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TOLERANCE IN ALGEBRAIC STRUCTURES II

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In [1] the concept of tolerance relation was studied. Here we shall add some new results on this topic.

A tolerance relation on a set is a binary relation which is reflexive and symmetric.

Let an algebraic structure $\mathfrak{A} = \langle A, \mathcal{F} \rangle$ be given, where A is the set of elements of this structure and \mathcal{F} is the set of operations. If a tolerance ξ on A is given, we say that ξ is compatible with \mathfrak{A} , if and only if for any n -ary (n positive integer) operation $f \in \mathcal{F}$ and any $2n$ elements $x_1, \dots, x_n, y_1, \dots, y_n$ of A such that $(x_i, y_i) \in \xi$ for $i = 1, \dots, n$ we have $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \xi$.

We shall prove some theorems.

Theorem 1. *Let $\mathfrak{A} = \langle A, \mathcal{F} \rangle$ be an algebraic structure and ξ a tolerance on A which is compatible with \mathfrak{A} . Then the transitive closure $T\xi$ of ξ is a congruence on \mathfrak{A} .*

Proof. Let $f \in \mathcal{F}$ be an n -ary operation of \mathfrak{A} , where n is a positive integer. Let $x_1, \dots, x_n, y_1, \dots, y_n$ be elements of A such that $(x_i, y_i) \in T\xi$ for $i = 1, \dots, n$. This means that for any $i = 1, \dots, n$ there exists a finite sequence $z_1^{(i)}, \dots, z_{k_i}^{(i)}$ such that $x_i = z_1^{(i)}, y_i = z_{k_i}^{(i)}$ and $(z_j^{(i)}, z_{j+1}^{(i)}) \in \xi$ for $j = 1, \dots, k_i - 1$. Let $k = \max_{1 \leq i \leq n} k_i$. If for some i the number $k_i < k$, we define $z_{k_i+1}^{(i)}, \dots, z_k^{(i)}$ so that all these elements are equal to y_i . Therefore for any $i = 1, \dots, n$ we have the sequence $z_1^{(i)}, \dots, z_k^{(i)}$ so that $z_1^{(i)} = x_i, z_k^{(i)} = y_i$ and $(z_j^{(i)}, z_{j+1}^{(i)}) \in \xi$ for $j = 1, \dots, k - 1$. As ξ is compatible with \mathfrak{A} , we have $(f(z_j^{(1)}, z_j^{(2)}, \dots, z_j^{(n)}), f(z_{j+1}^{(1)}, z_{j+1}^{(2)}, \dots, z_{j+1}^{(n)})) \in \xi$ for $j = 1, \dots, k - 1$ and therefore $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) = (f(z_1^{(1)}, \dots, z_n^{(1)}), f(z_1^{(k)}, \dots, z_n^{(k)})) \in T\xi$. This can be done for any n -ary operation $f \in \mathcal{F}$, where n is a positive integer, therefore $T\xi$ is a congruence on \mathfrak{A} .

Theorem 2. *Let \mathfrak{G} be a group, let ξ be a tolerance which is compatible with \mathfrak{G} as with a semigroup. Then ξ is a congruence on \mathfrak{G} .*

Remark. We say that ξ is compatible with \mathfrak{G} as with a semigroup, if $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$ imply $(x_1x_2, y_1y_2) \in \xi$, saying nothing about the inversion operation.

Proof. In [1] it was proved that if ξ is compatible with \mathfrak{G} as with a group, i.e. if $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$ imply not only $(x_1x_2, y_1y_2) \in \xi$, but also $(x_1^{-1}, y_1^{-1}) \in \xi$, $(x_2^{-1}, y_2^{-1}) \in \xi$, the tolerance ξ is a congruence. Therefore it remains to prove that each tolerance ξ which is compatible with \mathfrak{G} as with a semigroup is compatible with \mathfrak{G} as with a group. Let $a \in \mathfrak{G}$, $b \in \mathfrak{G}$, $(a, b) \in \xi$. Let a^{-1}, b^{-1} be the inverse elements to a, b respectively. As ξ is reflexive, we have $(a^{-1}, a^{-1}) \in \xi$, $(b^{-1}, b^{-1}) \in \xi$. The relations $(a, b) \in \xi$, $(a^{-1}, a^{-1}) \in \xi$ imply $(e, a^{-1}b) \in \xi$, where e is the unit element of \mathfrak{G} . This relation together with $(b^{-1}, b^{-1}) \in \xi$ implies $(b^{-1}, a^{-1}) \in \xi$ and the symmetry of ξ gives $(a^{-1}, b^{-1}) \in \xi$, q.e.d.

