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INTERSECTION GRAPHS OF FINITE ABELIAN GROUPS

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In [1] B. CsÁKÁNY and G. POLLÁK have defined the intersection graphs of groups. (This study was inspired by the definition of intersection graphs of semigroups due to J. BoSÁK.)

Let \( G \) be a group. The intersection graph \( G(G) \) of \( G \) is the undirected graph (without loops and multiple edges) whose vertices are in a one-to-one correspondence with all proper non-trivial subgroups of \( G \) and two vertices are joined by an edge, if and only if the corresponding subgroups of \( G \) have a non-trivial intersection (i.e., an intersection containing a non-unit element).

Here we shall study the intersection graphs of finite Abelian groups. Our main goal is to find out how much information about the structure of such a group can be obtained from its intersection graph.

First we shall prove some lemmas.

**Lemma 1.** Any finite non-trivial Abelian group contains a cyclic subgroup whose order is a prime number.

**Proof.** Any finite Abelian group can be expressed as a direct product of primary cyclic groups, i.e., cyclic groups of the order equal to a power of a prime number. If \( a \) is the generator and \( p^x \) the order of any of these primary cyclic groups, then its subgroup generated by \( a^{p^x-1} \) is cyclic and has the order \( p \), which is a prime number.

Evidently a primary cyclic group can contain only one such subgroup.

**Lemma 2.** The vertex independence number of the graph \( G(G) \) is equal to the maximal number of prime order subgroups of \( G \).

**Proof.** Two distinct prime order subgroups of \( G \) have always a trivial intersection, because such groups contain only one proper subgroup, namely the trivial one. Therefore any system of prime order subgroups of \( G \) corresponds to an independent set in \( G(G) \). Now let us have a maximal independent set in \( G(G) \). Any vertex of this
set corresponds to a subgroup of $G$; this subgroup has a prime order subgroup (Lemma 1). As any two subgroups of $G$ corresponding to vertices of this independent set have trivial intersection, the prime order subgroups in subgroups of $G$ corresponding to distinct vertices of this set must be distinct. This implies that an independent set in $G(G)$ cannot have more elements than the number of prime order subgroups of $G$. Moreover, if some vertex of an independent set in $G(G)$ corresponds to a subgroup of $G$ containing more than one prime order subgroup, the cardinality of this independent set is less than the independence number of the graph $G(G)$.

**Corollary of Lemma 2.** A vertex of $G(G)$ corresponds to a primary cyclic subgroup of $G$, if and only if it belongs to some independent set of $G(G)$ of maximal cardinality.

**Lemma 3.** Let $G$ be a finite Abelian group which is not a direct product of two prime order groups. Let $u, v$ be two vertices of $G(G)$ not joined by an edge and corresponding to primary cyclic subgroups $U, V$ of $G$. Then the orders of $U$ and $V$ are powers of different prime numbers, if and only if there exists a vertex $w$ in $G(G)$ joined with both $u$ and $v$ and with no vertex which is not joined with $u$ and $v$.

**Proof.** Let the orders of $U$ and $V$ be powers of different prime numbers. Let $w$ be the subgroup of $G$ generated by the prime order subgroups of $U$ and $V$; the subgroup $w$ is a proper subgroup of $G$, because $G$ is not a direct product of two prime order groups. The vertex $w$ of $G(G)$ corresponding to $w$ is evidently joined with both $u$ and $v$. Now let some vertex $x$ of $G(G)$ be joined with $w$. This means that $x$ corresponds to a subgroup $X$ of $G$ such that $X \cap W \neq \{e\}$. Let $e \neq a \in X \cap W$; then $a = b^m c^n$, where $b, c$ are generators of $U, V$ respectively. If $p, q$ are orders of $b, c$ respectively, take $a^p = b^{mp} c^{np}$. This is equal to $c^{np}$, because $b^{mp} = e$. According to the assumption, $p, q$ are relatively prime, therefore $c^{np} = e$ implies $np \equiv 0 \pmod{q}$ and $n \equiv 0 \pmod{q}$ which means $c^n = e$ and $a = b^m$. We have either $a = b^m$, or $a^p = c^{np} = e$. As both $a$ and $a^p$ are in $X$, this means that either $X \cap U \neq \{e\}$, or $X \cap V \neq \{e\}$ and $x$ is joined either with $u$, or with $v$.

Now let the orders of $U$ and $V$ be powers of the same prime number $p$; let the order of $U$ be $p^r$, the order of $V$ be $p^s$. Without loss of generality let $r \leq s$. Let $b, c$ be the generators of $U$ and $V$ respectively. Then $c^{ps-rs}$ has the same order $p^r$ as $b$ and the product $bc^{ps-rs}$ has also this order. The primary cyclic subgroup generated by $bc^{ps-rs}$ will be denoted by $W$; evidently it has trivial intersections with $U$ and $V$. Let $X$ be a subgroup of $G$ which has non-trivial intersections with both $U$ and $V$; thus $X \cap U \ni b^r$, $X \cap V \ni c^s$, where $r, s$ are positive integers, $r \equiv 0 \pmod{p^r}$, $s \equiv 0 \pmod{p^s}$. Then $X$ contains also the product $(bc^{ps-rs})^t$, where $t$ is the least common multiple of $r$ and of the greatest common divisor of $p^r - s$ and $s$. This element is evidently different from $e$ and belongs to $W$. Therefore $X \cap W \neq \{e\}$ and $x$ is joined also with with $w$ (which is joined neither with $u$, nor with $v$). As $X$ was chosen arbitrarily, the assertion is proved.
Lemma 4. Let \( \mathcal{G} \) be a direct product of two prime order groups. If these groups have different orders, the graph \( G(\mathcal{G}) \) consists of two isolated vertices. If these groups have equal orders, the graph \( G(\mathcal{G}) \) contains more than two vertices.

