Dalibor Klucký
Ternary rings with zero associated to Desarguesian and Pappian planes


Persistent URL: http://dml.cz/dmlcz/101279

Terms of use:

© Institute of Mathematics AS CR, 1974

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
TERNARY RINGS WITH ZERO ASSOCIATED TO DESARGUESIAN AND PAPPIAN PLANES

DALIBOR KLUCKY, OLOMOUC

(Received November 19, 1973)

The article is an immediate continuation of [6]. Let us recall some necessary notions introduced there. Under coordinate system in an affine plane $P(n)$ we understand any couple of bijections $(\pi, \lambda)$

$$
\pi : S^2 \to A, \quad \lambda : S^2 \to B,
$$

where $S$ is a set whose cardinality equals the order of $P(n)$, $A$ is the set of proper points of $P(n)$, $B$ is the set of lines of $P(n)$ which do not pass through the given direction (improper point) $V$. The direction $V$ will be called vertical direction of the coordinate system $(\pi, \lambda)$. If a ternary operation $t$ on $S$ is given so, that $(S, t)$ is a planar ternary ring (abb. PTR) in sense of [5] or [6] we shall always require

$$(x, y)^e \in (a, b)^2 \iff y = t(x, a, b).$$

In this case we say that PTR $(S, t)$ coordinatizes the affine plane $P(n)$ or that $(S, t)$ is a PTR of $P(n)$. We also say that $(S, t)$ corresponds to the coordinate system $(\pi, \lambda)$. An affine plane $P(n)$ is said to be a translation plane, if the group of translations operates transitively on $A$.

Let $(S, t)$ be an arbitrary PTR of a given affine plane $P(n)$. The following statement is proved in [6]:

$P(n)$ is a translation plane if and only if the following conditions are fulfilled:

(A) $(S, +)$ is a group (in this case, $(S, +)$ is abelian).

---

1) $n$ is the improper line (the line in infinity) of $P(n)$.
2) i.e. with a zero element 0, but not necessarily with the unit.
(B) \( \forall a, b, c \in S \),
\[
\mathbf{t}(a, b, c) = a \cdot b + c^3
\]
holds.

(C) For arbitrary \( a, b, c \in S \) the equation
\[
a \cdot m + b \cdot m = c \cdot m
\]
either has only the trivial solution or it is fulfilled identically (cf. [6] Corollary 4 of Theorem 2).

The purpose of this article is to formulate and prove analogous conditions for \( P(n) \) to be desarguesian or pappian plane.

For any proper point \( B \) of \( P(n) \) let us denote by \( H(B) \) the group of all (regular) homotheties with fixed point \( B \).

**Definition.** An affine plane \( P(n) \) will be called a desarguesian plane, if for any proper point \( B \) and for any affine line \( p \ni B \) the group \( H(B) \) operates transitively on \( p \setminus \{B\} \). Moreover, if \( H(B) \) is an abelian group for each point \( B \), then \( P(n) \) will be called a pappian plane.

The following facts are well known:

(a) Every desarguesian plane (and, consequently, every pappian plane) is a translation plane.

(b) A translation plane is desarguesian if and only if there exists a proper point \( B \) and an affine line \( p \ni B \) such that the group \( H(B) \) operates transitively on \( p \setminus \{B\} \).

(c) A desarguesian plane is pappian if and only if there exists a proper point \( B \) such that the group \( H(B) \) is abelian.

**Theorem 1.** Let \( P(n) \) be a desarguesian plane and let \((S, \mathbf{t})\) be its arbitrary PTR. Then the following condition is fulfilled:

(D) For arbitrary \( a, b, c \in S \) the equation
\[
m \cdot a + m \cdot b = m \cdot c
\]
either has only the trivial solution or it is fulfilled identically.

**Proof.** Let \( m \) be a non-trivial solution of (1) and \( m \in S \setminus \{0\} \). We may assume that \( a \neq 0, \ b \neq 0 \). Consider the homothety \( \kappa \in H(O) \), \( \kappa : (m, 0)^* \mapsto (m, 0)^* \). Let \( \bar{p}, p \) be two parallel lines with the same first coordinate \( a \) such that \( \bar{p} \ni (m, 0)^* \).

\[3^) \] + and \( \cdot \) are binary operations on \( S \) defined by \( a \cdot b = \mathbf{t}(a, b, 0); \ a + b = \mathbf{t}(a, e_a, b) \), where \( e_0 = 0 \) and for \( a \neq 0 \), \( e_a \) is the solution of the equation \( a \cdot x = a \).

\[4^) \] In the sequel we shall denote by \( O \) the point \( (0, 0)^* \).

