

John J. Buoni

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A TRACE INEQUALITY FOR FUNCTIONS OF TRIANGULAR
HILBERT-SCHMIDT OPERATORS

JOHN J. BUONI, Youngstown

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Introduction. CHANDLER DAVIS [1] has found trace inequalities for functions of matrices¹⁾. This note gives an extension of a variation of a result of C. Davis. Section 1 develops a trace inequality for simultaneously triangularizable matrices and section 2 extends these results to simultaneously triangularizable Hilbert-Schmidt operators.

Section 1. Two matrices X and Y are said to be simultaneously triangularizable iff there exists a unitary matrix U such that UXU^* and UYU^* are upper triangular. A matrix N shall be called strictly upper triangular iff N is upper triangular and nilpotent. Now if N is strictly upper triangular and S is a diagonal matrix then the following are clear:

- (i) NS and SN are nilpotent;
- (ii) if $f(\lambda)$ can be expanded in a power series in the circle $|\lambda - \lambda_0| < r$;

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda_n (\lambda - \lambda_0)^n,$$

then this expansion remains valid when the scalar argument λ is replaced by the matrix $S + N$ whose spectrum, $\sigma(S + N)$, lies within the circle of convergence;

(iii) $f(S + N)$ can be written as a diagonal plus a strictly nilpotent where the diagonal part is precisely $f(S)$.

Let us adhere to the following notations: set $f^1[\beta, \alpha] = (f(\beta) - f(\alpha))/(\beta - \alpha)$ for $\beta \neq \alpha$, $\phi^*\psi = (\psi, \phi)$, and let $\psi\phi^*$ be the linear operator defined by $\psi\phi^*\theta = (\theta, \phi)\psi$. In this notation, we find that the trace $(XY) = \sum_{i=1}^n \psi_i^*XY\psi_i$ where $\{\psi_i \mid i =$

¹⁾ The author has pointed out to Chandler Davis that the proof of Theorem 3 in [1] is incorrect.

$= 1, \dots, n\}$ is any orthonormal basis and X and Y are matrices. We can now obtain a trace inequality for a function of a pair of diagonal matrices.

Lemma 1. *Let X and Y be a pair of diagonal matrices, K an open disc of radius $r > 0$ centered at the origin, and $f(\lambda)$ have a power series whose region of convergence contains $\sigma(x)$ and $\sigma(x + y)$. If $f^1[\xi, \alpha] \in K$ for $\xi \in \sigma(x + y)$ and $\alpha \in \sigma(x)$, then $\{\text{trace}(YY^*)\}^{-1} \cdot \text{trace}(Y(f(X + Y) - f(X))) \in K$.*

Proof. Since X and Y are a diagonal pair of matrices, there exists an orthonormal basis $\{\psi_i \mid i = 1, \dots, n\}$ such that:

$$(1) \quad X = \sum_{i=1}^n \alpha_i \psi_i \psi_i^*, \quad Y = \sum_{i=1}^n \beta_i \psi_i \psi_i^* ;$$

$$(2) \quad f(X + Y) = \sum_{i=1}^n f(\alpha_i + \beta_i) \psi_i \psi_i^*, \quad f(X) = \sum_{i=1}^n f(\alpha_i) \psi_i \psi_i^* ;$$

and

$$(3) \quad \text{trace } Y(f(X + Y) - f(X)) = \sum_{i=1}^n \psi_i^* Y(f(X + Y) - f(X)) \psi_i .$$

If we replace (1) and (2) into (3), (3) then becomes

$$\sum_{i=1}^n \beta_i (f(\alpha_i + \beta_i) - f(\alpha_i)) = \sum_{i=1}^n |\beta_i|^2 (f(\alpha_i + \beta_i) - f(\alpha_i)) / \beta_i .$$

