Dalibor Klucký

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TERNARY RINGS WITH ZERO ASSOCIATED TO DESARGUESIAN AND PAPPIAN PLANES

DALIBOR KLUCKÝ, Olomouc

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The article is an immediate continuation of [6]. Let us recall some necessary notions introduced there. Under coordinate system in an affine plane $\boldsymbol{P}(n)$ [1]) we understand any couple of bijections (π, λ)

$$\pi : \mathbf{S}^2 \to \mathbf{A}\,, \quad \lambda : \mathbf{S}^2 \to \mathbf{B}\,,$$

where $\mathbf{S}$ is a set whose cardinality equals the order of $\boldsymbol{P}(n)$, $\mathbf{A}$ is the set of proper points of $\boldsymbol{P}(n)$, $\mathbf{B}$ is the set of lines of $\boldsymbol{P}(n)$ which do not pass through the given direction (improper point) V. The direction V will be called vertical direction of the coordinate system (π, λ). If a ternary operation $\mathbf{t}$ on $\mathbf{S}$ is given so, that $(\mathbf{S}, \mathbf{t})$ is a planar ternary ring (abb. PTR) in sense of [5] or [6] [2]) we shall always require

$$(x, y)^\pi \in (a, b)^\lambda \Leftrightarrow y = t(x, a, b)\,.$$

In this case we say that PTR $(\mathbf{S}, \mathbf{t})$ coordinatizes the affine plane $\boldsymbol{P}(n)$ or that $(\mathbf{S}, \mathbf{t})$ is a PTR of $\boldsymbol{P}(n)$. We also say that $(\mathbf{S}, \mathbf{t})$ corresponds to the coordinate system (π, λ). An affine plane $\boldsymbol{P}(n)$ is said to be a translation plane, if the group of translations operates transitively on $\mathbf{A}$.

Let $(\mathbf{S}, \mathbf{t})$ be an arbitrary PTR of a given affine plane $\boldsymbol{P}(n)$. The following statement is proved in [6]:

$\boldsymbol{P}(n)$ is a translation plane if and only if the following conditions are fulfilled:

(A) $(\mathbf{S}, +)$ is a group (in this case, $(\mathbf{S}, +)$ is abelian).

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[1]) n is the improper line (the line in infinity) of $\boldsymbol{P}(n)$.

[2]) i.e. with a zero element 0, but not necessarily with the unit.

(B) $\forall a, b, c \in \mathbf{S}$,

$$\mathbf{t}(a, b, c) = a \cdot b + c\ ^{3)}$$

holds.

(C) For arbitrary $a, b, c \in \mathbf{S}$ the equation

$$a \cdot m + b \cdot m = c \cdot m$$

either has only the trivial solution or it is fulfilled identically (cf. [6] Corollary 4 of Theorem 2).

The purpose of this article is to formulate and prove analogous conditions for $\boldsymbol{P}(n)$ to be desarguesian or pappian plane.

For any proper point B of $\boldsymbol{P}(n)$ let us denote by $\boldsymbol{H}(B)$ the group of all (regular) homotheties with fixed point B.

Definition. An affine plane $\boldsymbol{P}(n)$ will be called a *desarguesian plane*, if for any proper point B and for any affine line $p \ni B$ the group $\boldsymbol{H}(B)$ operates transitively on $p \setminus \{B\}$. Moreover, if $\boldsymbol{H}(B)$ is an abelian group for each point B, then $\boldsymbol{P}(n)$ will be called a *pappian plane*.

The following facts are well known:

(a) *Every desarguesian plane (and, consequently, every pappian plane) is a translation plane.*

(b) *A translation plane is desaguesian if and only if there exists a proper point B and an affine line $p \ni B$ such that the group $\boldsymbol{H}(B)$ operates transitively on $p \setminus \{B\}$.*

(c) *A desarguesian plane is pappian if and only if there exists a proper point B such that the group $\boldsymbol{H}(B)$ is abelian.*

Theorem 1. *Let $\boldsymbol{P}(n)$ be a desarguesian plane and let $(\mathbf{S}, \mathbf{t})$ be its arbitrary* PTR. *Then the following condition is fulfilled:*

(D) *For arbitrary $a, b, c \in \mathbf{S}$ the equation*

$$m \cdot a + m \cdot b = m \cdot c \tag{1}$$

either has only the trivial solution or it is fulfilled identically.

