

Darald J. Hartfiel; Carlton J. Maxson

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## A MATRIX CHARACTERIZATION OF THE MAXIMAL GROUPS IN $\beta_X$

D. J. HARTFIEL and C. J. MAXSON, College Station

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In establishing some setting of this note in the currently published research, we cite that recently, much work has been done on  $\beta_X$ , the semigroup of relations on a set  $X$ . SCHWARZ [6], characterizes the idempotents in this semigroup. Each of these idempotents is then in some maximal group of  $\beta_X$ . SCHWARZ [5] questions whether these groups are in fact isomorphic to symmetric groups on some subset of  $X$ . MONTAGUE and PLEMMONS [3] answer this question in the negative by proving the remarkable result that every finite group is isomorphic to a maximal group of  $\beta_X$  for some  $X$ . PLEMMONS and SCHEIN [4], as well as CLIFFORD [2], extend the result to arbitrary groups.

An essential tool in the arguments of the above results is the Theorem of BIRKHOFF [1] which states that every group is isomorphic to a group of automorphisms on some partially ordered set  $(X, \alpha)$  where  $\alpha$  is the partial order on the set  $X$ . The pivotal point of argument hinges on showing that  $\text{Auto}(X, \alpha)$  is isomorphic to the maximal group in  $\beta_X$  containing  $\alpha$  as its identity.

This paper then provides a matrix characterization of the maximal groups of  $\beta_X$ . This characterization may be utilized to given an alternate proof of the Montague-Plemmons result and in fact the characterization yields a clear view of the interplay of the roles of the automorphisms of  $(X, \alpha)$  and the members of the maximal group in  $\beta_X$  containing  $\alpha$ .

**Results.** Let  $n$  be a positive integer and  $X = \{1, 2, \dots, n\}$ . It is well known that the semigroup of relations on  $X$ , i.e.,  $\beta_X$ , is isomorphic to  $\mathcal{M}$ , the matrices of order  $n$  over a Boolean algebra  $\mathcal{B}$  of order two. This isomorphism maps the relation  $R$  to the matrix  $A$  where  $a_{ij} = 1$  if and only if  $(i, j) \in R$ . For the work herein we consider the equivalent matrix problem of characterizing the maximal groups of matrices in  $\mathcal{M}$ .

Let  $\mathcal{G}$  be a maximal group in  $\mathcal{M}$  with  $I$ , an idempotent, as its identity. Properties concerning  $I$  are contained in the following Theorem of Schwarz [6].

**Theorem.** If  $I$  is idempotent then there is a permutation matrix  $P$  so that

$$P^t I P = \begin{pmatrix} A_1 & 0 & \dots & 0 & 0 \\ A_{2,1} & A_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_{s-1,1} & A_{s-1,2} & \dots & A_{s-1} & 0 \\ A_{s,1} & A_{s,2} & \dots & A_{s,s-1} & A_s \end{pmatrix}$$

where

- (1)  $A_k$  is composed entirely of ones or  $A_k = (0)$ , the 0-matrix of order one.
- (2) Each  $A_{k,j}$  is composed entirely of ones or entirely of zeros.

- (3) The columns of  $\begin{pmatrix} 0 \\ \dots \\ 0 \\ A_k \\ A_{k+1,k} \\ \dots \\ A_{s,k} \end{pmatrix}$  are identical.

- (4) If  $A_{k,j} > 0$  and  $a_k$  a column in  $\begin{pmatrix} 0 \\ \dots \\ 0 \\ A_k \\ \dots \\ A_{s,k} \end{pmatrix}$ ,  $a_j$  a column in  $\begin{pmatrix} 0 \\ \dots \\ 0 \\ A_j \\ \dots \\ A_{s,j} \end{pmatrix}$ ,

then  $a_j \geq a_k$ .

Without loss of generality, we assume  $I$  has the form specified in the above idempotent theorem.

Our characterization of  $\mathcal{G}$  is accomplished through a sequence of lemmas. The first of these lemmas utilizes the following notations.

Let  $E = \{(x_1, x_2, \dots, x_n)'$  where  $x_k \in \mathcal{B}$  for each  $k\}$ . For  $A \in \mathcal{M}$ , let  $\mathcal{N}(A) = \{x \in E \text{ where } Ax = 0\}$  and  $R(A) = \{y \in E \text{ where } Ax = y \text{ for some } x \in E\}$ . An elementary argument provides the initial result.

**Lemma 1.** If  $A \in \mathcal{G}$  then  $\mathcal{N}(A) = \mathcal{N}(I)$  and  $R(A) = R(I)$ . Moreover, if  $x \leq y$  then  $Ax \leq Ay$ .

