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NOTE ON THE THEORY OF INDEPENDENCE IN CONTINUOUS GEOMETRIES

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Let L be a reducible or irreducible complete complemented modular lattice. Let 0 be the zero element of L. Let L satisfy the axiom of continuity of addition and multiplication. (See (II), Part I, Axiom III.) Let I be a set of things, and let ϑ be the complete Boolean algebra of the subsets of I. Let a_σ $(\sigma \in I)$ be elements of L. Following (II), Part I, Definition 2.1, let us write $(a_\sigma;\ \sigma \in I) \perp$ if and only if

$$\Sigma\{a_\sigma;\ \sigma \in J\}\ \Sigma\{a_\sigma;\ \sigma \in K\} = 0$$

for any two disjoint subsets, J and K, of I.

By the following Theorem 2.7, VON NEUMANN wished to state a necessary and sufficient condition for $(a_\sigma;\ \sigma \in I) \perp$. It should, however, be observed that, in (I) as well as in (II), Theorem 2.7 was improperly formulated. The author of the present note and L. ROY WILCOX, the writer of (I), exchanged some letters about this matter in 1967. We came to the conclusion that Theorem 2.7 should have been formulated as follows:

Theorem 2.7′. *If $a_\sigma \neq 0$ for every element σ of I, then $(a_\sigma;\ \sigma \in I) \perp$ if and only if $(J \to \Sigma\{a_\sigma;\ \sigma \in J\};\ J \in \vartheta)$ is a one-to-one mapping of ϑ into L carrying the operations $\mathfrak{S}$, $\mathfrak{P}$ into Σ, Π, respectively.*

This is actually proved in (II), p. 14, lines 3 to 12.

On the other hand, it would be wrong to interpret Theorem 2.7 as follows:

Theorem 2.7″. *If $a_\sigma \neq 0$ for every element σ of I, then $(a_\sigma;\ \sigma \in I) \perp$ if and only if there exists a one-to-one mapping of ϑ onto $\{\Sigma\{a_\sigma;\ \sigma \in J\};\ J \in \vartheta\}$ carrying the operations $\mathfrak{S}$, $\mathfrak{P}$ into Σ, Π, respectively.*

It is the main purpose of this note to show that Theorem 2.7″ is false. We shall indeed show that Theorem 2.7″ does hold if I is finite, or if L is of finite length. On the other hand, we shall see that Theorem 2.7″ does not hold generally. More in detail, if L is not of finite length, and if I is countably infinite, then non-zero elements a_σ $(\sigma \in I)$ of L can be chosen so that the backward implication of Theorem 2.7″ does not hold. Hence Theorem 2.7″ does not hold for an L which is not of finite length.

Explaining all this, we shall write $\bigvee$ and $\bigwedge$ instead of Σ and Π, respectively, as is customary to-day; if K is any set, let $\mathfrak{B}(K)$ be the complete Boolean algebra of the subsets of K.

Theorem 1. *Let I be finite. Let there exist a mapping T of $\mathfrak{B}(I)$ onto*

$$\{\bigvee\{a_\sigma;\ \sigma \in J\};\ J \in \mathfrak{B}(I)\}$$

which is a lattice monomorphism of $\mathfrak{B}(I)$ into L. Then $a_\sigma \neq 0$ for every element σ of I, and $(a_\sigma;\ \sigma \in I) \perp$.

Proof. Let $\sigma \in I$. Let J be a subset of I such that

$$T\{\sigma\} = \bigvee\{a_\tau;\ \tau \in J\}\ .$$

Let ϱ be an element of J such that $a_\varrho \neq 0$. It is obvious that this implies that

$$T\{\sigma\} = a_\varrho\ .$$

Let t be a mapping of I into I such that $T\{\sigma\} = a_{t\sigma}$ for every element σ of I. Then $a_{t\sigma} \neq 0$ for $\sigma \in I$. Because T is a monomorphism, t is one-to-one. Because I is finite, t is a mapping of I onto I,

$$\{a_{t\sigma};\ \sigma \in I\} = \{a_\varrho;\ \varrho \in I\}\ ,$$

and

$$a_\varrho \neq 0 \quad \text{for} \quad \varrho \in I\ .$$

Also,

$$a_\varrho = T\{t^{-1}\varrho\} \quad \text{for} \quad \varrho \in I\ .$$

Hence

$$\bigvee\{a_\varrho;\ \varrho \in J\} = T\{t^{-1}\varrho;\ \varrho \in J\}$$

if $J \subset I$, and

$$\bigvee\{a_\varrho;\ \varrho \in J_1\} \wedge \bigvee\{a_\varrho;\ \varrho \in J_2\} = 0$$

if J_1 and J_2 are any disjoint subsets of I. Hence $(a_\sigma;\ \sigma \in I) \perp$, completing the proof.

