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A GENERAL COORDINATIZATION PRINCIPLE FOR PROJECTIVE PLANES WITH COMPARISON OF HALL AND HUGHES FRAMES AND WITH EXAMPLES OF GENERALIZED OVAL FRAMES

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In this Note we shall suggest a general way of coordinate labeling of points and lines of a given projective plane by couples of elements of some set S. We shall denote such a labeling rule a frame. Every frame determines a "coordinative" consisting of one quaternary relation on the set S and of two equivalence relations on the set S^2. Conversely, every coordinative determines, up to isomorphisms, just one projective plane. Further we shall investigate special frames as a halfcartesian frame, a cartesian frame, a Hall frame and a Hughes frame. These special frames give rise to planar ternary rings. An intimate connection between Hall and Hughes planar ternary rings (corresponding to the cases of Hall and Hughes frames) is described and explained. Finally generalized oval frames are studied. For Hall's approach, cf. [3]–[4] and for the Hughes' one, see [5]–[6]. Planar ternary rings with zero are for the first time investigated in [10]. For general planar ternary rings see [8]–[9]. The references of articles on oval coordinates are not given except the stimulating paper [1] of R. Artzy.

Let $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ be a projective plane of order n, $(P_\infty, l_\infty) \in \mathrm{I}$ a fixed flag, S a set of cardinality n and $\pi : S^2 \to \mathscr{P} \setminus \tilde{l}_\infty$, $\lambda : S^2 \to \mathscr{L} \setminus \tilde{P}_\infty$ bijections.*) For every $(u, v) \in S^2$, u is called the *slope* and v the *intercept* of the line $(u, v)^\lambda$. Then a quaternary relation Q in S such that $(x, y, u, v) \in Q :\Leftrightarrow (x, y)^\pi \, \mathrm{I}(u, v)^\lambda$ and equivalence relations l, p on S^2 such that

$$(x_1, y_1)\, l\, (x_2, y_2) :\Leftrightarrow \exists l \in \tilde{P}_\infty \setminus \{l_\infty\} \quad (x_1, y_1)^\pi, (x_2, y_2)^\pi \, \mathrm{I} l\,,$$
$$(u_1, v_1)\, p\, (u_2, v_2) :\Leftrightarrow \exists P \in \tilde{l}_\infty \setminus \{P_\infty\} \quad P\mathrm{I}(u_1, v_1)^\lambda, (u_2, v_2)^\lambda$$

*) If $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ is a projective plane then denote by $A \sqcup B$ $(a \sqcap b)$ the line joining two distinct points A, B (the point at which two distinct lines a, b intersect). Further let $\tilde{A}$ $(\tilde{a})$ denote the set $\{l \in \mathscr{L} \mid A \mathrm{I} l\}$ $(\{P \in \mathscr{P} \mid P \mathrm{I} a\})$ for every $A \in \mathscr{P}$ $(a \in \mathscr{L})$. It is well known that any set S the cardinality of which is equal to the order of the given plane satisfies $\# S^2 = \# (\mathscr{P} \setminus \tilde{l}_\infty) = \# (\mathscr{L} \setminus \tilde{P}_\infty)$.

are well-determined. It can be readily shown that

(i) $\forall (c, d) \in S^2,\ \mathfrak{l} \in S^2/\boldsymbol{l} \quad \exists!\,(a, b) \in S^2 \quad (a, b, c, d) \in \boldsymbol{Q},\ (a, b) \in \mathfrak{l}$,

(ii) $\forall (a, b) \in S^2,\ \mathfrak{p} \in S^2/\boldsymbol{p} \quad \exists!\,(c, d) \in S^2 \quad (a, b, c, d) \in \boldsymbol{Q},\ (c, d) \in \mathfrak{p}$,

(iii) $\forall (a_1, b_1), (a_2, b_2) \in S^2;\ (a_1, b_1) \operatorname{non} \boldsymbol{l}(a_2, b_2) \quad \exists!\,(c, d) \in S^2$
$(a_1, b_1, c, d), (a_2, b_2, c, d) \in \boldsymbol{Q}$,

(iv) $\forall (c_1, d_1), (c_2, d_2) \in S^2;\ (c_1, d_1) \operatorname{non} \boldsymbol{p}(c_2, d_2) \quad \exists!\,(a, b) \in S^2$
$(a, b, c_1, d_1), (a, b, c_2, d_2) \in \boldsymbol{Q}$.