Since $f^1[\alpha_i + \beta_i, \alpha_i] \in K$ and K is an open disc of radius r , we find that

$$|(f(\alpha_i + \beta_i) - f(\alpha_i)) / \beta_i| = |(f(\alpha_i + \beta_i) - f(\alpha_i)) / \beta_i| \cdot \left| \frac{\beta_i}{\beta_i} \right| < r .$$

Thus,

$$\begin{aligned} & \{ \text{trace}(YY^*) \}^{-1} \text{trace} \{ Y(f(X + Y) - f(X)) \} = \\ & = \left| \left\{ \sum_{i=1}^n |\beta_i|^2 (f(\alpha_i + \beta_i) - f(\alpha_i)) / \beta_i \right\} / \sum_{i=1}^n |\beta_i|^2 \right| < r , \end{aligned}$$

and this yields the result.

We can now use the lemma to obtain a similar identity for matrices which are simultaneously triangularizable.

Theorem 1. *Let X and Y be simultaneously upper triangularizable, K an open disc centered at the origin of radius r , and $f(\lambda)$ have a power series whose region of convergence contains $\sigma(X)$ and $\sigma(X + Y)$. If $f^1[\gamma, \alpha] \in K$ for $\gamma \in \sigma(X + Y)$ and $\alpha \in \sigma(X)$, then $\{\text{trace } YY^*\}^{-1} \cdot \text{trace } Y(f(X + Y) - f(X)) \in K$.*

Proof. Since X and Y can be simultaneously upper triangularized, then there exists U unitary, S_1 and S_2 diagonal, and N_1 and N_2 strictly upper triangular, such

that $UXU^* = S_1 + N_1$ and $UYU^* = S_2 + N_2$. Since the trace is invariant under similarity, it follows that

$$\begin{aligned}
 (4) \quad & \{\text{trace } YY^*\}^{-1} \text{trace } Y(f(X + Y) - f(Y)) = \\
 & = \{\text{trace } UYY^*U^*\}^{-1} \text{trace } UYU^*U(f(X + Y) - f(Y))U^* = \\
 & = \{\text{trace } UYU^*UYU^*\}^{-1} \text{trace } UYU^*(f(U(X + Y)U^*) - f(UXU^*)) = \\
 & = \{\text{trace } (S_2 + N_2)(S_2^* + N_2^*)\}^{-1} \text{trace } (S_2 + N_2) \\
 & \quad (f(S_1 + S_2 + N_1 + N_2) - f(S_1 + N_1)).
 \end{aligned}$$

Since the trace is linear, and since $N_2S_2^*$, $S_2N_2^*$ and $N_2(f(S_1 + S_2 + N_1 + N_2) - f(S_1 + N_1))$ are nilpotent, one finds that (4) equals

$$\begin{aligned}
 (5) \quad & \{\text{trace } (S_2S_2^* + N_2N_2^*)\}^{-1} \text{trace } S_2(f(S_1 + S_2 + N_1 + N_2) - f(S_1 + N_1)) = \\
 & = \{\text{trace } (S_2S_2^* + N_2N_2^*)\}^{-1} \text{trace } (S_2(f(S_1 + S_2) - f(S_1)) + S_2T)
 \end{aligned}$$

where T is strictly upper triangular and nilpotent. Thus (5) equals

$$(6) \quad \{\text{trace } (S_2S_2^* + N_2N_2^*)\}^{-1} \text{trace } S_2(f(S_1 + S_2) - f(S_1)).$$

Since $\text{trace } S_2S_2^*$ and $\text{trace } N_2N_2^*$ are positive, the absolute value of (6) is less than or equal to

$$\left| \{\text{trace } S_2S_2^*\}^{-1} \text{trace } S_2(f(S_1 + S_2) - f(S_1)) \right|.$$

The result now follows immediately upon application of Lemma 1.