608
\[ p \ni (m, 0)^z. \text{ Hence, it follows: } \bar{p} = (a, -(\bar{m} \cdot a))^z, \ p = (a, -(m \cdot a))^z. \text{ Put } \bar{q} = (c, -(\bar{m} \cdot a))^z, \ q = (c, -(m \cdot a))^z. \text{ Clearly } \kappa(\bar{p}) = p \ni \kappa : (0, -(\bar{m} \cdot a))^z \ni \rightarrow (0, -(m \cdot a))^z \ni \kappa(\bar{q}) = q. \text{ Let } Y \text{ be the point lying on } \bar{q} \text{ with the first coordinate } \bar{m} \text{ and let } Y \text{ be the point lying on } q \text{ with the first coordinate } m. \text{ It is obvious that } \\
\kappa : Y \ni Y, \text{ hence } Y, Y \text{ lie on the same line } (r, 0)^z \text{ passing through } O. \text{ We have } \{Y\} = \{\bar{q} \cap (r, 0)^z\}, \{Y\} = \{q \cap (r, 0)^z\} \text{ which implies } \\
(2) \quad \bar{m} \cdot r = \bar{m} \cdot c - (\bar{m} \cdot a), \ m \cdot r = m \cdot c - (m \cdot a). \]

By the assumption \( \bar{m} \cdot b = \bar{m} \cdot c - (\bar{m} \cdot a) \Rightarrow r = b \) and the second relation of (2) yields (1).

**Proposition 1.** Let \( P(n) \) be a desarguesian plane and let \( (S, t) \) be its arbitrary \( \text{PTR. Let } \kappa \in H(O) \text{ be given by} \\
(3) \quad \kappa : (u, 0)^z \ni (\bar{u}, 0)^z, \quad u, \bar{u} \in S \setminus \{0\}.

If \( \kappa : (w, 0)^z \rightarrow (\bar{w}, 0)^z \), \( w, \bar{w} \in S \setminus \{0\} \), then for any \( m \in S \)

(4) \quad w\,(u \cdot m) = \bar{w}\,(\bar{u} \cdot m) \quad \text{5)}

holds.

**Proof.** We may assume \( m \neq 0 \). Putting \( k = w\,(u \cdot m) \) we get \( w \cdot k = u \cdot m \) and we have to prove

\[ (4a) \quad \bar{w} \cdot k = \bar{u} \cdot m. \]

Let \( p, \bar{p} \) be two parallel lines with the same first coordinate \( m \) such that \( p \ni (u, 0)^z \), \( \bar{p} \ni (\bar{u}, 0)^z \Rightarrow p = (m, -(u \cdot m))^z \), \( \bar{p} = (m, -(\bar{u} \cdot m))^z \). Put \( q = (k, -(u \cdot m))^z \), \( \bar{q} = (k, -(\bar{u} \cdot m))^z \). Clearly \( \kappa(p) = \bar{p} \Rightarrow \kappa : (0, -(u \cdot m))^z \ni (0, (\bar{u} \cdot m))^z \Rightarrow \kappa(\bar{q}) = \bar{q} \). Using \( w \cdot k = u \cdot m \) we obtain \( (w, 0)^z \in q \), hence \( (\bar{w}, 0)^z \in \bar{q} \Rightarrow (4a) \).

**Proposition 2.** Let \( P(n) \) be a desarguesian plane and let \( (S, t) \) be its arbitrary \( \text{PTR. Let } \kappa \in H(O) \text{ be given by} (3) \) and let \( w, \bar{w}, m \) be three non-zero elements of \( S \). If \( (4) \) is true, then \( \kappa : (w, 0)^z \ni (\bar{w}, 0)^z \).

**Proof.** Suppose \( \kappa : (w, 0)^z \ni (w', 0)^z \) and put

\[ k = w\,(u \cdot m). \]

By the assumption \( \bar{w} \cdot k = \bar{u} \cdot m \), by Proposition 1 \( w' \cdot k = \bar{u} \cdot m \Rightarrow \bar{w} = w' \).

---

5) For any couple \( (a, b) \in S^2 \), \( b \neq 0 \) we shall denote by \( b\,a \) the solution of the equation \( b \cdot x = a \) and by \( a/b \) the solution of the equation \( x \cdot b = a \).
Proposition 3. Let $P(n)$ be a desarguesian plane and let $(S, t)$ be its arbitrary PTR. Then the following condition is fulfilled:

(E) For arbitrarily given $u, \bar{u}, w, \bar{w} \in S \setminus \{0\}$ the equation

\[(5) \quad w \mid (u . m) = \bar{w} \mid (\bar{u} . m)\]

either has only the trivial solution or it is fulfilled identically.