A quadruple $(P_\infty, l_\infty, \pi, \lambda)$ will be called a *frame* and the quadruple $(S, \boldsymbol{Q}, \boldsymbol{p}, \boldsymbol{l})$ is said to be *corresponding* to it.

Now define a *coordinative* as a quadruple $(S, \boldsymbol{Q}, \boldsymbol{p}, \boldsymbol{l})$, where S is a set with $\# S \geqq 2$, $\boldsymbol{Q}$ is a quaternary relation in S and $\boldsymbol{p}, \boldsymbol{l}$ are equivalence relations on S^2 satisfying (i) to (iv).

To every coordinative $(S, \boldsymbol{Q}, \boldsymbol{p}, \boldsymbol{l})$ there exists a projective plane $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ together with its frame $\mathscr{F} = (P_\infty, l_\infty, \pi, \lambda)$ such that $(S, \boldsymbol{Q}, \boldsymbol{p}, \boldsymbol{l})$ corresponds to $\mathscr{F}$.

Proof. First define $\mathscr{P} := S^2 \cap S^2/\boldsymbol{p} \cup \{\infty\}$, $\mathscr{L} := S^2 \cup S^2/\boldsymbol{l} \cup \{\infty\}$ with disjoint summands. Secondly define $\mathrm{I} \subseteq \mathscr{P} \times \mathscr{L}$ by the rules

$$\begin{array}{lll}
(x, y)\,\mathrm{I}\,(u, v) :\Leftrightarrow (x, y, u, v) \in \boldsymbol{Q} & \forall x, y, u, v \in S\,, \\
(x, y)\,\mathrm{I}\,\mathfrak{l} \quad :\Leftrightarrow (x, y) \in \mathfrak{l} & \forall \mathfrak{l} \in S^2/\boldsymbol{l}\,, \\
\mathfrak{p}\,\mathrm{I}\,\infty & \forall \mathfrak{p} \in S^2/\boldsymbol{p}\,, \\
\infty\,\mathrm{I}\,\mathfrak{l} & \forall \mathfrak{l} \in S^2/\boldsymbol{l}\,, \\
\mathfrak{p}\,\mathrm{I}\,(u, v) :\Leftrightarrow (u, v) \in \mathfrak{p} & \forall \mathfrak{p} \in S^2/\boldsymbol{p}\,, \\
\infty\,\mathrm{I}\,\infty\,. &
\end{array}$$

It can be easily verified that $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ is a projective plane. If we choose $\pi : S^2 \to \to \mathscr{P} \setminus \widetilde{\infty}$, $\lambda : S^2 \to \mathscr{L} \setminus \widetilde{\infty}$ to be equal to identity mapping id_{S^2} then $(\infty, \infty, \mathrm{id}_{S^2}, \mathrm{id}_{S^2})$ is a frame of $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ and the corresponding coordinative is equal to the original one. ∎

We shall call the projective plane from the proof a plane *over* a given coordinative.

Now we shall introduce some important kinds of frames for a given projective plane $(\mathscr{P}, \mathscr{L}, \mathrm{I})$. A frame $(P_\infty, l_\infty, \pi, \lambda)$ is said to be *halfcartesian* if in the corresponding coordinative $(S, \boldsymbol{Q}, \boldsymbol{p}, \boldsymbol{l})$, $\boldsymbol{p}$ and $\boldsymbol{l}$ satisfy

($\boldsymbol{l}$) $\quad (x_1, y_1)\, \boldsymbol{l}\, (x_2, y_2) :\Leftrightarrow x_1 = x_2$,

($\boldsymbol{p}$) $\quad (u_1, v_1)\, \boldsymbol{p}\, (u_2, v_2) :\Leftrightarrow u_1 = u_2$.

Then conditions (i)–(iv) assume the following form:

(i′) $\forall a, c, d \in S \quad \exists!\, b \in S \quad (a, b, c, d) \in \boldsymbol{Q}$,

(ii′) $\forall a, b, c \in S \quad \exists!\, d \in S \quad (a, b, c, d) \in \boldsymbol{Q}$,

(iii′) $\forall (a_1, b_1), (a_2, b_2) \in S^2;\ a_1 \neq a_2 \quad \exists!\, (c, d) \in S^2$
$(a_1, b_1, c_1, d_1), (a_2. b_2, c_2, d_2) \in \boldsymbol{Q}$,

(iv′) $\forall (c_1, d_1), (c_2, d_2) \in S^2;\ c_1 \neq c_2 \quad \exists!\, (a, b) \in S^2$
$(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in \boldsymbol{Q}$.

