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## ON THE SILOV BOUNDARY INDUCED BY CERTAIN SEMIGROUP ALGEBRAS

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1. A locally compact commutative semigroup  $S$  is a locally compact Hausdorff space together with a continuous associative binary operation. We will assume that on such a semigroup  $S$  there is a non negative regular Borel measure,  $m$ , satisfying the condition that for each Borel set  $E$  in  $S$  with  $m(E) = 0$ ,  $m(x^{-1}E) = 0$ , where  $x^{-1}E = [y : xy \in E]$ . The semigroup algebra of the locally compact commutative semigroup  $S$  is taken as those finite regular Borel measures (linear functionals) in  $M(S)$  ( $=C_0(S)^*$ ) which are absolutely continuous with respect to the measure  $m$ , and is denoted by  $L^1(S, m)$ . Addition and scalar multiplication are defined pointwise and multiplication (convolution) is given by  $\mu * \nu(f) = \iint f(xy) \mu(dx) \nu(dy)$ , where  $f \in C_0(S)$  and  $\mu, \nu \in M(S)$ .

A multiplicative function  $\tau$  on  $S$  is a complex valued function on  $S$  satisfying  $\tau(xy) = \tau(x)\tau(y)$  for all  $x$  and  $y$  in  $S$  and with  $\tau \not\equiv 0$ . A semicharacter on  $S$  is a bounded  $m$ -measurable multiplicative function on  $S$ . We denote by  $S^*$  the set of semicharacters on  $S$ . If  $S$  possesses an identity element, then  $S^*$  is a semigroup under multiplication  $\tau\theta(x) = \tau(x)\theta(x)$ . If  $S$  is a locally compact abelian group and  $m$  is Haar measure on  $S$  then the semigroup algebra is the known  $L^1$  algebra and  $S^*$  is the dual group of continuous characters on  $S$ .

Some of the results of this paper are dependent on the work in [2], [4] and [6]. Furthermore, the work of COMFORT [3] on discrete semigroups is a base upon which this paper is built. It was shown in [5] that the set  $\mathcal{A}$  of non trivial multiplicative linear functionals on  $L^1(S, m)$  is in one to one correspondence with the elements of  $S^*$  (identified modulo equal almost everywhere). In particular, if  $\tau \in S^*$ , the linear functional  $h_\tau(\mu) = \int \tau d\mu$  is a multiplicative linear functional and  $\tau(x) = h_\tau(\mu * \bar{x})/h_\tau(\mu)$  ( $h_\tau(\mu) \neq 0$  and  $\mu * \bar{x} \in L^1(S, m)$ ) is such that the mapping of  $\tau \rightarrow h$  is the desired correspondence.

The topology on  $S^*$  will be taken to be the Gelfand topology that  $S^*$  inherits via the above correspondence with the maximal ideal space of the Banach algebra

$L^1(S, m)$ . Thus a net  $\{\tau_\alpha\}$  in  $S^*$  converges to  $\tau$  in  $S^*$  if and only if for each  $\mu$  in  $L^1(S, m)$ ,  $\hat{\mu}(\tau_\alpha) \rightarrow \hat{\mu}(\tau)$  ( $h_{\tau_\alpha}(\mu) \rightarrow h_\tau(\mu)$  or  $\int \tau_\alpha d\mu \rightarrow \int \tau d\mu$ ).

A fundamental neighborhood of  $\tau$  in  $S^*$  is a set

$$U = \left\{ \theta \in S^* : \left| \int (\theta - \tau) d\mu_j \right| < \varepsilon, \quad \{\mu_j\}_1^n \subset L^1(S, m) \right\}$$

where  $\varepsilon > 0$ . In this topology,  $S^*$  is locally compact and is compact if and only if 0 is not in the closure of the multiplicative linear functionals in the  $\omega^*$  topology on  $L^1(S, m)^*$ .

It will be assumed from here on that  $S$  has an identity element, so that  $S^*$  is a semigroup. It follows readily from the above remarks on the topology of  $S^*$  that multiplication is continuous in  $S^*$ .

The algebra of Gelfand transforms of  $L^1(S, m)$  is a separating subalgebra of the space of continuous functions vanishing at infinity on the locally compact Hausdorff space  $S^*$ , hence the Silov boundary  $\partial$  induced by  $S^*$  exists and satisfies

- (i)  $\partial$  is a closed subset of  $S^*$
- (ii) if  $\mu \in L^1(S, m)$  then  $|\hat{\mu}|$  assumes its maximum on  $\partial$
- (iii) no proper closed subset of  $\partial$  satisfies (ii).