Proof follows from the well-known properties of direct products of cyclic groups.

Lemma 5. Let \( \mathcal{G} \) be a finite Abelian group whose order is a power of a prime number \( p \). Then the vertex independence number of \( G(\mathcal{G}) \) is equal to \( \sum_{i=0}^{n-1} p^i \), where \( n \) is the number of direct factors in the expression of \( \mathcal{G} \) as a direct product of primary cyclic groups.

Proof. Let \( \mathcal{G}_1, \ldots, \mathcal{G}_n \) be the factors in the mentioned direct product. Evidently \( \mathcal{G}_i \) contains exactly one prime order subgroup \( S_i \) for \( i = 1, \ldots, n \); therefore it contains \( p - 1 \) elements of prime order. All elements of the order \( p \) (elements of another prime order evidently cannot exist) are products of these elements; thus their number is \( p^n - 1 \). As any prime order subgroup of \( \mathcal{G} \) has the order \( p \) and thus \( p - 1 \) non-unit elements which are all of the order \( p \) and as any two of such subgroups have trivial intersection, there are \( (p^n - 1)/(p - 1) = \sum_{i=0}^{n-1} p^i \) prime order subgroups of \( \mathcal{G} \). According to Lemma 2 this is also the vertex independence number of the graph \( G(\mathcal{G}) \).

Theorem. Let \( \mathcal{G} \) be a finite Abelian group, let \( G(\mathcal{G}) \) be its intersection graph. Knowing the graph \( G(\mathcal{G}) \), we can determine the number of factors in the expression of \( \mathcal{G} \) as a direct product of Sylow groups and the intersection graph of any of these Sylow groups. Moreover, for any of these Sylow subgroups of \( \mathcal{G} \) we can determine the number \( \sum_{i=0}^{n-1} p^i \), where \( p \) is the prime number whose power is the order of this group and \( n \) the number of factors in its expression as a direct product of primary cyclic groups.

Proof. Let \( G(\mathcal{G}) \) be given. We find an independent set \( A \) of vertices in \( G(\mathcal{G}) \) of the maximal cardinality; it corresponds to a system of primary cyclic subgroups of \( \mathcal{G} \) with pairwise trivial intersections (Lemma 2 and its Corollary). According to Lemma 3 (or Lemma 4) we shall decide for any pair of vertices of \( A \) whether the orders of the subgroups of \( \mathcal{G} \) corresponding to these vertices are powers of the same prime number or not. Now let \( B \) be a subset of \( A \) such that all vertices of \( B \) correspond to the subgroups of \( \mathcal{G} \) whose orders are powers of the same prime number \( p \) and any vertex of \( A \setminus B \) corresponds to a subgroup whose order is a power of another prime number. The subgraphs of \( \mathcal{G} \) corresponding to vertices of \( B \) belong to the same Sylow subgroup of \( \mathcal{G} \), the subgroups corresponding to vertices of \( A \setminus B \) belong to other Sylow subgroups. The mentioned Sylow subgroup contains as its non-trivial subgroups exactly all subgroups of \( \mathcal{G} \) which have a non-trivial intersection with at least
one subgroup corresponding to a vertex of \( B \) and have trivial intersections with all subgroups corresponding to vertices of \( A \triangleleft B \). This can be proved simply. The subgroups corresponding to vertices of \( B \) contain as their subgroups all subgroups of \( \mathcal{G} \) of the order \( p \) (any of them contains exactly one such subgroup); therefore any subgroup of \( \mathcal{G} \) of the order equal to a power of \( p \) must have a non-trivial intersection with some of them. Now if a subgroup of \( \mathcal{G} \) has a non-trivial intersection with a subgroup corresponding to a vertex of \( A \triangleleft B \), this intersection contains an element whose order is equal to a power of a prime number different from \( p \) and thus this subgroup is not a subgroup of the mentioned Sylow subgroup. The intersection graph of this Sylow subgroup is therefore the subgraph of \( G(\mathcal{G}) \) induced by the vertex set consisting of \( B \) and all vertices of the vertex set of \( G(\mathcal{G}) \) which are joined with at least one vertex of \( B \) and with no vertex of \( A \triangleleft B \). In this way we can construct intersection graphs of all Sylow subgroups of \( \mathcal{G} \) and thus also recognize the number of these subgroups. According to Lemma 5 we can find \( \sum_{i=0}^{n-1} p^i \) for any of these Sylow subgroups.

**Remark.** By the number \( \sum_{i=0}^{n-1} p^i \) neither \( p \) nor \( n \) is uniquely determined. For example, \( 31 = \sum_{i=0}^{4} 2^i = \sum_{i=0}^{2} 5^i \).

We shall express a conjecture.

**Conjecture.** Two finite Abelian groups with isomorphic intersection graphs are isomorphic.

If this conjecture is true, it suffices to prove it for the groups whose orders are powers of prime numbers.

**Reference**


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