $K_{1,p-1} \cup \ldots \cup K_{1,p-1}$ as follows. If $v'_i v'_j \in E(G)$ choose $v_i^k, v_j \in V(K_{1,p-1} \cup \ldots \cup K_{1,p-1})$ so that $v_i^k v_i, v_j^l v_j$ are $E(K_{1,p-1} \cup \ldots \cup K_{1,p-1})$. Then transfer $v_i^k v_i$ and $v_j^l v_j$ for $v_i v_j$ and $v_i^k v_j^l$. Since $\deg v_i = \deg v_j^l = p - 1$ the choice of v_i^k and v_j is always possible. After such a transfer is made for each $e \in E(G)$ the resulting graph H has an isomorphic copy of G on $\{v_1, \ldots, v_p\}$ (or $\langle \{v_1, \ldots, v_p\} \rangle \cong G$); if $\deg v'_i = t$ then in H, v_i is also adjacent to $p - 1 - t$ vertices of degree one and, finally, there are $|E(G)|$ components of H which are K_2 graphs.

By Theorem 1.1 of [3] any realization of S_p can be obtained from $K_{1,p-1} \cup \ldots \cup K_{1,p-1}$ by a finite sequence of transfers. This implies that if H is such a realization then it consists of a graph on p points, $H' = \langle \{v_1, \ldots, v_p\} \rangle$, where $\deg v_i = k$ *in* H' implies v_i is adjacent to $p - 1 - k$ points of degree one and H has $|EH'|$ components are K_2. Note that the realization H of S_p uniquely determines the H' uniquely and vice versa.

The above means that there is a one-to-one correspondence between realizations of S_p and graphs on p points. This gives the result.

Corollary 10. *Let S be a sequence of kp parts such that $d_1 = \ldots = d_p = k$, $d_{p+1} = \ldots = d_p = 1$, $1 \leqq k \leqq p - 1$. Then the number of realizations of S is the number of graphs on p points where each point has degree less than or equal to k (all p point graphs G with $\Delta(G) \leqq k$).*

Proof. The proof uses the same technique as Proposition 9.

Remark. The above proofs go through only because the star graphs used have the same size. Otherwise the graphs obtained from transfers are not uniquely determined by the adjacency of the centers of the star graph.

Proposition 11. *Let S_p be a sequence on $p^2(p - 1) + p$ parts such that*

$$d_1 = (2p - 1)(p - 1),\ d_2 = (2(p - 1) - 1)(p - 1), \ldots, d_p = 1(p - 1)$$

and $d_{p+1} = \ldots = d_{p^2(p+1)+p} = 1$. Then the number of realizations of S_p is $2^{\binom{p}{2}}$, the number of labelled graphs on p points.

Proof. One realization of s_p consists of the p star graphs $K_{1,(2p-1)(p-1)}, \ldots, K_{1,p-1}$. Any other realization of S_p can be obtained by joining the centers of two of the above stars and then detaching two points of degree one, one each from the centers of the chosen stars and joining them – the same procedure given in 9. However because of the differing degrees of the centers of stars it follows that graph determined by the centers is a *labelled* graph, the labelling induced by the differing degrees of the centers. Again it can be verified that there is a one-to-one correspondence between labelled graphs on p points and realizations of S_p.

C. BOUNDS FOR THE NUMBER OF REALIZATIONS OF GRAPHICAL SEQUENCES

We now apply some of the results of the previous two sections to the problem of estimating the number of realizations of a graphical sequence.

Lemma 12. *Given a constant C there exists a natural number p for which the number of connected realizations of the p-part sequence $(2t, 2, \ldots, 2)$ is greater than Cp for at least C positive integers t.*

Proof. Given C choose p so that both $p/4 > C + 3$ and $p^2/48 - 1 > Cp$ are satisfied.

By Lemma 6 the number of connected realizations of the p-part $(2t, 2, \ldots, 2)$ is the coefficient a_p of x^p in $x^{2t+1}/(1 - x) \ldots (1 - x^t)$. But the latter is also the o.g.f. of the number of partitions of $p-1-2t$ into parts of t or less. Assume $3 \leqq t \leqq$ $\leqq C+3$ and let $p_t = p-1-2t$. Then the number of partitions of p_t into parts no greater than t is the same as the number of partitions of p_t into t or less parts, $t \geqq 3$. Then by the way p was chosen we have

$$p/4 > C + 3 \geqq t \geqq 3$$

Hence $p/4 > t$ and this in turn implies that

$$(*) \qquad p_t = (p-1) - 2t \geqq p/2\,.$$

Now if $p_3(n)$ denotes the number of partitions of n into three parts we have by HALL [2, p. 30] that

$$({*\atop *}) \qquad p_3(n) > n^2/12 - 1\,.$$

Since the number of partitions of p_t into t or fewer parts, a_p is greater than or equal to $p_3(p_t)$ we have (by $(*)$ and $({*\atop *})$)

$$a_p \geqq p_3(p_t) \geqq p_3(p/2) > (p/2)^2/12 - 1 > p^2/48 - 1$$
$$> Cp \quad \text{by choice of} \quad p\,.$$

Hence $a_p > Cp$ for $t = 3, \ldots, C+3$. This yields the lemma.