In this paper, the boundary will be determined for a class of semigroups. In particular, for compact linearly-quasi ordered semigroups [4] we will show that

$$\partial = [\tau \in S^* : |\tau| \text{ is idempotent}] .$$

**2.** Let  $S$  be a compact commutative topological semigroup with identity element 1 and let  $m$  be a non negative regular Borel measure on  $S$  such that for each set  $E$  of  $m$ -measure zero,

$$x^{-1}E = [y : xy \in E]$$

is also of  $m$ -measure 0. Let  $K$  be the minimal ideal of  $S$ . Since  $K$  is a compact abelian topological group, for each  $\tau \in \hat{K}$ ,  $\tau$  a continuous character on  $K$ , the mapping  $\tau'(x) = \tau(xe)$  ( $e$  the identity of  $K$ ) is a continuous semicharacter on  $S$  and thus Borel measurable. We identify  $\hat{K}$  then as a subset of  $S^*$ . It is clear that

$$\hat{K} = [\tau \in S^* : |\tau| \equiv 1] .$$

**Lemma 2.1.** *Let  $S$  be as above and let  $\partial$  denote the Silov boundary of the maximal ideal space of  $L^1(S, m)$ , i.e.  $S^*$ , then  $\hat{K} \subset \partial$ .*

Proof. Let  $\chi \in \hat{K}$  and let  $U$  be a neighborhood of  $\chi$  in  $S^*$ . Without loss of generality we let

$$U = [\tau : |\hat{\mu}_i(\tau) - \hat{\mu}_i(\chi)| < \varepsilon, \quad \mu_i \in L^1(S, m) \quad \text{and} \quad 1 \leq i \leq n]$$

and also assume that  $\varepsilon < \frac{1}{2}$ .

We need to find  $v \in L^1(S, m)$  such that  $|\hat{v}|$  assumes its maximum only on  $U$ . We shall find a complex valued measurable function  $\alpha$ , with  $m$ -finite support, on  $S$  and take  $dv = \alpha dm$ .

For each  $\mu_i$ , let  $A_i$  be the compact support of  $\mu_i$  and let  $B_i = [x^2 : x \in A_i]$ , note that  $B_i$  is the image under multiplication of  $A(S \times S) \cap (A_i \times A_i)$  and is thus compact and hence a Borel set. Let  $A = \bigcup_{i=1}^n (A_i \cup B_i)$  and define

$$\alpha(x) = \chi_A(x) \overline{\chi(x)} \quad \text{for all } x \text{ in } S.$$

Then  $\alpha$  is Borel measurable with compact support and  $\alpha dm \in L^1(S, m)$ . Let  $v$  be this measure. Then

$$m(A) = \int \chi_A(x) dm = \int \chi_A(x) \overline{\chi(x)} \chi(x) dm = \int \chi(x) dv = \hat{v}(\chi).$$

If  $\theta \notin U$  then there is a  $j$ ,  $1 < j < n$ , such that  $|\hat{\mu}_j(\theta) - \hat{\mu}_j(\chi)| \geq \varepsilon$ , that is

$$\int |\theta - \chi| d\mu_j \geq \int |(\theta - \chi) \chi| d\mu_j \geq \varepsilon,$$

hence there is a subset  $B$  of  $A_j$  such that  $m(B) > 0$  and  $|\theta - \chi| > \delta > 0$  on  $B$ , i.e.  $\theta(x) \neq \chi(x)$  for all  $x$  in  $B$ .

If  $\theta(x) = 0$  on a set of positive measure  $C \subset B$  then

$$|\hat{v}(\theta)| = \left| \int \theta(x) \alpha(x) dm \right| = \left| \int_{A \setminus C} \theta(x) \alpha(x) dm \right| < m(A \setminus C) < m(A) = |\hat{v}(\chi)|.$$

On the other hand, if  $\theta|_B$  is zero only on a set of measure zero, then without loss of generality  $\theta(x) \neq 0$  for all  $x$  in  $B$ . If  $|\hat{v}(\theta)| = m(A)$ , then  $\hat{v}(\theta) = m(A) e^{i\varphi}$ ,  $0 < \varphi < 2\pi$ . Let  $d\lambda = e^{-i\varphi} \alpha^2(x) \theta(x) dm$  and  $d\gamma = e^{-i\varphi} \alpha(x) dm$  and let  $\mu = \gamma + \lambda$ . Then  $|\hat{\mu}(\theta)| = \left| \int e^{-i\theta} (\alpha\theta + \alpha^2\theta^2) dm \right| < 2m(A)$  since  $|\alpha + \alpha^2\theta^2| < 2$  and  $|\hat{\mu}(\chi)| = \left| \int e^{-i\varphi} (\alpha\chi + \chi^2\chi) dm \right| = |e^{-i\varphi} m(A) + \int e^{-i\varphi} \alpha\theta dm| = 2m(A)$ . Hence for  $\theta \notin U$ ,  $|\hat{\mu}(\theta)| < |\hat{\mu}(\chi)|$  and  $\chi \in \partial$ .