Section 2. We would like to recall a few facts about Hilbert-Schmidt and trace class operators on a separable Hilbert space H . For complete information about these operators, the reader is referred to [2]. Let $\{\psi_i\}_{i=1}^\infty$ be a complete orthonormal set for H . A bounded linear operator X is said to be a Hilbert-Schmidt (H. S.) operator in case the quantity $\|X\| = \{\sum_i |X\psi_i|^2\}^{1/2}$ is finite where $|X\psi_i|$ is the norm of the vector $X\psi_i$. The number $\|\cdot\|$ is sometimes referred to as the Hilbert-Schmidt norm and is independent of the orthonormal basis chosen. Every H.S. operator is compact and is the limit in the $\|\cdot\|$ norm of a sequence of operators with finite range. If X is an H.S. operator and f is a singlevalued analytic function on a domain containing $\sigma(X)$ vanishing at zero, then $f(X)$ is an H.S. operator and the map $X \rightarrow f(X)$ is continuous in the $\|\cdot\|$ norm.

Let $\{\lambda_i\}_{i=1}^\infty$ be the eigenvalues repeated according to multiplicity of the H.S. operator X . X is said to be a trace class if $\sum_{i=1}^\infty |\lambda_i| < \infty$. The trace X of an operator of trace class is defined to be $\text{trace } X = \sum_{i=1}^\infty \lambda_i$. Although an H.S. operator X need not be of trace class, the product of two H.S. operators are of trace class [2, p. 1093].

Two compact operators X and Y on H will be said to be simultaneously triangularized iff there exists an orthonormal bases $\{\psi_i\}_{i=1}^\infty$ and orthogonal projections P_n onto the subspace determined by $\psi_1, \psi_2, \dots, \psi_n$ such that $P_n X P_n = X P_n$ and $P_n Y P_n = Y P_n$.

In view of the above statements, it would be desirable to extend Theorem 1 to H.S. operators. With this in mind the following lemma is useful.

Lemma 2. *Let X and Y be a simultaneous triangularizable pair of Hilbert-Schmidt operators, then there exists two sequences of operators $\{X_n\}, \{Y_n\}$ such that the following hold:*

(i) *for each n , X_n and Y_n are simultaneously triangularizable operators of finite rank,*

(ii) *the sequence $\{X_n\}$ and $\{Y_n\}$ converge in the Hilbert-Schmidt norm to X and Y respectively.*

Proof. By the hypothesis there exists an orthonormal basis $\{\psi_i\}_{i=1}^\infty$ and a sequence of orthogonal projections $\{P_n\}$ onto the subspace spanned by $\psi_1, \psi_2, \dots, \psi_n$ such that $P_n X P_n = X P_n$ and $P_n Y P_n = Y P_n$ for each n . Set $X_n = P_n X P_n$ and $Y_n = P_n Y P_n$. Clearly (i) is satisfied.

To show (ii), observe that

$$\|X - X_n\|^2 = \sum_{i=1}^{\infty} |(X - P_n X P_n) \psi_i|^2 = \sum_{i=1}^{\infty} |(X - X P_n) \psi_i|^2 = \sum_{i=n+1}^{\infty} |X \psi_i|^2.$$

Since $\sum_{i=1}^{\infty} |X \psi_i|^2$ equals $\|X\|^2$ and converges, the proof of the lemma is completed.

The above lemma, together with the earlier discussion, the continuity of trace [2, p. 1100], and Theorem 1 yields the following theorem.

Theorem 2. *Let X and Y be simultaneously triangularizable Hilbert-Schmidt operators on a separable Hilbert space, K an open disc, and $f(\lambda)$ an analytic function vanishing at zero with a power series whose region of convergence contains $\sigma(X)$ and $\sigma(X + Y)$. If $f'[\gamma, \alpha] \in K$ for $\gamma \in \sigma(X + Y)$ and $\alpha \in \sigma(X)$, then $\{\text{trace}(YY^*)\}^{-1} \text{trace}\{Y(f(X + Y) - f(X))\} \in K$.*

References

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Author's address: Department of Mathematics, Youngstown State University, Youngstown, Ohio 44503, U.S.A.