A set S, $\# S \geqq 2$, together with a quaternary relation $\boldsymbol{Q}$ in S satisfying (i′) –(iv′) will be called a *planar relation system*. Any given planar relation system $(S, \boldsymbol{Q})$ determines automatically both the equivalence relations $\boldsymbol{l}, \boldsymbol{p}$ on S^2 satisfying $(\boldsymbol{l}), (\boldsymbol{p})$ so that we can consider $(S, \boldsymbol{Q})$ itself a coordinative.

For every planar relation system $(S, \boldsymbol{Q})$ construct a projective plane isomorphic to the projective plane over it substituting every $\mathfrak{l} \in S^2/\boldsymbol{l}$ by the element $x \in S$ such that $(x, y) \in \mathfrak{l}\ \forall y \in S$ and similarly substituting every $\mathfrak{p} \in S^2/\boldsymbol{p}$ by the element $u \in S$ such that $(u, v) \in \mathfrak{p}\ \forall v \in S$. We get the projective plane $(S^2 \cup S \cup \{\infty\}, S^2 \cup S \cup \{\infty\}, \mathrm{I})$ where

$$(x, y)\, \mathrm{I}\, (u, v) :\Leftrightarrow (x, y, u, v) \in \boldsymbol{Q}\,, \quad (x, y)\, \mathrm{I}\, x \quad \forall x, y \in S\,,$$
$$u\, \mathrm{I}\, (u, v) \quad \forall u, v \in S\,, \quad u\, \mathrm{I}\, \infty \quad \forall u \in S\,, \quad \infty\, \mathrm{I}\, x \quad \forall x \in S\,, \quad \infty\, \mathrm{I}\, \infty\,.$$

For simplicity we shall say again that this plane is *over* $(S, \boldsymbol{Q})$; it will be always clear from the context which of both cases is meant.

If $(P_\infty, l_\infty, \pi, \lambda)$ is a halfcartesian frame and $(S, \boldsymbol{Q})$ its corresponding relation system, then define ternary operations T, T^* over S by virtue of

$$(a, T(a, c, d), c, d) \in \boldsymbol{Q} \quad \forall a, c, d \in S\,; \quad (a, b, c, T^*(c, a, b)) \in \boldsymbol{Q} \quad \forall a, b, c \in S\,.$$

Conditions (i′)–(iv′) are then

($\mathrm{i}_{\mathrm{Hall}}$) T is well defined;

($\mathrm{ii}_{\mathrm{Hall}}$) $\forall a, b, c \in S \quad \exists!\, d \in S \quad T(a, c, d) = b$;

($\mathrm{iii}_{\mathrm{Hall}}$) $\forall a_1, b_1, a_2, b_2 \in S;\ a_1 \neq a_2 \quad \exists!\, c, d \in S\ T(a_1, c, d) = b_1,\ T(a_2, c, d) = b_2$;

($\mathrm{iv}_{\mathrm{Hall}}$) $\forall c_1, d_1, c_2, d_2 \in S;\ c_1 \neq c_2 \quad \exists!\, a \in S \quad T(a, c_1, d_1) = T(a, c_2, d_2)$;

or, respectively,

($\mathrm{i}_{\mathbf{Hughes}}$) $\forall a, c, d \in S \quad \exists!\, b \in S \quad T^*(c, a, b) = d$,

($\mathrm{ii}_{\mathbf{Hughes}}$) T^* is well defined;

($\mathrm{iii}_{\mathbf{Hughes}}$) $\forall a_1, b_1, a_2, b_2 \in S;\ a_1 \neq a_2 \quad \exists!\, c \in S\ T^*(c, a_1, b_1) = T^*(c, a_2, b_2)$;

($\mathrm{iv}_{\mathbf{Hughes}}$) $\forall c_1, d_1, c_2, d_2 \in S;\ c_1 \neq c_2 \quad \exists!\, (a, b) \in S^2 \quad T^*(c_1, a, b) = d_1$,
$T^*(c_2, a, b) = d_2$.