Proposition 13. *Let C be a natural number. There is a natural number p for which there is a graphical sequence S on p-parts with $|S| > p^C$.*

Proof. Let C be given. By Lemma 12 there is a p_0 such that the number of connected realizations of the p_0-part $(2t, 2, \ldots, 2)$ is greater than Cp_0 for at least C distinct values of t: $t_1, \ldots, t_C$. Now consider the graphs on p_0C points obtained by taking the union of C connected graphs — one for each $(2t_i, 2, \ldots, 2)$, $i = 1, \ldots, C$. It is clear that these are more than $(Cp_0)^C$ non-isomorphic ways of constructing such a union. Moreover each such union is a realization of the p_0C — part sequence $(2t_C, \ldots, 2t_1, 2, \ldots, 2)$. Hence we take s to be the latter sequence and $p = p_0C$ to obtain the result.

Corollary 14. *For a given integer c there exists an integer p and a sequence S of length p such that the number of connected realizations, S_c, of S exceeds p^c.*

Proof. Applying the previous proposition to $2c$ we get an m-part sequence which is realized by a union of $2c$ roses with $t_{2c}, \ldots, t_2$ and t_1 petals. To each such realization adjoin a new point adjacent to the points of degree $2t_{2c}, \ldots, 2t_1$. Hence there is an $m+1$ part sequence with more than m^{2c} distinct connected realizations. But for $m \geqq 2$, $m^{2c} = (m^2)^c > (m+1)^c$. Taking $p = m+1$ gives the result.

Lemma 15. *For $p \geqq 6$ we have* $\binom{p}{2}^2 > p^3 - p^2 + p.$

Proof. If the left hand side of above is squared out we get

$$\tfrac{1}{4}p^4 - \tfrac{1}{2}p^3 + \tfrac{1}{4}p^2 > p^3 - p^2 + p$$

or $p^3 - 6p^2 + 5p - 4 > 0$ or $p(p-5)(p-1) - 4 > 0$. Thus the lemma is equivalent to the latter being true and if $p \geqq 6$ the latter is trivially true.

Proposition 16. *Let n be a positive integer of the form $p^3 - p^2 + p$, $p \geqq 6$. There is a sequence of length n which has more than $2^{\sqrt{n}}$ realizations.*

Proof. By Proposition 11 there is a sequence on $n = p^2(p-1) + p = p^3 - - p^2 + p$ parts which has $2^{\binom{p}{2}}$ realizations. By the lemma

$$\tfrac{1}{2}(p^2 - p)^2 > p^3 - p^2 + p = n$$

or

$$\binom{p}{2} > p^3 - p^2 + p = \sqrt{n}\,.$$

This gives the result.

The meaning of propositions 13 and 16 is not only that there are sequences with an arbitrarily large number of realizations but also that the 'order' of largeness is itself large. Again, very roughly, there can be no useful bound b_n for an arbitrary positive integer n such that $|s| < b_n$ for sequence s on n parts.

Turning in another direction we look briefly at another question raised by Senior [5] in 1951: For which n does there exist a graphical sequence S such that $|S| = n$, such that S has n realizations. Senior raised the question with respect to multigraphs with loops and found no solution. The status of the problem has remained essentially the same and has not been raised with respect to ordinary graphs. It is conjectured here that the equation $|S| = n$ has solutions for all $n = 1, 2, \dots$. First note that the conjecture has been verified for $n = 1, \dots, 8$. We now give a few simple results in the direction of the conjecture.

Proposition 17. *Let n be a positive integer. If there is one graphical sequence $S = (s_1, \dots, s_p)$ such that*

$$(*) \qquad |S| = n$$

then there are an infinite number of graphical sequences which also satisfy $(*)$.

Proof. If S satisfies $(*)$ then the sequence $S' = (p, s_1 + 1, \dots, s_p + 1)$, also satisfies $(*)$. Obviously this process of 'coning' can be continued indefinitely so that $(*)$ must have an infinite number of solutions.

Notation. If s is a graphical sequence let $|S|_c$ denote the number of connected realizations of s.

Proposition 18. For any positive integer n the equation $|S|_c = n$ has a solution.

P r o o f. Consider the proof of Proposition 3. It was proven there that if f_n denotes the number of 'figure eights' on n-points, then the o.g.f. for $\langle f_n \rangle$ is $x^5/(1 - x) \cdot (1 - x^2)$. It is easily verified that $f_n = \frac{1}{2}(n - 5) + 1$ if $n \geqq 5$ and $f_n = 0$ for $1 \leqq n \leqq 4$. Now notice that f_n assumes all non-negative integer values.

References

[1] *Chartrand, Gary* and *Behzad Mehdi,* Introduction to the Theory of Graphs, Allyn and Bacon 1971.

[2] *Hall, Marshall,* Combinatorial Theory, Ginn and Bloisdell 1967.

[3] *Johnson, R. H.,* Ph. D. Thesis.

[4] *Riordan, John,* An Introduction to Combinatorial Analysis, John Wiley and Sons 1958.

[5] *Senior, J. K.,* Partitions and their Representative Graphs, American Journal of Mathematics *73* (1951), p. 663—689.

Author's address: 2337 California Street, Berkeley, California 94703, U.S.A.