Theorem 3. Let $\mathfrak{A} = \langle A, \mathcal{F} \rangle$ be an algebra, let ξ_1, ξ_2 be two tolerances on A which are compatible with \mathfrak{A} . Then the relation $\xi_1 \cap \xi_2$ is also a tolerance compatible with \mathfrak{A} .

Proof. The intersection of two reflexive and symmetric relations is evidently again reflexive and symmetric. Now let $f \in \mathcal{F}$ be an n -ary operation on \mathfrak{A} and let $x_1, \dots, x_n, y_1, \dots, y_n$ be elements of A such that $(x_i, y_i) \in \xi_1 \cap \xi_2$ for $i = 1, \dots, n$. Then $(x_i, y_i) \in \xi_1$ and therefore $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \xi_1$. But also $(x_i, y_i) \in \xi_2$ for $i = 1, \dots, n$ and thus $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \xi_2$. We have obtained that $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \xi_1 \cap \xi_2$. As f was chosen arbitrarily, this holds for any $f \in \mathcal{F}$ and $\xi_1 \cap \xi_2$ is a compatible tolerance on \mathfrak{A} .

This theorem enables us to formulate the following definition.

Let ξ_0 be a tolerance on the set of elements of an algebra \mathfrak{A} . Then the tolerance generated by ξ_0 on \mathfrak{A} is the intersection of all tolerances which are compatible with \mathfrak{A} and contain ξ_0 .

In the case when \mathfrak{A} is a semigroup, it is easy to prove that such a tolerance consists of all pairs $(x_1x_2 \dots x_n, y_1y_2 \dots y_n)$, where n is a positive integer and $(x_i, y_i) \in \xi_0$ for $i = 1, \dots, n$.

Theorem 4. Let \mathfrak{S} be a semigroup with at least three elements. If \mathfrak{S} contains a proper two-side ideal \mathfrak{I} , then there exists a tolerance compatible with \mathfrak{S} which is not a congruence on \mathfrak{S} .

Remark. We must suppose that \mathfrak{S} has at least three elements, because each tolerance on a set with less than three elements is either the equality, or the universal relation, which both are equivalences.

Proof. At first assume that $\mathfrak{S} \div \mathfrak{I}$ contains at least two elements. Choose two distinct elements a, c of $\mathfrak{S} \div \mathfrak{I}$ and one element $b \in \mathfrak{I}$. Let ξ_0 consist of the pairs $(a, b), (b, c)$ and of all pairs of equal elements; this is a tolerance on \mathfrak{S} . Let ξ be the

tolerance on \mathfrak{S} generated by ξ_0 . This tolerance ξ contains obviously (a, b) and (b, c) ; we shall prove that it does not contain (a, c) . Assume that $(a, c) \in \xi$. Then $a = x_1 x_2 \dots x_n$, $c = y_1 y_2 \dots y_n$, where $(x_i, y_i) \in \xi_0$ for $i = 1, \dots, n$ and n is a positive integer. This means that either $x_i = y_i$, or one of these two elements is equal to b . But if $x_i = b$, then $a \in \mathfrak{I}$, because $b \in \mathfrak{I}$ and \mathfrak{I} is an ideal of \mathfrak{S} . As we have supposed that $a \in \mathfrak{S} \setminus \mathfrak{I}$, we obtain a contradiction. Similarly $y_i = b$ implies $c \in \mathfrak{I}$. Thus $x_i = y_i$ must hold for all $i = 1, \dots, n$, which implies $a = c$, which is also a contradiction. Therefore ξ is not transitive and it is not a congruence.