A ternary system (S, T) will be called *planar ternary ring* if $\# S \geqq 2$ and if $(\mathrm{i}_{\text{Hall}})-(\mathrm{iv}_{\text{Hall}})$ hold. The above planar ternary rings (S, T), (S, T^*) are said to be *Hall-corresponding* or *Hughes-corresponding* to the given frame, respectively. We see at once that they are isomorphic because $(\mathrm{ii}_{\text{Hall}})$ coincides with $(\mathrm{i}_{\text{Hughes}})$, $(\mathrm{iii}_{\text{Hall}})$ with $(\mathrm{iv}_{\text{Hughes}})$ and $(\mathrm{iv}_{\text{Hall}})$ with $(\mathrm{iii}_{\text{Hughes}})$ if we replace T by T^*.

A *cartesian frame* (*with zero*) is defined as such a half-cartesian frame $(P_\infty, l_\infty, \pi, \lambda)$ for which there is an element $0 \in S$ such that

$$(\mathrm{v}') \qquad (0, b, c, b), (a, b, 0, b) \in \boldsymbol{Q} \quad \forall a, b, c \in S .$$

In virtue of (v') it follows

$$\forall a, b, d \in S;\ a \neq 0 \quad \exists!\, c \in S \quad (a, b, c, d) \in \boldsymbol{Q},$$

$$\forall b, c, d \in S;\ c \neq 0 \quad \exists!\, a \in S \quad (a, b, c, d) \in \boldsymbol{Q}.$$

Thus we can introduce mappings $T_1 : S \times (S \setminus \{0\}) \times S \to S$, $T_3 : (S \setminus \{0\}) \times S \times S \to S$ by virtue of $(T_1(b, c, d), b, c, d) \in \boldsymbol{Q}$ $\forall b, c, d \in S;\ c \neq 0$; $(a, b, T_3(c, a, b), d) \in \boldsymbol{Q}$ $\forall a, b, c \in S;\ a \neq 0$ and denote the preceding ternary operations T, T^* in this context also by T_1, T_4.

Condition (v') can be re-written as

$$(\mathrm{v}_{\text{Hall}}) \quad T(0, c, b) = b,\ T(a, 0, b) = b \quad \forall a, b, c \in S,$$

or

$$(\mathrm{v}_{\text{Hughes}}) \quad T^*(a, 0, b) = b,\ T^*(0, c, b) = b \quad \forall a, b, c \in S.$$

Thus $(\mathrm{v}_{\text{Hall}})$ coincides with $(\mathrm{v}_{\text{Hughes}})$ if we replace T by T^*. The planar ternary ring (S, T) possessing an element $0 \in S$ with $(\mathrm{v}_{\text{Hall}})$ is said to be *with zero*. A frame $(P_\infty, l_\infty, \pi, \lambda)$ is called *Hall frame* if it is cartesian (with zero 0) and if there exists an element $1 \in S \setminus \{0\}$ such that:

(a) $(x, x)^\lambda \mathrm{I} (0, 0)^\pi \sqcup (1, 1)^\pi$ $\forall x \in S$ (Fig. 1),

(b) the slope u of any line $l \in \mathscr{L} \setminus \tilde{P}_\infty$ is given by $(1, u)^\pi = ((1, 1)^\pi \sqcup P_\infty) \sqcap \sqcap ((0, 0)^\pi \sqcup (l \sqcap l_\infty))$, whereas its intercept v is determined by $(0, v)^\pi \mathrm{I}\ l$ (Fig. 2).

What additional algebraic condition corresponds to the case considered?

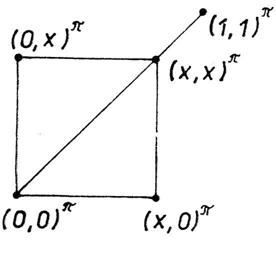

Fig. 1.

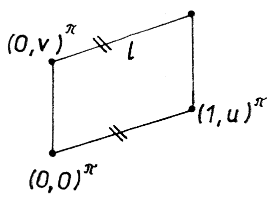

Fig. 2.

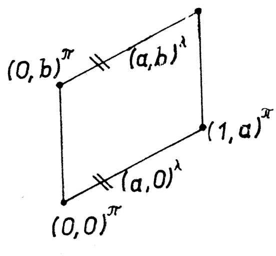

Fig. 3.

