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NOTE ON FUNCTIONAL-DIFFERENTIAL EQUATIONS
WITH INITIAL FUNCTIONS OF BOUNDED VARIATION

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In this note we shall deal with the standard functional-differential equation of
retarded type

(1) \[ \dot{x}(t) = \int_{t-r}^{t} \left[ d_s P(t, s) \right] x(t + s) + f(t) \text{ a.e. on } [a, b], \]

(2) \[ x(t) = u(t) \text{ on } [a - r, a], \]

where \(-\infty < a < b < +\infty\) and the initial functions \(u(t)\) are of bounded variation
on \([a - r, a]\). We assume that \(P(t, s)\) is a Borel measurable in \((t, s) \in [a, b] \times \times (-\infty, +\infty)\) \(n \times n\)-matrix function such that \(P(t) = \text{var}^0, P(t, s) < \infty\) for all \(t \in [a, b]\) and

\[ \int_{a}^{b} p(t) \, dt < \infty, \]

\(f(t)\) is an \(n\)-vector function Lebesgue integrable on \([a, b]\) \((f(t) \in \mathcal{L}_n(a, b))\). We shall
suppose also \(P(t, s) = P(t, -r)\) for \(s \leq -r\) and \(P(t, s) = P(t, 0)\) for \(s \geq 0\). Without
any loss of generality we may suppose furthermore that \(P(t, \cdot)\) is right continuous
on \((-r, 0)\) and \(P(t, 0) = 0\) for all \(t \in [a, b]\).

Let \(\mathcal{BV}_n(a - r, a)\) denote the space of (column) \(n\)-vector functions with bounded
variation on \([a - r, a]\). \(\mathcal{AC}_n(a, b)\) is the space of \(n\)-vector functions which are
absolutely continuous on \([a, b]\). The introduced spaces are equipped with the
usual norms

\[ u \in \mathcal{BV}_n(a - r, a) \rightarrow \|u\|_{\mathcal{BV}} = \|u(a)\| + \text{var}^a_{a-r} u, \]

\[ x \in \mathcal{AC}_n(a, b) \rightarrow \|x\|_{\mathcal{AC}} = \|x(a)\| + \text{var}^a_{a} x, \]

\[ f \in \mathcal{L}_n(a, b) \rightarrow \|f\|_{\mathcal{L}} = \int_{a}^{b} \|f(t)\| \, dt. \]
Proposition 1. There exists a unique \( n \times n \)-matrix function \( Y(t, s) \) defined on \([a, b] \times [a, b]\) and such that

\[
Y(t, s) = \begin{cases}
I - \int_s^t Y(t, \sigma) P(\sigma, s - \sigma) \, d\sigma & \text{for } a \leq t \leq b, \ a \leq s \leq t, \\
I & \text{for } a \leq t \leq b, \ t \leq s \leq b,
\end{cases}
\]

where \( I \) is the identity \( n \times n \)-matrix. Given \( t \in [a, b] \), \( Y(t, \cdot) \) is of bounded variation on \([a, b]\) and given \( s \in [a, b] \), \( Y(\cdot, s) \) is absolutely continuous on \([a, b]\).

(For the proof of a slightly modified assertion see J. K. Hale [2], Theorem 32,2.)

The following representation of solutions of the system (1), (2) is well known (cf. H. T. Banks [1] or J. K. Hale [2], Theorems 16,1 and 32,2):

Proposition 2. Given \( u \in \mathcal{Y}_n(a - r, a) \), there exists a unique \( n \)-vector function \( x(t) \) defined on \([a - r, b]\) and absolutely continuous on \([a, b]\) and such that (1) and (2) hold. This function \( x(t) \) is on \([a, b]\) given by

\[
x(t) = \Phi u + \Psi f,
\]

where

\[
\Phi : u \in \mathcal{Y}_n(a - r, a) \rightarrow Y(t, a) u(a) + \int_{a-r}^a \left[ \int_a^t Y(t, \sigma) P(\sigma, s - \sigma) \, d\sigma \right] u(s) \in \mathcal{A}\mathcal{C}_n(a, b),
\]

\[
\Psi : f \in \mathcal{L}_n(a, b) \rightarrow \int_a^t Y(t, s) f(s) \, ds \in \mathcal{A}\mathcal{C}_n(a, b)
\]

and \( Y(t, s) \) is defined by Proposition 1.