Theorem 2. *Let L be of finite length. Then Theorem 2.7″ holds for L.*

Proof. Let there exist a mapping of $\mathfrak{B}(I)$ onto $\{\bigvee\{a_\sigma;\ \sigma \in J\};\ J \in \mathfrak{B}(I)\}$ which is a complete lattice monomorphism of $\mathfrak{B}(I)$ into L. Then the $\bigvee\{a_\sigma;\ \sigma \in J\}$, $J \in \mathfrak{B}(I)$, form a sublattice of L which is isomorphic to $\mathfrak{B}(I)$. Because L is of finite length, this sublattice is of finite length. Hence $\mathfrak{B}(I)$ is of finite length. Hence I is finite. By Theorem 1, $(a_\sigma;\ \sigma \in I) \perp$. This result and Theorem 2.7′ imply Theorem 2.7″.

Corollary. *If L is a desarguesian or non-desarguesian projective geometry then Theorem 2.7″ holds for L.*

If A is a subset of L, let us call A *independent* if $\bigvee X \wedge \bigvee Y = 0$ for any two disjoint subsets, X and Y, of A. This implies that A is independent if and only if $(\sigma; \sigma \in A) \perp$.

Lemma 1. *Let A be a set of non-zero elements of L which is countably infinite and independent. Then there exists a non-independent set B of joins of non-void finite subsets of A with the property that there exists a mapping of* $\mathfrak{B}(B)$ *onto*

$$\{\bigvee J;\ J \in \mathfrak{B}(B)\}$$

which is a complete lattice monomorphism of $\mathfrak{B}(B)$ *into L.*

Proof. By Theorem 2.7′, there exists a mapping of $\mathfrak{B}(A)$ onto $\{\bigvee J;\ J \in \mathfrak{B}(A)\}$ which is a complete lattice monomorphism of $\mathfrak{B}(A)$ into L. Let B be a set of joins of non-void finite subsets of A which is a proper superset of A. (E.g., $B = A \cup \cup \{a \vee b\}$ where a and b are two different elements of A.) Then B is non-independent and countably infinite. Because of the latter property, $\mathfrak{B}(B)$ is isomorphic to $\mathfrak{B}(A)$. Also,

$$\{\bigvee J;\ J \in \mathfrak{B}(B)\} = \{\bigvee J;\ J \in \mathfrak{B}(A)\}\ .$$

Hence B has all the properties required.

Theorem 3. *Let L be not of finite length. Let I be countably infinite. Then non-zero elements* a_σ $(\sigma \in I)$ *of L can be chosen so that there exists a mapping of* $\mathfrak{B}(I)$ *onto* $\{\bigvee\{a_\sigma;\ \sigma \in J\},\ J \in \mathfrak{B}(I)\}$ *which is a complete lattice monomorphism of* $\mathfrak{B}(I)$ *into L, while* $(a_\sigma;\ \sigma \in I) \perp$ *does not hold.*

Proof. It is easily seen that there exists a set A of non-zero elements of L which is countably infinite and independent. Let B be a subset of L of the sort considered in Lemma 1. Choose the a_σ $(\sigma \in I)$ so that $(\sigma \to a_\sigma;\ \sigma \in I)$ is a one-to-one mapping of I onto B. Then the a_σ $(\sigma \in I)$ obviously have the property required.

Corollary. *If L is irreducible, and if Case* ∞ *in the sense of Lemma* 7.4 of *Part* I *of* (II) *occurs for L, then Theorem* 2.7″ *does not hold for L.*

References

(I) Continuous geometry. Lecture held by John von Neumann in the Institute for Advanced Study, Princeton, New Jersey, U.S.A., in the academic year 1935—36. Notes by L. Roy Wilcox.

(II) Continuous geometry. By John von Neumann. Edited by Israel Halperin. Princeton University Press, Princeton, 1960.

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