It is easily seen that this will be the condition

$(\mathrm{vi}_{\mathrm{Hall}})$ $T(a, 1, 0) = T(1, a, 0) = a \quad \forall a \in S.$

Conversely, if we start from a cartesian frame $(P_\infty, l_\infty, \pi, \lambda)$ and suppose the validity of $(\mathrm{vi}_{\mathrm{Hall}})$ for the Hall-corresponding planar ternary ring (S, T) then for every $a \in S$ we have $(a, a)^\pi \mathrm{I} (1, 0)^\lambda$, $(1, a)^\pi \mathrm{I} (a, 0)^\lambda$. This together with $(\mathrm{v}_{\mathrm{Hall}})$ implies $(0, 0)^\pi$, $(1, a)^\pi \mathrm{I} (a, 0)^\lambda$ and $(0,b)^\pi \mathrm{I} (a, b)^\lambda$. Here the lines $(a, 0)^\lambda$, $(a, b)^\lambda$ are distinct and without intersection point in $\mathscr{P} \setminus \bar{l}_\infty$ (Fig. 3).

The planar ternary ring (S, T) is said to be *with zero and unity* or more briefly a *Hall ring* if there exist elements $0, 1 \in S$; $0 \neq 1$ such that $(\mathrm{v}_{\mathrm{Hall}})$ and (vi) are valid.

Condition equivalent to $(\mathrm{vi}_{\mathrm{Hall}})$ expressed in terms of T^* is

$(\mathrm{vi}_{\mathrm{Hughes}})$ $T^*(1, a, 0) = T^*(a, 1, 0) = a \quad \forall a \in S.$

Let (S, T) be a Hall ring. Then it is usual to introduce two induced binary operations $+_T$, $\cdot_T$ over S by virtue of

$$x +_T v := T(x, 1, v) \quad \forall x, v \in S\,,$$

$$x \cdot_T u := T(x, u, 0) \quad \forall x, u \in S\,.$$

These operations play an important role in the study of Hall rings and connect them with the theory of loops (geometrically: this will be a connection between projective planes and nets).

To describe the definition of $+_T$ and $\cdot_T$ geometrically let us introduce the following notation:

$(0, 0)^\pi =: 0$	(origin)
$(0, 0)^\lambda \sqcap l_\infty =: X$	(the improper point of x-axis),
$P_\infty =: Y$	(the improper point of y-axis),
$(1, 1)^\pi =: U$	(unity point),
$(1, 0)^\lambda \sqcap l_\infty =: Z$	(the improper diagonal point),
$((1, 0)^\pi \sqcup (0, 1)^\pi) \sqcap l_\infty =: Z^*$	(the improper antidiagonal point)
$(0, x)^\pi =: P_x \ \forall x \in S$	(the representing point of $x \in S$).

Then for all $x, v \in S$ we have the well-known expressions $P_{x +_T v} := (((((P_x \sqcup X) \sqcap \sqcap (0 \sqcup Z)) \sqcup X) \sqcap (P_v \sqcup Z)) \sqcup X) \sqcap (0 \sqcup Y)$ and for all $x, u \in S$ we have

$$P_{x \cdot_T u} := (((((P_x \sqcup X) \sqcap (0 \sqcup Z)) \sqcup Y) \sqcap$$

$$\sqcap (((P_u \sqcup X) \sqcap (U \sqcup Y)) \sqcup 0) \sqcup X) \sqcap (0 \sqcup Y)\,.$$

(Fig. 4–5).

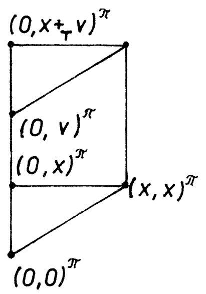

Fig. 4.

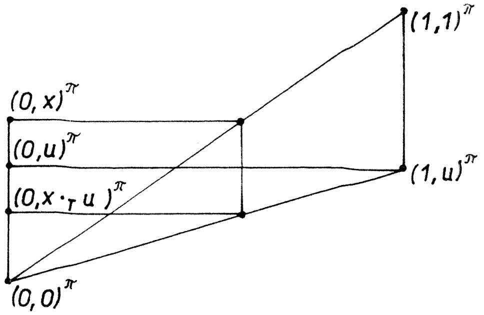

Fig. 5.

Pose a question: What does $(\mathrm{vi}_{\mathrm{Hughes}})$ mean geometrically for a given cartesian frame $(P_\infty, l_\infty, \pi, \lambda)$?

