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INTERSECTION GRAPHS OF FINITE ABELIAN GROUPS

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In [1] B. CSÁKÁNY and G. POLLÁK have defined the intersection graphs of groups. (This study was inspired by the definition of intersection graphs of semigroups due to J. BOSÁK.)

Let $\mathfrak{G}$ be a group. The intersection graph $G(\mathfrak{G})$ of $\mathfrak{G}$ is the undirected graph (without loops and multiple edges) whose vertices are in a one-to-one correspondence with all proper non-trivial subgroups of $\mathfrak{G}$ and two vertices are joined by an edge, if and only if the corresponding subgroups of $\mathfrak{G}$ have a non-trivial intersection (i.e., an intersection containing a non-unit element).

Here we shall study the intersection graphs of finite Abelian groups. Our main goal is to find out how much information about the structure of such a group can be obtained from its intersection graph.

First we shall prove some lemmas.

Lemma 1. *Any finite non-trivial Abelian group contains a cyclic subgroup whose order is a prime number.*

Proof. Any finite Abelian group can be expressed as a direct product of primary cyclic groups, i.e., cyclic groups of the order equal to a power of a prime number. If a is the generator and p^α the order of any of these primary cyclic groups, then its subgroup generated by $a^{p^{\alpha-1}}$ is cyclic and has the order p, which is a prime number.

Evidently a primary cyclic group can contain only one such subgroup.

Lemma 2. *The vertex independence number of the graph $G(\mathfrak{G})$ is equal to the maximal number of prime order subgroups of $\mathfrak{G}$.*

Proof. Two distinct prime order subgroups of $\mathfrak{G}$ have always a trivial intersection, because such groups contain only one proper subgroup, namely the trivial one. Therefore any system of prime order subgroups of $\mathfrak{G}$ corresponds to an independent set in $G(\mathfrak{G})$. Now let us have a maximal independent set in $G(\mathfrak{G})$. Any vertex of this set corresponds to a subgroup of $\mathfrak{G}$; this subgroup has a prime order subgroup (Lemma 1). As any two subgroups of $\mathfrak{G}$ corresponding to vertices of this independent set have trivial intersection, the prime order subgroups in subgroups of $\mathfrak{G}$ corresponding to distinct vertices of this set must be distinct. This implies that an independent set in $G(\mathfrak{G})$ cannot have more elements than the number of prime order subgroups of $\mathfrak{G}$. Moreover, if some vertex of an independent set in $G(\mathfrak{G})$ corresponds to a subgroup of $\mathfrak{G}$ containing more than one prime order subgroup, the cardinality of this independent set is less than the independence number of the graph $G(\mathfrak{G})$.

Corollary of Lemma 2. *A vertex of $G(\mathfrak{G})$ corresponds to a primary cyclic subgroup of $\mathfrak{G}$, if and only if it belongs to some independent set of $G(\mathfrak{G})$ of maximal cardinality.*

Lemma 3. *Let $\mathfrak{G}$ be a finite Abelian group which is not a direct product of two prime order groups. Let u, v be two vertices of $G(\mathfrak{G})$ not joined by an edge and corresponding to primary cyclic subgroups $\mathfrak{U}$, $\mathfrak{V}$ of $\mathfrak{G}$. Then the orders of $\mathfrak{U}$ and $\mathfrak{V}$ are powers of different prime numbers, if and only if there exists a vertex w in $G(\mathfrak{G})$ joined with both u and v and with no vertex which is not joined with u and v.*