Proof. Let $\overline{m}$ be a non – trivial solution of (1) and $m \in \mathbf{S} \setminus \{0\}$. We may assume that $a \neq 0$, $b \neq 0$. Consider the homothety $\varkappa \in \boldsymbol{H}(O)$ [4], $\varkappa : (\overline{m}, 0)^\pi \mapsto (m, 0)^\pi$. Let $\bar{p}$, p be two parallel lines with the same first coordinate a such that $\bar{p} \ni (\overline{m}, 0)^\pi$,

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[3]) $+$ and $.$ are binary operations on $\mathbf{S}$ defined by $a \cdot b = \mathbf{t}(a, b, 0)$; $a + b = \mathbf{t}(a, e_a, b)$, where $e_0 = 0$ and for $a \neq 0$, e_a is the solution of the equation $a \cdot x = a$.

[4]) In the sequel we shall denote by O the point $(0, 0)^\pi$.

$p \ni (m, 0)^\pi$. Hence, it follows: $\bar{p} = (a, -(\bar{m} . a))^\lambda$, $p = (a, -(m . a))^\lambda$. Put $\bar{q} = (c, -(\bar{m} . a))^\lambda$, $q = (c, -(m . a))^\lambda$. Clearly $\varkappa(\bar{p}) = p \Rightarrow \varkappa : (0, -(\bar{m} . a))^\pi \mapsto (0, -(m . a))^\pi \Rightarrow \varkappa(\bar{q}) = q$. Let $\bar{Y}$ be the point lying on $\bar{q}$ with the first coordinate $\bar{m}$ and let Y be the point lying on q with the first coordinate m. It is obvious that $\varkappa : \bar{Y} \mapsto Y$, hence $\bar{Y}$, Y lie on the same line $(r, 0)^\lambda$ passing through O. We have $\{\bar{Y}\} = \bar{q} \cap (r, 0)^\lambda$, $\{Y\} = q \cap (r, 0)^\lambda$ which implies

$$\bar{m} . r = \bar{m} . c - (\bar{m} . a), \quad m . r = m . c - (m . a). \tag{2}$$

By the assumption $\bar{m} . b = \bar{m} . c - (\bar{m} . a) \Rightarrow r = b$ and the second relation of (2) yields (1).

Proposition 1. *Let $P(n)$ be a desarguesian plane and let $(\mathbf{S}, \mathbf{t})$ be its arbitrary* PTR. *Let $\varkappa \in H(O)$ be given by*

$$\varkappa : (u, 0)^\pi \mapsto (\bar{u}, 0)^\pi, \quad u, \bar{u} \in \mathbf{S} \setminus \{0\}. \tag{3}$$

If $\varkappa : (w, 0)^\pi \to (\bar{w}, 0)^\pi$, $w, \bar{w} \in \mathbf{S} \setminus \{0\}$, then for any $m \in \mathbf{S}$

$$w\backslash(u . m) = \bar{w}\backslash(\bar{u} . m) \tag{4}$$ [5])

holds.

Proof. We may assume $m \neq 0$. Putting $k = w\backslash(u . m)$ we get $w . k = u . m$ and we have to prove

$$\bar{w} . k = \bar{u} . m. \tag{4a}$$

Let p, $\bar{p}$ be two parallel lines with the same first coordinate m such that $p \ni (u, 0)^\pi$, $\bar{p} \ni (\bar{u}, 0)^\pi \Rightarrow p = (m, -(u . m))^\lambda$, $\bar{p} = (m, -(\bar{u} . m))^\lambda$. Put $q = (k, -(u . m))^\lambda$, $\bar{q} = (k, -(\bar{u} . m))^\lambda$. Clearly $\varkappa(p) = \bar{p} \Rightarrow \varkappa : (0, -(u . m))^\pi \mapsto (0, (\bar{u} . m))^\pi \Rightarrow \varkappa(q) = \bar{q}$. Using $w . k = u . m$ we obtain $(w, 0)^\pi \in q$, hence $(\bar{w} . 0)^\pi \in \bar{q} \Rightarrow$ (4a).