The answer is: For every $a \in S$ the equation $T^*(1, a, 0) = a$ means $(a, 0)^\pi \mathrm{I} (1, a)^\lambda$ and the equation $T^*(a, 1, 0) = a$ means $(1, 0)^\pi \mathrm{I} (a, a)^\lambda$. Thus the line $(1, 0)^\pi \sqcup \sqcup (0, 1)^\pi$ must have the slope 1 and for $a \neq 0$ also $(a, 0)^\pi \sqcup (0, a)^\pi$ has the slope 1 (because of $(a, 0)^\pi$, $(a, 0)^\pi \mathrm{I} (1, a)^\lambda$ since $(0, a)^\pi \mathrm{I} (1, a)^\lambda)$ and again for every $a \in S$ (without exception) we obtain similarly $(1, 0)^\pi \sqcup (0, a) = (a, a)^\lambda$.

Thus we have (Fig. 6–7):

(1) the line $(a, 0)^\pi \sqcup (0, a)^\pi$ has the slope 1 for all $a \in S \setminus \{0\}$,

(2) if the line $l \in \mathscr{L} \setminus \tilde{P}_\infty$ intersects the line $(0, 0)^\pi \sqcup P_\infty$ at the point $(0, b)^\pi$ then the intercept of l will be b,

(3) the line $(1, 0)^\pi \sqcup (0, a)^\pi$ has the slope a for all $a \in S$,

(4) any two distinct lines from $\mathscr{L} \setminus \tilde{P}_\infty$ which intersect at the point of $l_\infty \setminus \{P_\infty\}$ have the same slope.

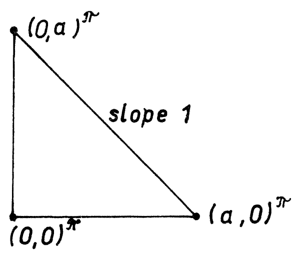

Fig. 6.

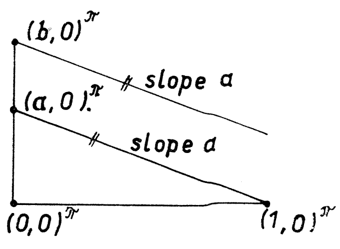

Fig. 7.

Conversely, if we realize bijections π, λ in a cartesian frame $(P_\infty, l_\infty, \pi, \lambda)$ with zero 0 so that (1)–(4) are fulfilled for an element $1 \in S \setminus \{0\}$ then $(\mathrm{vi}_{\mathrm{Hughes}})$ holds. We speak then of *Hughes frame*.

The geometric sense of operations $+_{T^*}$, $\cdot_{T^*}$, where (S, T^*) is the Hughes-corresponding planar ternary ring to a Hughes frame, is the following: Write $u +_{T^*} y := T^*(u, 1, y)$ $\forall u, y \in S$; $u \cdot_{T^*} x := T^*(u, x, 0)$ $\forall u, x \in S$ and denote $P_a := (0, a)^\pi$ for every $a \in S$. Then we have at once $P_{u+_{T^*}y} := ((P_y \sqcup x) \sqcap (U \sqcup Y)) \sqcup (l_\infty \sqcap (P_u \sqcup ((U \sqcup Y) \sqcap (0 \sqcup X)))) \sqcap (0 \sqcup Y)$ $\forall u, y \in S$ (Fig. 8–9) as well as $P_{u \cdot_{T^*} x} := (((P_x \sqcup Z^*) \sqcap (0 \sqcup X)) \sqcup ((((U \sqcup X) \sqcup P_u) \sqcap l_\infty)) \sqcap (0 \sqcup Y)$ $\forall u, x \in S$.

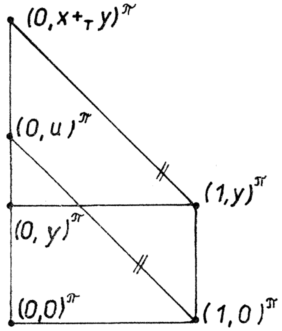

Fig. 8.

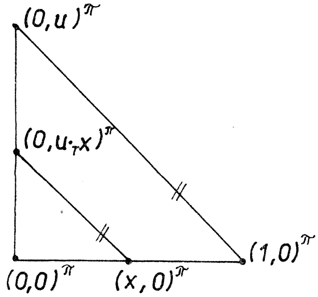

Fig. 9.