Proof. Let the orders of $\mathfrak{U}$ and $\mathfrak{V}$ be powers of different prime numbers. Let $\mathfrak{W}$ be the subgroup of $\mathfrak{G}$ generated by the prime order subgroups of $\mathfrak{U}$ and $\mathfrak{V}$; the subgroup $\mathfrak{W}$ is a proper subgroup of $\mathfrak{G}$, because $\mathfrak{G}$ is not a direct product of two prime order groups. The vertex w of $G(\mathfrak{G})$ corresponding to $\mathfrak{W}$ is evidently joined with both u and v. Now let some vertex x of $G(\mathfrak{G})$ be joined with w. This means that x corresponds to a subgroup $\mathfrak{X}$ of $\mathfrak{G}$ such that $\mathfrak{X} \cap \mathfrak{W} \neq \{e\}$. Let $e \neq a \in \mathfrak{X} \cap \mathfrak{W}$; then $a = b^m c^n$, where b, c are generators of $\mathfrak{U}$, $\mathfrak{V}$ respectively. If p, q are orders of b, c respectively, take $a^p = b^{mp} c^{np}$. This is equal to c^{np}, because $b^{mp} = e$. According to the assumption, p, q are relatively prime, therefore $c^{np} = e$ implies $np \equiv 0 \pmod{q}$ and $n \equiv 0 \pmod{q}$ which means $c^n = e$ and $a = b^m$. We have either $a = b^m$, or $a^p = c^{np} \neq e$. As both a and a^p are in $\mathfrak{X}$, this means that either $\mathfrak{X} \cap \mathfrak{U} \neq \{e\}$, or $\mathfrak{X} \cap \mathfrak{V} \neq \{e\}$ and x is joined either with u, or with v.

Now let the orders of $\mathfrak{U}$ and $\mathfrak{V}$ be powers of the same prime number p; let the order of $\mathfrak{U}$ be p^α, the order of $\mathfrak{V}$ be p^β. Without loss of generality let $\alpha \leqq \beta$. Let b, c be the generators of $\mathfrak{U}$ and $\mathfrak{V}$ respectively. Then $c^{p^{\beta-\alpha}}$ has the same order p^α as b and the product $bc^{p^{\beta-\alpha}}$ has also this order. The primary cyclic subgroup generated by $bc^{p^{\beta-\alpha}}$ will be denoted by $\mathfrak{W}$; evidently it has trivial intersections with $\mathfrak{U}$ and $\mathfrak{V}$. Let $\mathfrak{X}$ be a subgroup of $\mathfrak{G}$ which has non-trivial intersections with both $\mathfrak{U}$ and $\mathfrak{V}$; thus $\mathfrak{X} \cap \mathfrak{U} \ni b^r$, $\mathfrak{X} \cap \mathfrak{V} \ni c^s$, where r, s are positive integers, $r \not\equiv 0 \pmod{p^\alpha}$, $s \not\equiv 0 \pmod{p^\beta}$. Then $\mathfrak{X}$ contains also the product $(bc^{p^{\beta-\alpha}})^t$, where t is the least common multiple of r and of the greatest common divisor of $p^{\beta-\alpha}$ and s. This element is evidently different from e and belongs to $\mathfrak{W}$. Therefore $\mathfrak{X} \cap \mathfrak{W} \neq \{e\}$ and x is joined also with with w (which is joined neither with u, nor with v). As $\mathfrak{X}$ was chosen arbitrarily, the assertion is proved.

Lemma 4. *Let $\mathfrak{G}$ be a direct product of two prime order groups. If these groups have different orders, the graph $G(\mathfrak{G})$ consists of two isolated vertices. If these groups have equal orders, the graph $G(\mathfrak{G})$ contains more than two vertices.*

Proof follows from the well-known properties of direct products of cyclic groups.

Lemma 5. *Let $\mathfrak{G}$ be a finite Abelian group whose order is a power of a prime number p. Then the vertex independence number of $G(\mathfrak{G})$ is equal to $\sum_{i=0}^{n-1} p^i$, where n is the number of direct factors in the expression of $\mathfrak{G}$ as a direct product of primary cyclic groups.*

Proof. Let $\mathfrak{G}_1, \ldots, \mathfrak{G}_n$ be the factors in the mentioned direct product. Evidently $\mathfrak{G}_i$ contains exactly one prime order subgroup $\mathfrak{H}_i$ for $i = 1, \ldots, n$; therefore it contains $p - 1$ elements of prime order. All elements of the order p (elements of another prime order evidently cannot exist) are products of these elements; thus their number is $p^n - 1$. As any prime order subgroup of $\mathfrak{G}$ has the order p and thus $p - 1$ non-unit elements which are all of the order p and as any two of such subgroups have trivial intersection, there are $(p^n - 1)/(p - 1) = \sum_{i=0}^{n-1} p^i$ prime order subgroups of $\mathfrak{G}$. According to Lemma 2 this is also the vertex independence number of the graph $G(\mathfrak{G})$.