Now assume that $\mathfrak{S} \setminus \mathfrak{I}$ consists only of one element a . If a is not idempotent, $a^2 \in \mathfrak{I}$ and a cannot be expressed as a product of two elements of \mathfrak{S} . We choose two elements b and c of \mathfrak{I} . Let ξ_0 consist of (a, b) , (b, c) and all pairs of equal elements of \mathfrak{S} , let ξ be the tolerance on \mathfrak{S} generated by ξ_0 . Then ξ cannot contain (a, c) , because a cannot be expressed as $x_1 x_2 \dots x_n$.

Now suppose that a is idempotent and $a\mathfrak{I}a$ is a proper subset of \mathfrak{I} . Choose $b \in a\mathfrak{I}a$, $c \in \mathfrak{I} \setminus a\mathfrak{I}a$ and let again ξ_0 consist of (a, b) , (b, c) and all pairs of equal elements. If $(a, c) \in \xi$, this means again $a = x_1 \dots x_n$, $c = y_1 \dots y_n$ and $(x_i, y_i) \in \xi_0$ for $i = 1, \dots, n$. As a is the unique element of \mathfrak{S} not belonging to the ideal \mathfrak{I} , we must have $x_i = a$ for $i = 1, \dots, n$ and therefore for each i either $y_i = a$, or $y_i = b$. As $b \in a\mathfrak{I}a$ and a is idempotent, we have $ab = ba = b$ and therefore $y_1 \dots y_n$ is equal to a or to a power of b . But as $ab = ba = b$, we have $b^k = ab^k a$ for any positive integer k and thus all powers of b are in $a\mathfrak{I}a$. None of them can be equal to c , because $c \in \mathfrak{I} \setminus a\mathfrak{I}a$; we have a contradiction.

Finally suppose that a is idempotent and $a\mathfrak{I}a = \mathfrak{I}$. Then a is a unit element for all elements of \mathfrak{I} . We choose two elements b, c of \mathfrak{I} so that c is no power of b . This can be always done. If \mathfrak{I} contains idempotents, then we choose b idempotent and c will be an arbitrary element of \mathfrak{I} different from b . If \mathfrak{I} does not contain idempotents, it is torsion-free and it suffices to choose c and put $b = c^2$. Now let again ξ_0 consist of (a, b) , (b, c) and all pairs of equal elements and let ξ be the tolerance on \mathfrak{S} generated by ξ_0 . If $(a, c) \in \xi$, this means again $a = x_1 \dots x_n$, $c = y_1 \dots y_n$ and $(x_i, y_i) \in \xi_0$ for $i = 1, \dots, n$. As in the preceding case we must have $x_i = a$ for $i = 1, \dots, n$ and therefore either $y_i = a$, or $y_i = b$. As a is the unit element for \mathfrak{I} , $y_1 \dots y_n$ is a or a power of b . Therefore it cannot be equal to c . Thus we have exhausted all cases and the proof is complete.

Theorem 5. *For each positive integer n there exists a semigroup with n elements such that each tolerance on its element set is compatible with it.*

Proof. We take an arbitrary set S with n elements and for any $x \in S$, $y \in S$ we put $xy = y$. It is easy to prove that the semigroup thus defined is the required semigroup.

This theorem shows that on a semigroup compatible tolerances which are not congruences can exist even if this semigroup has no proper two-side ideal.

Problem 1. Does there exist a semigroup with more than two elements which is not a group and on which each compatible tolerance is a congruence?*)

Problem 2. Does there exist a commutative semigroup such that each tolerance on its element set is compatible with it?

Reference

[1] *B. Zelinka:* Tolerance in algebraic structures. Czech. Math. J. 20 (1970), 179—183.

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*) Added in proof: In the author's paper „*Tolerance on periodic commutative semigroups*“ (to appear in this Journal) this question is answered negatively for periodic commutative semigroups and affirmatively for periodic non-commutative ones.