The operators \( \Phi, \Psi \) in (4) are obviously linear and bounded. The aim of this note is to show that \( \Phi \) is even completely continuous. By Theorem 3,1 of St. Schwabik [5] it suffices to show that the function

\[
K(t, s) = \int_a^t Y(t, \sigma) P(\sigma, s - \sigma) \, d\sigma, \quad (t, s) \in [a, b] \times [a - r, a]
\]

is of bounded two-dimensional variation (according to Vitali) on \([a, b] \times [a - r, a]\) \((v(K) < \infty)\) and \( \text{var}_{a-r}^a K(a, \cdot) + \text{var}_b^a K(\cdot, a) < \infty \). Such functions are said to be of strongly bounded variation on \([a, b] \times [a - r, a]\). (For the definition and basic properties of functions of bounded two-dimensional variation see T. H. Hildebrandt [4]).

Lemma 1. The fundamental matrix solution \( Y(t, s) \) defined by Proposition 1 is of strongly bounded variation on \([a, b] \times [a, b]\).

Proof. Analogously to J. K. Hale in the proof of Theorem 32,2 in [2] we shall introduce the function \( W(t, s) \) fulfilling the matrix Volterra integral equation

\[
W(t, s) = \begin{cases}
-P(t, s - t) - \int_s^t W(t, \sigma) P(\sigma, s - \sigma) \, d\sigma & \text{for } a \leq t \leq b, \ a \leq s \leq t, \\
0 & \text{for } a \leq t \leq b, \ t \leq s \leq b.
\end{cases}
\]
The existence of such a function $W(t, s)$ follows from the contraction mapping principle. Moreover, given $t \in [a, b]$, the function $W(t, \cdot)$ is of bounded variation on $[a, b]$. Now, let $s, t \in [a, b]$, $s \leq t$ and let $\{s = s_0 < s_1 < \ldots < s_m = t\}$ be an arbitrary subdivision of the interval $[s, t]$. Then

$$\sum_{j=1}^{m} \left\| W(t, s_j) - W(t, s_{j-1}) \right\| \leq \sum_{j=1}^{m} \left\| P(t, s_j - t) - P(t, s_{j-1} - t) \right\| +$$

$$+ \sum_{j=1}^{m} \left\{ \int_{s_j}^{t} \left\| W(t, \sigma) \right\| \left\| P(\sigma, s_j - \sigma) - P(\sigma, s_{j-1} - \sigma) \right\| d\sigma \right\} \leq p(t) + 2 \int_{s}^{t} \left( \text{var}_{s} W(t, \cdot) \right) p(\sigma) d\sigma ,$$

where $p(t) = \text{var}_{s} P(t, \cdot)$ for $t \in [a, b]$. Gronwall's inequality yields

$$\left\| W(t, s) \right\| \leq \text{var}_{s} W(t, \cdot) \leq p(t) \exp \left( 2 \int_{s}^{t} p(\sigma) d\sigma \right) < \infty$$

for all $t, s \in [a, b]$, $t \geq s$. It is easy to verify (cf. [2], proof of Theorem 32.2) that for all $t, s \in [a, b]$

$$Y(t, s) = I + \int_{s}^{t} W(\tau, s) d\tau .$$

Furthermore, let $v = \{a = t_0 < t_1 < \ldots < t_p = b; \ a = s_0 < s_1 < \ldots < s_q = b\}$ be an arbitrary net type subdivision of $[a, b] \times [a, b]$. Then according to (6)

$$\sum_{j=1}^{p} \sum_{k=1}^{q} \Delta \Delta_{j,k} Y = \sum_{j=1}^{p} \sum_{k=1}^{q} \left\| Y(t_j, s_k) - Y(t_{j-1}, s_k) - Y(t_j, s_{k-1}) + Y(t_{j-1}, s_{k-1}) \right\| \leq$$