It is interesting that Hughes himself introduces another addition (which we shall denote here by $\oplus_{T^*}$) for which $x \oplus_{T^*} y := T^*(1, x, y)$ $\forall x, y \in S$. This has a very simple geometric meaning, namely (Fig. 10)

$$P_{x \oplus_{T^*} y} := ((((P_x \sqcup Z^*) \sqcap (0 \sqcup X) \sqcup Y) \sqcap (P_y \sqcup X)) \sqcup Z^*) \sqcap (0 \sqcup Y) \quad \forall x, y \in S .$$

Now what is the geometric meaning of $\oplus_T$ with respect to Hall frame?
Write $u \oplus_T v := T(1, u, v)$ $\forall u, v \in S$ and this means (Fig. 11)

$$P_{u \oplus_T v} = ((((((P_v \sqcup X) \sqcap (U \sqcup Y)) \sqcup 0) \sqcap l_\infty) \sqcup P_u) \sqcap (U \sqcup Y)) \sqcup X) \sqcap (0 \sqcup Y) \quad \forall u, v \in S .$$

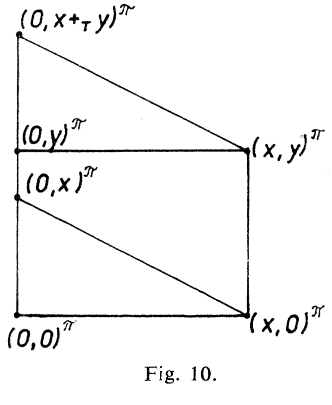

Fig. 10.

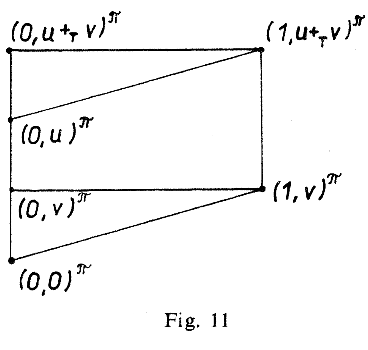

Fig. 11

Let $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a projective plane and $\mathcal{O} \subseteq \mathcal{P}$. We shall say that the point $A \in \mathcal{O}$ is oval point of $\mathcal{O}$ if there exists just one $a \in \mathcal{L}$ with $A \,\mathrm{I}\, a$, $\#(\mathcal{O} \cap a) = 1$ (the so called *tangent line*) and if $\#(l \cap \mathcal{O}) = 2$ for all $l \in \mathcal{L} \setminus \{a\}$ with $A \,\mathrm{I}\, l$. Now let $\mathcal{O} \subseteq \mathcal{P}$ have three pairwise distinct oval points $1, X, Y$ (set also $Y =: 0$) with tangent lines t, t_X, t_Y. As $t_X \neq t_Y$, the point $t_X \sqcap t_Y$ must exist (Fig. 12). Define a frame $(P_\infty, l_\infty, \pi, \lambda)$ as follows (Fig. 13–14): $P_\infty := Y$, $l_\infty := X \sqcup Y$, $S := \mathcal{O} \setminus \{X\}$; further choose two permutations α, β of S such that α interchanges X, Y and fixes 1 and β fixes X, Y. Then for all $x, y \in S$ put $(x, y)^\pi := (y \sqcup X) \sqcap (x^\alpha \sqcup Y)$ and for all $u, v \in S$; $u \neq 1$ put $(u, v)^\lambda := (0, v)^\pi \sqcup ((1 \sqcup u^\beta) \sqcap l_\infty)$ whereas for every $u \in S$ we put $(u, 1)^\lambda := := (0, v)^\pi \sqcup (t \sqcap l_\infty)$. It can be readily verified that $(P_\infty, l_\infty, \pi, \lambda)$ is a cartesian frame, so that its Hall-corresponding planar ternary ring (S, T) is with zero and the set $\{(x, x^\alpha) \mid x \in S\} \cup \{0, \infty\}$ of points of the projective plane over (S, T) contains three mutually distinct oval points $0, \infty, (1, 1)$ the tangent lines at the first two being $(0, 0)$ and 0. We shall speak of a *generalized oval frame*. In case that $\mathcal{O}$ is oval this frame is well-known, see for instance [1].