Proof. If $\bar{m}$ is a non trivial solution of (5), then the homothety $\kappa \in H(O)$ given by (3) maps $(u, 0)^*$ into $(\bar{w}, 0)^*$ (Proposition 2). It follows then from Proposition 1, that (5) is fulfilled identically.

Lemma. Let PTR $(S, t)$ satisfy the condition (A)–(D). Then for arbitrary $a, b, c \in S$, $c \neq 0$

\[(6) \quad a \cdot (c \mid (-b)) = -(a \cdot (c \mid b))\]

holds.

Proof. Put

\[k = c \mid (-b), \quad s = c \mid b\]

then $c \cdot k + c \cdot s = 0$. The condition (D) gives $a \cdot k + a \cdot s = 0$ which implies (6).

Consider an arbitrary PTR $(S, t)$. Let $u, \bar{u}, m$ be three non-zero elements of $S$.
Let $f : S \to S$, $g : S \to S$ be functions defined by

\[(7) \quad f(x) = (\bar{u} . m)((x \mid (u . m)), \text{ if } x \neq 0, \quad f(0) = 0\]

\[g(y) = \bar{u} \cdot (u \mid y).\]

Proposition 4. Let $P(n)$ be an affine plane and let $(S, t)$ be its arbitrary PTR satisfying the conditions (A)–(E). Then the mapping $\kappa : A \to A$ defined by

\[\kappa : (x, y)^* \mapsto (f(x), g(y))^*\]

is a homothety of $H(O)$ such that $\kappa : (u, 0)^* \mapsto (\bar{u}, 0)^*$.

Proof. $\kappa$ is obviously a non-singular transformation (permutation) of $A$ with the fixed point $O$ carrying $(u, 0)^*$ into $(\bar{u}, 0)^*$. Furthermore, $\kappa$ maps every vertical affine line\(^6\) again onto such a line. It remains to prove that for every non-vertical affine line $p$ its map $\tilde{p}$ is a affine line parallel with $p$.

Consider a point $(x, y)^* \neq O$. It follows from (7) that

\[(8) \quad x \mid (u . m) = f(x) \mid (\bar{u} . m).\]

\(^6\) i.e. the affine line with vertical direction.
According to the condition (E), we obtain from (8) that for any $\bar{m} \in S$

\[(9) \quad x'(u \cdot \bar{m}) = f(x)'(\bar{u} \cdot \bar{m}).\]

Choose

\[(10a) \quad \bar{m} = u'(-\bar{y})\]

and put

\[(10b) \quad b = x'(u \cdot \bar{m})\]

(10a) and (10b) imply

\[(11) \quad y = -(x \cdot b).\]

Using Lemma, (9), (10a), (10b) and the definition of the function $g$ we get

\[(12) \quad g(y) = -(f(x) \cdot b).\]

Let $p = (r, q)^x$ be an arbitrary non-vertical affine line. We shall prove that $\kappa$ maps $p$ onto $\bar{p} = (r, g(q))^x$.

A. First suppose that $q = 0$ and $(x, y)^x \in p$. Then $g(q) = 0$ and $y = x \cdot r$. We obtain from this and (11)

\[(13) \quad x \cdot b + x \cdot r = 0 ,\]

the condition (D) gives

\[(14) \quad f(x) \cdot b + f(x) \cdot r = 0 .\]

Finally, (12) and (14) imply $g(y) = f(x) \cdot r \Rightarrow (f(x), g(y))^x \in p$.

Conversely, if $(f(x), g(y))^x \in p$, then $g(y) = f(x) \cdot r$. We may assume $(f(x), g(y))^x \neq 0$. Now, (12) implies (14) and according to the condition (D) we get (13). (13) and (11) give $y = x \cdot r \Rightarrow (x, y)^x \in p$.

B. Let $q$ be an arbitrary element of $S$ and let $(x, y)^x \in p$. If $x = 0$ then $f(x) = 0$, $y = q$ and $(f(x), g(y))^x = (0, g(q))^x$ lies on $\bar{p}$. Let $x \neq 0$. As $y = x \cdot r + q$ putting

\[c = x'(-q)\]

and using (11) we obtain

\[(15) \quad x \cdot r + x \cdot b = x \cdot c .\]

As the condition (D) is fulfilled, it also holds

\[(16) \quad f(x) \cdot r + f(x) \cdot b = f(x) \cdot c .\]
Furthermore, \(-q = x . c \Leftrightarrow (x, -q)^x \in (c, 0)^x\) and according to part A we get \((f(x), g(-q))^x \in (c, 0)^x\). As \(g(-q) = -g(q)\), we have \(-g(q) = f(x) . c\). Finally, (12) and (16) imply that \(g(y) = f(x) . r + g(q) \Rightarrow (f(x), g(y))^x \in \bar{p}\).