$$\leq \sum_{j=1}^{p} \sum_{k=1}^{q} \left\| \int_{t_{j-1}}^{t_j} (W(\tau, s_k) - W(\tau, s_{k-1})) d\tau \right\| \leq \int_{a}^{b} \sum_{k=1}^{q} \left\| W(\tau, s_k) - W(\tau, s_{k-1}) \right\| d\tau \leq$$

$$\leq \int_{a}^{b} \text{var}_{s} W(\tau, \cdot) d\tau = \int_{a}^{b} \text{var}_{s} W(\cdot, s) = \int_{a}^{b} \text{var}_{s} W(t, \cdot) \exp \left( 2 \int_{s}^{t} p(\sigma) d\sigma \right) d\tau = M < \infty .$$

Thus

$$v(Y) = \sup_{j=1}^{p} \sum_{k=1}^{q} \Delta \Delta_{j,k} Y \leq M < \infty$$

which completes the proof.

**Corollary 1.** There exists $M < \infty$ such that for all $t, s \in [a, b]$

$$\left\| Y(t, s) \right\| + \text{var}_{s} Y(t, \cdot) + \text{var}_{s} Y(\cdot, s) + v(Y) \leq M .$$

**Lemma 2.** The function $K(t, s)$ defined by (5) is of strongly bounded variation on $[a, b] \times [a - r, a]$. 69
Proof. a) $K(a, \cdot) = 0$ on $[a - r, a]$.

b) Let $\{a = t_0 < t_1 < \ldots < t_m = b\}$ be an arbitrary subdivision of $[a, b]$. Then by Corollary 1

$$
\sum_{j=1}^{m} \| K(t_j, a) - K(t_{j-1}, a) \| = \sum_{j=1}^{m} \left\| \int_{t_{j-1}}^{t_j} Y(t_j, \sigma) P(\sigma, a - \sigma) \, d\sigma + \int_{a}^{t_{j-1}} (Y(t_j, \sigma) - Y(t_{j-1}, \sigma)) P(\sigma, a - \sigma) \, d\sigma \right\| \leq M \int_{a}^{b} p(\sigma) \, d\sigma < \infty.
$$

Hence $\var{b} K(\cdot, a) < \infty$.

c) Given a net type subdivision $\{a = t_0 < t_1 < \ldots < t_p = b; a - r = s_0 < s_1 < \ldots < s_q = a\}$ of $[a, b] \times [a - r, a]$, we have by Corollary 1

$$
\sum_{j=1}^{p} \sum_{k=1}^{q} \| K(t_j, s_k) - K(t_{j-1}, s_k) - K(t_j, s_{k-1}) + K(t_{j-1}, s_{k-1}) \| = \\
\sum_{j=1}^{p} \sum_{k=1}^{q} \left\| \int_{a}^{t_{j-1}} (Y(t_j, \sigma) - Y(t_{j-1}, \sigma)) (P(\sigma, s_k - \sigma) - P(\sigma, s_{k-1} - \sigma)) \, d\sigma + \int_{t_{j-1}}^{t_j} Y(t_j, \sigma) (P(\sigma, s_k - \sigma) - P(\sigma, s_{k-1} - \sigma)) \, d\sigma \right\| \\
\leq \int_{a}^{b} \var{b} Y(\cdot, \sigma) + \sup_{\tau \in [a, b]} \| Y(\tau, \sigma) \| \var{a} P(\sigma, \cdot) \, d\sigma \leq M \int_{a}^{b} p(\sigma) \, d\sigma < \infty.
$$

Consequently, $v(K) < \infty$ and this completes the proof of Lemma 2.

The following theorem is a direct consequence of Theorem 3.1 from [5] and of Lemma 2.

**Theorem.** The Cauchy operator $\Phi$ in the variation $-$ of $-$ constants formula (4) is completely continuous.

**References**


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