Theorem. *Let $\mathfrak{G}$ be a finite Abelian group, let $G(\mathfrak{G})$ be its intersection graph. Knowing the graph $G(\mathfrak{G})$, we can determine the number of factors in the expression of $\mathfrak{G}$ as a direct product of Sylow groups and the intersection graph of any of these Sylow groups. Moreover, for any of these Sylow subgroups of $\mathfrak{G}$ we can determine the number $\sum_{i=0}^{n-1} p^i$, where p is the prime number whose power is the order of this group and n the number of factors in its expression as a direct product of primary cyclic groups.*

Proof. Let $G(\mathfrak{G})$ be given. We find an independent set A of vertices in $G(\mathfrak{G})$ of the maximal cardinality; it corresponds to a system of primary cyclic subgroups of $\mathfrak{G}$ with pairwise trivial intersections (Lemma 2 and its Corollary). According to Lemma 3 (or Lemma 4) we shall decide for any pair of vertices of A whether the orders of the subgroups of $\mathfrak{G}$ corresponding to these vertices are powers of the same prime number or not. Now let B be a subset of A such that all vertices of B correspond to the subgroups of $\mathfrak{G}$ whose orders are powers of the same prime number p and any vertex of $A \dot{-} B$ corresponds to a subgroup whose order is a power of another prime number. The subgraphs of $\mathfrak{G}$ corresponding to vertices of B belong to the same Sylow subgroup of $\mathfrak{G}$, the subgroups corresponding to vertices of $A \dot{-} B$ belong to other Sylow subgroups. The mentioned Sylow subgroup contains as its non-trivial subgroups exactly all subgroups of $\mathfrak{G}$ which have a non-trivial intersection with at least one subgroup corresponding to a vertex of B and have trivial intersections with all subgroups corresponding to vertices of $A \dot{-} B$. This can be proved simply. The subgroups corresponding to vertices of B contain as their subgroups all subgroups of $\mathfrak{G}$ of the order p (any of them contains exactly one such subgroup); therefore any subgroup of $\mathfrak{G}$ of the order equal to a power of p must have a non-trivial intersection with some of them. Now if a subgroup of $\mathfrak{G}$ has a nontrivial intersection with a subgroup corresponding to a vertex of $A \dot{-} B$, this intersection contains an element whose order is equal to a power of a prime number different from p and thus this subgroup is not a subgroup of the mentioned Sylow subgroup. The intersection graph of this Sylow subgroup is therefore the subgraph of $G(\mathfrak{G})$ induced by the vertex set consisting of B and all vertices of the vertex set of $G(\mathfrak{G})$ which are joined with at least one vertex of B and with no vertex of $A \dot{-} B$. In this way we can construct intersection graphs of all Sylow subgroups of $\mathfrak{G}$ and thus also recognize the number of these subgroups. According to Lemma 5 we can find $\sum_{i=0}^{n-1} p^i$ for any of these Sylow subgroups.

Remark. By the number $\sum_{i=0}^{n-1} p^i$ neither p nor n is uniquely determined. For example, $31 = \sum_{i=0}^{4} 2^i = \sum_{i=0}^{2} 5^i$.

We shall express a conjecture.

Conjecture. *Two finite Abelian groups with isomorphic intersection graphs ar isomorphic.*

If this conjecture is true, it suffices to prove it for the groups whose orders ar powers of prime numbers.

Reference

[1] *B. Csákány, G. Pollák:* О графе подгрупп конечной группы. Czech. Math. J. *19* (1969), 241–247.

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