Conversely, let \((f(x), g(y))^x \in \bar{p}\). If \(x = 0\), then \(f(x) = 0\) and \(g(y) = g(q) \Rightarrow y = -q \Rightarrow (x, y)^x = (0, q)^x \in p\). Assume \(x \neq 0 \Rightarrow f(x) \neq 0\). Putting
\[c = f(x) \backslash g(-q)\]
we have \(f(x) . c = g(-q) \Rightarrow (f(x), -g(q))^x \in (c, 0)^x\) and with respect to part A \((x, -q)^x \in (c, 0)^x\) which implies \(x . c = -q\). On the other hand, it is
\[f(x) . r + g(q) = g(y)\]

Then we obtain from (12) and from \(f(x) . c = -g(q)\) that (16) holds. As the condition (D) is fulfilled, it holds (15). Finally, using (11) and \(x . c = -q\) we have \(y = x . r + q \Rightarrow (x, y)^x \in p\).

Combining the above mentioned result of [6], Theorem 1 and Propositions 3,4 we obtain

**Theorem 2.** Let \(P(n)\) be an affine plane and let \((S, t)\) be its arbitrary PTR. Then \(P(n)\) is desarguesian if and only if \((S, t)\) satisfies the conditions (A)–(E).

**Remark.** Suppose that \((S, t)\) has a unity \(e\) and that \((S, t)\) satisfies the conditions (A)–(E). As
\[a . e + b . e = (a + b) . e\]
and
\[e . a + e . b = e . (a + b)\]
the validity of both distributive laws follows from (C) and (D). Let \(a, b, c\) be arbitrary elements of \(S, a \neq 0\). As
\[a \backslash ((a . b) . e) = e \backslash (b . e)\]
is true, then it also holds by (E)
\[a \backslash ((a . b) . c) = e \backslash (b . c)\].

The last relation gives \((a . b) . c = a . (b . c)\) i.e., the associative law for multiplication (which holds obviously also in the case \(a = 0\)).

**Proposition 5.** Let \(P(n)\) be a desarguesian plane and let \((S, t)\) be its arbitrary PTR. Then \(P(n)\) is pappian if and only if the following condition is fulfilled:

(F) For arbitrary \(a, b, c \in S, c \neq 0\) the identity
\[a . (c \backslash (b . c)) = b . (c \backslash (a . c))\]
holds.

612
Proof. (17) is obviously fulfilled for \( a = 0 \) or \( b = 0 \). Suppose \( a \neq 0 \), \( b \neq 0 \) and consider homotheties \( \alpha_1, \alpha_2 \in H(0) \) given by
\[
\alpha_1 : (c, 0)^e \mapsto (b, 0)^e, \quad \alpha_2 : (c, 0)^e \mapsto (a, 0)^e.
\]
Put \( z = (0, c . c)^e \) and denote by \( d_{21} \) the second coordinate of \((\alpha_2 \circ \alpha_1)(z)\) and by \( d_{12} \) the second coordinate of \((\alpha_1 \circ \alpha_2)(z)\). By Proposition 4 we get
\[
d_{21} = a \cdot (c \cdot (b \cdot c)), \quad d_{12} = b \cdot (c \cdot (a \cdot c)).
\]

\( P(n) \) is pappian if and only if for every \( a, b, c \in S \setminus \{0\}, \alpha_1, \alpha_2 \) commute \( \iff d_{21} = d_{12} \) (for every \( a, b, c \in S \setminus \{0\} \)) \( \iff (17) \) is true (again for every \( a, b, c \in S \setminus \{0\} \)).

Summarizing the results of theorem 2 and Proposition 5 we obtain

**Theorem 3.** Let \( P(n) \) be an affine plane and let \((S, t)\) be its arbitrary PTR. Then \( P(n) \) is pappian if and only if \((S, t)\) satisfies the conditions (A)–(F).

**Remark.** If PTR \((S, t)\) has a unity \( e \) and satisfies the conditions (A)–(F), then putting \( c = e \) in (17) we obtain \( a \cdot b = b \cdot a \) for any \( a, b \in S \), i.e., the commutative law for multiplication.

**References**


**Author’s address:** 771 46 Olomouc, Leninova 26, ČSSR (Palackého universita).