**Lemma 2.** If  $A \in \mathcal{G}$  then  $A$  is a permutation on  $R(I) = R(A)$ .

**Proof.** If  $Ax = Ay$  for  $x, y \in R(I)$  then  $Ix = A^{-1}Ax = A^{-1}Ay = Iy$ . Since  $x, y \in R(I)$ ,  $x = y$ .

The next lemma utilizes an elementary result concerning the algebraic system  $E$ . For the sake of completeness, we include the necessary background for this result.

If  $\mathcal{A} = \{\alpha_1, \dots, \alpha_r\} \subseteq E$  is such that  $\lambda_i \alpha_i = \sum_{k \neq i} \lambda_k \alpha_k$ , where each  $\lambda_k \in \mathcal{B}$  implies that  $\lambda_1 = \lambda_2 = \dots = \lambda_r = 0$ , then  $\mathcal{A}$  is said to be *independent*. If  $\mathcal{S} \subseteq E$  is closed under addition and  $\mathcal{S}$  contains an independent set  $\mathcal{A}$  such that  $\mathcal{S} = \left\{ \sum_{k=1}^r \lambda_k \alpha_k \mid \lambda_k \in \mathcal{B} \text{ and } \alpha_k \in \mathcal{A} \right\}$ , then  $\mathcal{A}$  is called a *basis* of  $\mathcal{S}$ . The aforementioned result may now be formulated as follows. The proof is left to the reader.

**Lemma 3.** *Every set  $\mathcal{S} \subseteq E$  which is closed under addition has a unique basis.*

Let  $I = (a_1, a_2, \dots, a_n)$ . From the above discussion then,  $R(I)$  being closed under addition has a unique basis, say  $\mathcal{A} = \{a_{i_1}, \dots, a_{i_s}\}$ . As any  $A \in \mathcal{G}$  is one-one and onto  $R(I)$ ,  $A$  must map  $\mathcal{A}$  onto  $\mathcal{A}$ . Thus there is a permutation  $\bar{\pi}$  on  $\{i_1, \dots, i_s\}$  such that  $Aa_{i_k} = a_{\bar{\pi}(i_k)}$ . Since  $A$  is order preserving on  $\mathcal{A}$ ,  $\bar{\pi}$  induces an order automorphism  $\pi$  on  $\mathcal{A}$  by defining  $\pi(a_{i_k}) = a_{\bar{\pi}(i_k)}$ .

**Lemma 4.** *If  $A \in \mathcal{G}$  then there is an order automorphism  $\pi$  of the poset  $\mathcal{A} = \{a_{i_1}, \dots, a_{i_s}\}$  such that  $Aa_{i_k} = a_{\bar{\pi}(i_k)}$ .*

Our next lemma allows us to determine a form for matrices  $A$  in  $\mathcal{G}$ . For this, let  $E_i = \{e_k \mid Ie_k = a_i\}$ .

**Lemma 5.** *Let  $A \in \mathcal{G}$  and let  $\pi$  be the order automorphism of  $\mathcal{A}$  determined by  $A$ . If  $a_j = a_{i_{k_1}} + \dots + a_{i_{k_r}}$  then  $E_j = \{e_k \mid Ae_k = a_{\bar{\pi}(i_{k_1})} + \dots + a_{\bar{\pi}(i_{k_r})}\}$ . In particular, if  $a_j \in \mathcal{A}$  then  $E_j = \{e_k \mid Ae_k = a_{\bar{\pi}(j)}\}$ .*

*Proof.* If  $e_k \in E_j$  then  $Ie_k = a_j$ . Hence  $Ae_k = AIe_k = Aa_j = Aa_{i_{k_1}} + \dots + Aa_{i_{k_r}} = a_{\bar{\pi}(i_{k_1})} + \dots + a_{\bar{\pi}(i_{k_r})}$ . On the other hand, if  $Ae_k = a_{\bar{\pi}(i_{k_1})} + \dots + a_{\bar{\pi}(i_{k_r})}$  then  $Ie_k = A^{-1}Ae_k = A^{-1}a_{\bar{\pi}(i_{k_1})} + \dots + A^{-1}a_{\bar{\pi}(i_{k_r})} = a_{i_{k_1}} + \dots + a_{i_{k_r}} = a_j$ .

Lemma 5 may be utilized to determine a form for each  $A \in \mathcal{G}$ . For this, partition the columns of  $I$  as in the idempotent theorem. Partition the columns of each  $A \in \mathcal{G}$  as those of  $I$ . Lemma 4 now implies that the columns in each partition of  $A$  are identical.