Proposition 2. *Let $P(n)$ be a desarguesian plane and let $(\mathbf{S}, \mathbf{t})$ be its arbitrary* PTR. *Let $\varkappa \in H(O)$ be given by (3) and let w, $\bar{w}$, m be three non-zero elements of $\mathbf{S}$. If (4) is true, then $\varkappa : (w, 0)^\pi \mapsto (\bar{w}, 0)^\pi$.*

Proof. Suppose $\varkappa : (w, 0)^\pi \mapsto (w', 0)^\pi$ and put

$$k = w\backslash(u . m).$$

By the assumption $\bar{w} . k = \bar{u} . m$, by Proposition 1 $w' . k = \bar{u} . m \Rightarrow \bar{w} = w'$.

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[5]) For any couple $(a, b) \in \mathbf{S}^2$, $b \neq 0$ we shall denote by $b\backslash a$ the solution of the equation $b . x = a$ and by a/b the solution of the equation $x . b = a$.

Proposition 3. *Let $P(n)$ be a desarguesian plane and let $(\mathbf{S}, \mathbf{t})$ be its arbitrary* PTR. *Then the following condition is fulfilled*:

(E) *For arbitrarily given $u, \bar{u}, w, \bar{w} \in \mathbf{S} \setminus \{0\}$ the equation*

$$w\backslash(u \,.\, m) = \bar{w}\backslash(\bar{u} \,.\, m) \tag{5}$$

either has only the trivial solution or it is fulfilled identically.

Proof. If $\bar{m}$ is a non trivial solution of (5), then the homothety $\varkappa \in H(O)$ given by (3) maps $(u, 0)^\pi$ into $(\bar{w}, 0)^\pi$ (Proposition 2). It follows then from Proposition 1, that (5) is fulfilled identically.

Lemma. *Let* PTR $(\mathbf{S}, \mathbf{t})$ *satisfy the condition* (A)–(D). *Then for arbitrary $a, b, c \in \mathbf{S}$, $c \neq 0$*

$$a \,.\, (c\backslash(-b)) = -(a \,.\, (c\backslash b)) \tag{6}$$

holds.

Proof. Put

$$k = c\backslash(-b)\,, \quad s = c\backslash b$$

then $c \,.\, k + c \,.\, s = 0$. The condition (D) gives $a \,.\, k + a \,.\, s = 0$ which implies (6).

Consider an arbitrary PTR $(\mathbf{S}, \mathbf{t})$. Let $u, \bar{u}, m$ be three non-zero elements of $\mathbf{S}$. Let $f : \mathbf{S} \to \mathbf{S}$, $g : \mathbf{S} \to \mathbf{S}$ be functions defined by

$$\begin{aligned} f(x) &= (\bar{u} \,.\, m)/(x\backslash(u \,.\, m))\,, \quad \text{if} \quad x \neq 0\,, \quad f(0) = 0 \\ g(y) &= \bar{u} \,.\, (u\backslash y)\,. \end{aligned} \tag{7}$$

Proposition 4. *Let $P(n)$ be an affine plane and let $(\mathbf{S}, \mathbf{t})$ be its arbitrary* PTR *satisfying the conditions* (A)–(E). *Then the mapping $\varkappa : \mathbf{A} \to \mathbf{A}$ defined by*

$$\varkappa : (x, y)^\pi \mapsto (f(x), g(y))^\pi$$

is a homothety of $H(O)$ such that $\varkappa : (u, 0)^\pi \mapsto (\bar{u}, 0)^\pi$.

Proof. $\varkappa$ is obviously a non-singular transformation (permutation) of $\mathbf{A}$ with the fixed point O carrying $(u, 0)^\pi$ into $(\bar{u}, 0)^\pi$. Furthermore, $\varkappa$ maps every vertical affine line[6]) again onto such a line. It remains to prove that for every non-vertical affine line p its map $\bar{p}$ is a affine line parallel with p.