Let  $\Gamma = [\chi \in S^* : |\chi| = 0 \text{ or } 1]$ . Note that  $\Gamma$  is the set of all those elements of  $S^*$  which have an inverse with respect to some idempotent in  $S^*$  and hence that  $\Gamma$  is a union of groups, the maximal groups containing each idempotent element in  $S^*$ . For  $\chi = \chi^2$  in  $S^*$  and  $H(\chi)$  the maximal group with identity  $\chi$ ,  $H(\chi)$  is a locally com-

compact topological group, since inversion in  $H(\chi)$  is complex conjugation,  $H(\chi)$  is closed in  $S^*$  and multiplication is continuous. Now  $\Gamma$  is closed since  $S^*$  is closed under complex conjugation and the idempotent elements of a semigroup are a closed set, thus  $\{\chi_\alpha\}$  a net in  $\Gamma$  with  $\chi_\alpha \rightarrow \chi$  implies  $\bar{\chi}_\alpha \rightarrow \bar{\chi}$  and hence  $\chi_\alpha \bar{\chi}_\alpha \rightarrow \chi \bar{\chi}$ . Thus since  $\chi_\alpha \bar{\chi}_\alpha$  is an idempotent,  $\chi \bar{\chi}$  is then an idempotent and  $\chi \in \Gamma$ .

**Lemma 2.2.** *Let  $S$ ,  $\partial$  and  $\Gamma$  be as above. Then  $\partial \subset \Gamma$ .*

*Proof.* We will show that for  $\psi \in S^*$  and  $\psi \notin \Gamma$  that  $\psi \notin \partial$ . Let  $\psi \in S^* \setminus \Gamma$ . Then there is an  $\varepsilon > 0$  and a Borel set  $B$  in  $S$  of finite positive measure ( $m$ ) such that  $2\varepsilon < |\psi(x)| < 1 - 2\varepsilon$  for all  $x \in B$ . Let  $\nu = \chi_B(x) dm$  ( $\chi_B(x)$  is the characteristic function of  $B$ ), then  $\nu \in L^1(S)$  and  $U = [\theta \in S^* : |\hat{\nu}(\theta) - \hat{\nu}(\bar{\psi})| < \varepsilon]$  is an open neighborhood of  $\psi$  in  $S^*$ . Note that  $\theta \in U$  implies that  $\varepsilon < \hat{\nu}(\theta) < 1 - \varepsilon$  and  $|\hat{\lambda}(\bar{\psi})| = \int |\psi| dm < (1 - 2\varepsilon) m(B)$ . In order to show that  $\psi \notin \partial$ , it suffices to show that for each  $\mu \in L^1(S, m)$ ,  $|\hat{\mu}|$  attains its maximum outside of  $U$ . Let  $\theta \in S^*$  such that

$$|\hat{\mu}(\theta)| \geq |\hat{\mu}(\varphi)|$$

for all  $\varphi \in S^*$ . If  $\theta \in S^* \setminus U$  there is nothing to prove hence we assume  $\theta \in U$ . If  $|\hat{\mu}(\theta)| = 0$  then  $|\hat{\mu}| \equiv 0$  and since the identically 1 valued semicharacter is not in  $U$  (i.e.  $U \neq S^*$ ) again we are finished. Thus we can assume that  $0 < |\hat{\mu}(\theta)|$  and  $\theta$  is an element of  $U$ . We will now construct a semicharacter not in  $U$  where  $|\hat{\mu}|$  also attains its maximum value.

Since  $\mu \in L^1(S, m)$ ,  $\mu$  is absolutely continuous with respect to  $m$  and hence  $d\mu = f(x) dm$ . Let

$$A = [x : f(x) \theta(x) \neq 0].$$

We wish to consider the function of a complex variable

$$\begin{aligned} g(z) &= \int \theta(x) e^{(1+z)\ln|\theta(x)|} d\mu = \int f(x) \theta(x) e^{(1+z)\ln|\theta(x)|} dm = \\ &= \int_A f(x) \theta(x) e^{(1+z)\ln|\theta(x)|} dm. \end{aligned}$$