Further, if  $a_j \in \mathcal{A}$  column  $j$  of  $A$  is  $a_{\bar{\pi}(j)}$ . If  $a_j \notin \mathcal{A}$  say  $a_j = a_{i_{k_1}} + \dots + a_{i_{k_r}}$ , then column  $j$  of  $A$  is  $a_{\bar{\pi}(i_{k_1})} + \dots + a_{\bar{\pi}(i_{k_r})}$ . We call any  $A$  so determined an order induced form of  $I$  or simply an  $I$ -form.

It is clear that  $I$  and any order automorphism  $\pi$  of  $\mathcal{A}$  uniquely determine an  $I$ -form  $A$ . The identity map on  $\mathcal{A}$  of course, uniquely determines  $I$ . These  $I$ -forms then provide the characterization of  $\mathcal{G}$ .

**Theorem 1.** *A matrix  $A \in \mathcal{G}$  if and only if  $A$  is an  $I$ -form.*

**Proof.** Let  $\mathcal{F} = \{A \mid A \text{ is an } I\text{-form}\}$ . From the above lemmas,  $\mathcal{G} \subseteq \mathcal{F}$ .

Conversely, pick  $A \in \mathcal{F}$  and let  $\pi$  be the order automorphism of  $\mathcal{A}$  associated with  $A$ . First note that as  $Ix = x$  for each  $x \in R(I)$ ,  $IA = A$ . Now pick  $a_i \in \mathcal{A}$ . Suppose  $a_i = e_{i_1} + e_{i_2} + \dots + e_{i_r}$ . Then  $a_i = Ie_{i_1} + Ie_{i_2} + \dots + Ie_{i_r}$ . Hence  $e_{i_k} \in \mathcal{N}(I)$  or  $e_{i_k} \in E_i$  which in turn implies that  $Ae_{i_k} = 0$  or  $Ae_{i_k} = a_{\pi(i_k)}$ . Thus  $Aa_i = a_{\pi(i)}$ . Hence if  $e_l \in E_i$ ,  $Ale_l = Ae_l$  and as  $A$  is an  $I$ -form,  $AI = A$ . Finally, let  $B$  be the  $I$ -form determined by  $\pi^{-1}$ . It follows that  $ABA_i = BAA_i$  for each  $a_i \in \mathcal{A}$ . Thus, as the product of two  $I$ -forms is an  $I$ -form,  $AB = BA = I$ . Hence  $\mathcal{F}$  is a group with  $I$  as identity which implies that  $\mathcal{F} \subseteq \mathcal{G}$ .

As examples of the utility of this characterization we provide the following.

**Examples.** Let

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = (a_1 a_2 a_3).$$

Note that  $a_1 > a_2$  and  $a_1 > a_3$ . Thus

$$\mathcal{G} = \left\{ (a_1 a_2 a_3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, (a_1 a_3 a_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}.$$

Let

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = (a_1 a_2 a_2 a_3).$$

Then  $a_1 > a_3$  and  $a_2 > a_3$ . Thus

$$\mathcal{G} = \left\{ (a_1 a_2 a_2 a_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, (a_2 a_1 a_1 a_3) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\}.$$

As an immediate consequence of Theorem 1 we have the following isomorphism result.

**Corollary 1.**  $\mathcal{G}$  is isomorphic to  $\text{Auto}(\mathcal{A}, \leq)$ .

From this corollary we see that the Montague-Plemmons result is also a consequence of our characterization by showing the following.

**Corollary 2.**  $\text{Auto}(\mathcal{A}, \leq)$  is isomorphic to  $\text{Auto}(X, \alpha)$  for any partial order  $\alpha$ .

*Proof.* First note that since  $\alpha$  is a partial order,  $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ , i.e. each column of  $I$  is a member of the basis. Suppose  $(i, j) \in \alpha$ . Then by the idempotent theorem  $a_j > a_i$ . Further if  $a_j > a_i$  then again by the idempotent theorem  $(i, j) \in \alpha$ . Thus  $(\mathcal{A}, \leq)$  is the transpose of  $(X, \alpha)$  and hence  $\text{Auto}(\mathcal{A}, \leq)$  is isomorphic to  $\text{Auto}(X, \alpha)$ .

This corollary, together with the characterization theorem, then give the reader some indication as to why  $\text{Auto}(X, \alpha)$  is isomorphic to  $\mathcal{G}$  for  $\alpha$  a partial order.

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*Author's address:* Mathematics Department, Texas AM University, College Station, Texas 77843, U.S.A.