Consider a point $(x, y)^\pi \neq O$. It follows from (7) that

$$x\backslash(u \,.\, m) = f(x)\backslash(\bar{u} \,.\, m)\,. \tag{8}$$

[6]) i.e. the affine line with vertical direction.

According to the condition (E), we obtain from (8) that for any $\bar{m} \in \mathbf{S}$

(9) $$x \backslash (u \,.\, \bar{m}) = f(x) \backslash (\bar{u} \,.\, \bar{m}) \,.$$

Choose

(10a) $$\bar{m} = u \backslash (-y)$$

and put

(10b) $$b = x \backslash (u \,.\, \bar{m})$$

(10a) and (10b) imply

(11) $$y = -(x \,.\, b) \,.$$

Using Lemma, (9), (10a), (10b) and the definition of the function g we get

(12) $$g(y) = -(f(x) \,.\, b) \,.$$

Let $p = (r, q)^{\lambda}$ be an arbitrary non-vertical affine line. We shall prove that $\varkappa$ maps p onto $\bar{p} = (r, g(q))^{\lambda}$.

A. First suppose that $q = 0$ and $(x, y)^{\pi} \in p$. Then $g(q) = 0$ and $y = x \,.\, r$. We obtain from this and (11)

(13) $$x \,.\, b + x \,.\, r = 0 \,,$$

the condition (D) gives

(14) $$f(x) \,.\, b + f(x) \,.\, r = 0 \,.$$

Finally, (12) and (14) imply $g(y) = f(x) \,.\, r \Rightarrow (f(x), g(y))^{\pi} \in p$.

Conversely, if $(f(x), g(y))^{\pi} \in p$, then $g(y) = f(x) \,.\, r$. We may assume $(f(x), g(y))^{\pi} \neq O$. Now, (12) implies (14) and according to the condition (D) we get (13). (13) and (11) give $y = x \,.\, r \Rightarrow (x, y)^{\pi} \in p$.

B. Let q be an arbitrary element of $\mathbf{S}$ and let $(x, y)^{\pi} \in p$. If $x = 0$ then $f(x) = 0$, $y = q$ and $(f(x), g(y))^{\pi} = (0, g(q))^{\pi}$ lies on $\bar{p}$. Let $x \neq 0$. As $y = x \,.\, r + q$ putting

$$c = x \backslash (-q)$$

and using (11) we obtain

(15) $$x \,.\, r + x \,.\, b = x \,.\, c \,.$$

As the condition (D) is fulfilled, it also holds

(16) $$f(x) \,.\, r + f(x) \,.\, b = f(x) \,.\, c \,.$$

Furthermore, $-q = x \, . \, c \Rightarrow (x, -q)^{\pi} \in (c, 0)^{\lambda}$ and according to part A we get $(f(x), g(-q))^{\pi} \in (c, 0)^{\lambda}$. As $g(-q) = -g(q)$, we have $-g(q) = f(x) \, . \, c$. Finally, (12) and (16) imply that $g(y) = f(x) \, . \, r + g(q) \Rightarrow (f(x), g(y))^{\pi} \in \bar{p}$.

Conversely, let $(f(x), g(y))^{\pi} \in \bar{p}$. If $x = 0$, then $f(x) = 0$ and $g(y) = g(q) \Rightarrow y = q \Rightarrow (x, y)^{\pi} = (0, q)^{\pi} \in p$. Assume $x \neq 0 \Rightarrow f(x) \neq 0$. Putting

$$c = f(x) \backslash g(-q)$$

we have $f(x) \, . \, c = g(-q) \Rightarrow (f(x), -g(q))^{\pi} \in (c, 0)^{\lambda}$ and with respect to part A $(x, -q)^{\pi} \in (c, 0)^{\lambda}$ which implies $x \, . \, c = -q$. On the other hand, it is

$$f(x) \, . \, r + g(q) = g(y) \, .$$

Then we obtain from (12) and from $f(x) \, . \, c = -g(q)$ that (16) holds. As the condition (D) is fulfilled, it holds (15). Finally, using (11) and $x \, . \, c = -q$ we have $y = x \, . \, r + q \Rightarrow (x, y)^{\pi} \in p$.