Let  $X_n = [x : |\theta(x)| > 1/n]$  and consider

$$g_n(z) = \int_{X_n} \theta(x) e^{(1+z)\ln|\theta(x)|} d\mu.$$

Then

$$g(z+h) - g(z) = \frac{1}{h} \int_{X_n} \theta(x) e^{(1+z)\ln|\theta(x)|} [e^{h\ln|\theta(x)|} - 1] d\mu.$$

For  $|h|$  small,  $\text{Re}(h)$  is small and the difference quotient is

$$\int_{X_n} \theta(x) e^{(1+z)\ln|\theta(x)|} [\ln|\theta(x)| + h[\dots]] d\mu$$

and thus

$$\lim_{|h| \rightarrow 0} \frac{g(z+h) - g(z)}{h} = \int_{X_n} \theta(x) e^{(1+z)\ln|\theta(x)|} \ln|\theta(x)| d\mu$$

and  $g_n$  is analytic. Now

$$\begin{aligned} |g(z) - g_n(z)| &= \left| \int_{S \setminus X_n} \theta(x) e^{(1+z)\ln|\theta(x)|} d\mu \right| < \int_{S \setminus X_n} |\theta(x)| |e^{(1+z)\ln|\theta(x)|}| d\mu = \\ &= \int_{S \setminus X_n} |\theta(x)| |e^{(1+\text{Re } z)\ln|\theta(x)|}| d\mu. \end{aligned}$$

Now for  $\text{Re } z > -1$ ,  $e^{(1+\text{Re } z)\ln|\theta(x)|} < 1$  and

$$|g_n(z) - g(z)| < \frac{1}{n} \|\mu\|$$

hence  $g_n(z)$  converges uniformly to  $g(z)$  for  $\text{Re } z > -1$  and  $g$  is analytic in  $\text{Re } z > -1$ .

We define semicharacters  $\theta_z$  on  $S$  for  $\text{Re } z > -1$  by

$$\begin{aligned} \theta_z(x) &= 0 \quad \text{if } \theta(x) = 0 \\ \theta_z(x) &= \theta(x) e^{(1+z)\ln|\theta(x)|} \quad \text{if } \theta(x) \neq 0. \end{aligned}$$

For  $a, b \in S$  if  $\theta(a) = 0$  or  $\theta(b) = 0$  then  $\theta(ab) = 0$  and  $\theta_z(ab) = 0 = \theta_z(a)\theta_z(b)$ .

If  $\theta(a) \neq 0$ , then  $\theta(ab) \neq 0$  and

$$\theta_z(ab) = \theta(ab) e^{(1+z)\ln|\theta(ab)|} = \theta(a) e^{(1+z)\ln|\theta(a)|} \theta(b) e^{(1+z)\ln|\theta(b)|}.$$

Further  $\text{Re}(z) > -1$  implies  $|\theta_z(x)| = |\theta(x) e^{(1+z)\ln|\theta(x)|}| < 1$  and  $\theta_z \in S^*$  since  $\theta_z$  is clearly measurable.

Note that  $\hat{\mu}(\theta_z) = g(z)$ , and that for  $\text{Re } z > -1$ ,  $\theta_z \in S^*$  and  $|\hat{\mu}|$  attains its maximum at  $\theta$  i.e.  $|\hat{\mu}(\theta)| = |g(0)|$  is maximum value of the analytic function  $g$  in a neighborhood of 0 and hence  $g$  is constant thus  $|\hat{\mu}|$  attains its maximum also at  $\theta_z$  for  $\text{Re } z > -1$  and we choose  $z$  real such that  $\theta_z \in S^*$  and  $|\theta|^{z+1} > (1-\varepsilon)m(B)$ , i.e.  $\theta_z \notin U$  and we see that  $\chi$  is not in the boundary  $\partial$ .

**3. Linearly Quasi-ordered semigroups.** A general discussion of linearly quasi-ordered semigroups can be found in [4] and [6]. The existence of a measure on such semigroups satisfying the conditions in section 1 is to be found in [6]. We show in

this section that the boundary  $\partial$  of the maximal ideal space  $S^*$  of  $L^1(S, m)$ , where  $S$  is a compact commutative linearly quasi-ordered topological semigroup with identity is  $\Gamma$  the set of those semicharacters (measurable) whose absolute values are idempotent elements of the semigroup  $S^*$  of all measurable semicharacters. From lemma 2.2 we know that  $\partial \subset \Gamma$  and need only the reverse inequality.