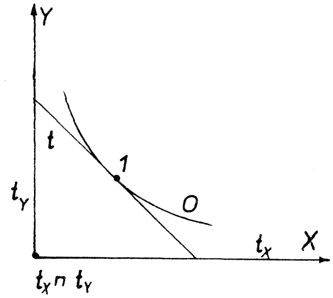

Fig. 12.

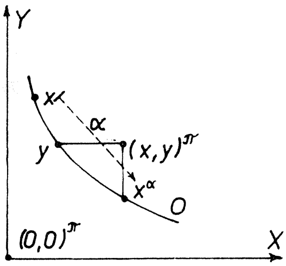

Fig. 13.

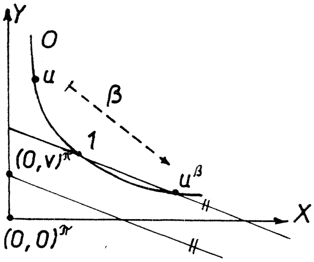

Fig. 14.

In the conclusion we introduce several examples of generalized oval frames for which the set $\mathcal{O}$ is not an oval.

1° (after oral communication given by G. Tallini in 1969). Let $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ be the projective plane over the planar ternary ring (R, T) with $T: (a, b, c) \mapsto a \cdot b + c$ where $(R, +, \cdot)$ is the field of all reals. Then $\hat{\mathcal{O}} := \{(x, 1/x) \mid x \in R\} \cup \{0, \infty\}$ is an oval. We shall change it into the set $\mathcal{O} := (\hat{\mathcal{O}} \setminus \{(-1, -1,)\}) \cup \{(0, 0)\}$. Now $0, \infty, (1, 1)$ are distinct oval points of $\mathcal{O}$ with tangent lines $(0, -1), -1, (-1, 2)$. The lines through the point $(0, 0)$ intersect $\mathcal{O}$ generally in three distinct points so that $\mathcal{O}$ is not an oval.

2° Start in the same plane with the same oval $\hat{\mathcal{O}}$. Leave $\hat{\mathcal{O}}_1 := \{(x, 1/x) \mid x \in R, x > 0\} \cup \{0, \infty\}$ unchanged but replace $\hat{\mathcal{O}}_2 := \{(x, 1/x) \mid x \in R, x < 0\}$ by a polygon with sides of the same length but non-parallel to the lines with slope -1 and with vertices on $\hat{\mathcal{O}}_2$. Denote this new set by $\mathcal{O}$. Then $0, \infty, (1, 1)$ are distinct oval points of $\mathcal{O}$ with tangent lines $(0, 0), 0, (-1, 2)$. Here $\mathcal{O}$ is not an oval because it contains segments.

3° Now let $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ be the projective plane over the planar ternary ring (Q, T) with $T: (a, b, c) \mapsto a \cdot b + c$ where $(Q, +, \cdot)$ is the field of all rationals. We use one result of A. Barlotti from [2] about the existence of 2-curves in $(\mathscr{P}, \mathscr{L}, \mathrm{I})$ with prescribed set $\mathscr{E}$ of all exterior lines and with prescribed set $\mathscr{T}$ of all tangent lines such that $\mathscr{E} \cap \mathscr{T} = \emptyset$ and such that for every $l \in \mathscr{L} \setminus \mathscr{E}$ there exist infinitely many points on l with the property that no line from $(\mathscr{E} \cap \mathscr{T}) \setminus \{l\}$ passes through them. Choose $\mathscr{E} := \{(u, 0) \mid u \in Q, u < 0\}$, $\mathscr{T} := \{(0, 0), 0, (-1, 2)\}$ (Fig. 15). Then the requirements of Barlotti are fulfilled so that there exists a 2-curve $\mathcal{O}$ in $(\mathscr{P}, \mathscr{L}, \mathrm{I})$. Tangent points on the lines $(0, 0), 0, (-1, 2)$ with respect to $\mathcal{O}$ are the only oval points of $\mathcal{O}$. The construction is complete.

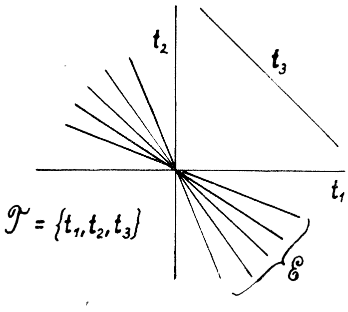

Fig. 15.

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