Combining the above mentioned result of [6], Theorem 1 and Propositions 3,4 we obtain

Theorem 2. *Let* $\boldsymbol{P}(n)$ *be an affine plane and let* $(\mathbf{S}, \mathbf{t})$ *be its arbitrary* PTR. *Then* $\boldsymbol{P}(n)$ *is desarguesian if and only if* $(\mathbf{S}, \mathbf{t})$ *satisfies the conditions* (A)–(E).

Remark. Suppose that $(\mathbf{S}, \mathbf{t})$ has a unity e and that $(\mathbf{S}, \mathbf{t})$ satisfies the conditions (A)–(E). As

$$a \, . \, e + b \, . \, e = (a + b) \, . \, e$$

and

$$e \, . \, a + e \, . \, b = e \, . \, (a + b)$$

the validity of both distributive laws follows from (C) and (D). Let a, b, c be arbitrary elements of S, $a \neq 0$. As

$$a \backslash ((a \, . \, b) \, . \, e) = e \backslash (b \, . \, e)$$

is true, then it also holds by (E)

$$a \backslash ((a \, . \, b) \, . \, c) = e \backslash (b \, . \, c) \, .$$

The last relation gives $(a \, . \, b) \, . \, c = a \, . \, (b \, . \, c)$ i.e., the associative law for multiplication (which holds obviously also in the case $a = 0$).

Proposition 5. *Let* $\boldsymbol{P}(n)$ *be a desarguesian plane and let* $(\mathbf{S}, \mathbf{t})$ *be its arbitrary* PTR. *Then* $\boldsymbol{P}(n)$ *is pappian if and only if the following condition is fulfilled*:

(F) *For arbitrary* $a, b, c \in \mathbf{S}$, $c \neq 0$ *the identity*

(17) $$a \, . \, (c \backslash (b \, . \, c)) = b \, . \, (c \backslash (a \, . \, c))$$

holds.

Proof. (17) is obviously fulfilled for $a = 0$ or $b = 0$. Suppose $a \neq 0$, $b \neq 0$ and consider homotheties $\varkappa_1, \varkappa_2 \in \boldsymbol{H}(O)$ given by

$$\varkappa_1 : (c, 0)^\pi \mapsto (b, 0)^\pi , \quad \varkappa_2 : (c, 0)^\pi \mapsto (a, 0)^\pi .$$

Put $z = (0, c \,.\, c)^\pi$ and denote by d_{21} the second coordinate of $(\varkappa_2 \,.\, \varkappa_1)(z)$ and by d_{12} the second coordinate of $(\varkappa_1 \,.\, \varkappa_2)(z)$. By Proposition 4 we get

$$d_{21} = a \,.\, (c \backslash (b \,.\, c)) , \quad d_{12} = b \,.\, (c \backslash (a \,.\, c)) .$$

$\boldsymbol{P}(n)$ is pappian if and only if for every $a, b, c \in \mathbf{S} \setminus \{0\}$, $\varkappa_1, \varkappa_2$ commute $\Leftrightarrow d_{21} = d_{12}$ (for every $a, b, c \in \mathbf{S} \setminus \{0\}$.) $\Leftrightarrow$ (17) is true (again for every $a, b, c \in \mathbf{S} \setminus \{0\}$).

Summarizing the results of theorem 2 and Proposition 5 we obtain

Theorem 3. *Let $\boldsymbol{P}(n)$ be an affine plane and let $(\mathbf{S}, \mathbf{t})$ be its arbitrary* PTR. *Then $\boldsymbol{P}(n)$ is pappian if and only if $(\mathbf{S}, \mathbf{t})$ satisfies the conditions* (A)–(F).

Remark. If PTR $(\mathbf{S}, \mathbf{t})$ has a unity e and satisfies the conditions (A)–(F), then putting $c = e$ in (17) we obtain $a \,.\, b = b \,.\, a$ for any $a, b \in \mathbf{S}$, i.e., the commutative law for multiplication.

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Author's address: 771 46 Olomouc, Leninova 26, ČSSR (Palackého universita).