Let  $S$  be a compact commutative linearly quasi-ordered topological semigroup and  $\chi \in S^*$ , the measurable semicharacters on  $S$ . Now  $\chi^{-1}(0)$  is a prime ideal of  $S$  and is identical with  $|\chi|^{-1}(0)$ . Since  $|\chi|$  can also be considered as a multiplicative function on  $S/\mathcal{Q}$ , it is readily seen that  $\chi^{-1}(0)$  either is equal to  $Se$  for some idempotent element  $e$  in  $S$  or is equal to  $Se \setminus H(e)$  for some idempotent element  $e$  in  $S$ . In the following  $\varphi$  is the natural mapping  $\varphi : S \rightarrow S/\mathcal{Q}$ .

**Lemma 3.1.** *Let  $S$  and  $\chi$  be as above and let  $e$  be such that  $\varphi(e)$  is the zero of a unit thread in  $S/\mathcal{Q}$ , then  $\chi$  in  $\Gamma$  implies  $\chi$  is in  $\partial$ .*

*Proof.* Let  $f$  be the idempotent element of  $S$  such that  $\varphi(f)$  is the identity of the unit thread for which  $\varphi(e)$  is the zero then since  $\chi$  is a measurable semicharacter on  $S$ ,  $\chi \upharpoonright Sf \setminus Se$  is a continuous semicharacter on  $Sf \setminus Se$  and hence is the restriction of a character on the group  $H(f) \times R$  to a subsemigroup and as such is in the boundary of the maximal ideal space of  $L^1(Sf \setminus Se)$  [1]. Since  $L^1(Sf \setminus Se)$  can be considered as a subalgebra of  $L^1(S)$ , we see that  $\chi \in \partial$ .

**Lemma 3.2.** *Let  $S$  and  $\chi$  be as above and let  $e$  be such that the connected component containing  $e$  in  $E$ , the set of all idempotent elements, is a point. If  $\varphi(e)$  is not the zero of a unit thread and  $\chi$  is in  $\Gamma$  then  $\chi$  is in  $\partial$ .*

*Proof.* Since the component of  $E$  containing  $e$  is  $\{e\}$ , there exists a linearly ordered net  $f_\alpha$  of idempotent elements  $f_\alpha \downarrow e$  such that each  $f_\alpha$  is such that  $\varphi(f_\alpha)$  is the zero of a unit thread. Define

$$\chi_\alpha = \begin{cases} \chi & \text{on } S \setminus Sf_\alpha, \\ 0 & \text{on } Sf_\alpha, \end{cases}$$

then by lemma 3.1  $\chi_\alpha \in \partial$ . Thus we need only show that  $\chi_\alpha \rightarrow \chi$  to obtain  $\chi \in \partial$ . Now  $\chi_\alpha \rightarrow \chi$  if and only if for each  $\nu \in L^1(S)$ ,  $\hat{\nu}(\chi_\alpha) \rightarrow \hat{\nu}(\chi)$ . We need consider only those  $\nu$  with support contained in  $S \setminus Se$ , thus

$$|\hat{\nu}(\chi_\alpha) - \hat{\nu}(\chi)| = \left| \int (\chi_\alpha - \chi) \, d\nu \right| = \left| \int_{Sf_\alpha \setminus Se} (\chi_\alpha - \chi) \, d\nu \right| < \nu(Sf_\alpha \setminus Se).$$

Since  $\nu$  is a regular Borel measure and  $f_\alpha \rightarrow e$ ,  $\nu(Sf_\alpha \setminus Se) \rightarrow 0$  and  $\chi \in \partial$ .

**Lemma 3.3.** *Let  $S$  and  $\chi$  be as above and let  $e$  be such that  $\varphi(e)$  belongs to a non-trivial idempotent interval in  $S/\mathcal{Q}$ , then  $\chi$  in  $\Gamma$  implies  $\chi \in \partial$ .*

Proof. Since  $\varphi(e)$  is in an idempotent interval  $\chi$  is equivalent to a semicharacter on  $S$  defined uniquely by extension of a character on  $H(e)$  to  $S$ . For any idempotent  $f$  with  $\varphi(f)$  in the same interval and  $\varphi(f) > \varphi(e)$ , there is an involution on  $L^1(Sf \setminus Se)$  so that the algebra is self-adjoint and hence the boundary of the maximal ideal space of this algebra is the whole maximal ideal space. It then follows that  $\chi$  is in  $\partial$ .

We thus have the following

**Theorem 3.4.** *Let  $S$  be a compact commutative linearly quasiordered topological semigroup such that  $S/\mathcal{Q}$  contains no nil thread. The natural measure  $m$  on  $S$  is such that  $L^1(S, m)$  is a Banach algebra and the Silov boundary of the maximal ideal space corresponds in a one to one fashion with those  $m$ -measurable semicharacters on  $S$  whose absolute values are idempotent semicharacters.*

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