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REMARK TO THE DEPENDENCE OF SOLUTION OF NONLINEAR OPERATOR EQUATION ON THE SPACE IN WHICH IT IS SOLVED

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In this paper, we shall consider reflexive Banach space S and its dual space S', with the norm $\|u\|$ and $|||f|||$, respectively, and with the pairing $\langle u, f \rangle$ $(u \in S, f \in S')$. Throughout the paper, we shall denote the weak convergence in these spaces by w-lim $v_n = v_0$ or $v_n \rightharpoonup v_0$; if the sequence $\{v_n\}$ tends to v_0 in the norm we shall write $v_0 = \lim v_n$ or $v_n \to v_0$.

Let us now introduce the notion of convergence of subspaces; cf. e.g. [3].

Definition 1. Let $H_n \subset S$, $n = 0, 1, 2, \ldots$ be closed linear subspaces of the space S. We say that $\lim H_n = H_0$ or $H_n \to H_0$ if the following conditions are fulfilled:

$$(n_k < n_{k+1},\ v_k \in H_{n_k},\ v_k \rightharpoonup v_0) \Rightarrow v_0 \in H_0\,, \tag{1, i}$$

$$\forall w_0 \in H_0\ \exists \{w_n\},\quad w_n \in H_n\,,\quad w_n \to w_0\,. \tag{1, ii}$$

Lemma 1. *Let* $H_0 = \bigcap_n^{\infty} \overline{[\bigcup_{i=n}^{\infty} H_i]}$. *Then* (1, i) *holds.* (By $[M]$ we denote the minimal linear space which contains M.)

Proof. Let $v_k \in H_{n_k}$, $v_k \rightharpoonup v_0$. Then there exists a constant K such that $\|v_k\| \leqq K$, $n = 0, 1, 2, \ldots$. The set $B_n = \{x \in \overline{[\bigcup_{i=n}^{\infty} H_i]} \mid \|x\| \leqq K\}$ is a convex closed set and hence it is weakly closed, too. This implies that $v_n \in B_n$ for arbitrary n, which proves the lemma.

Example 1. Let $\Omega \subset R_n$ be a bounded domain with infinitely differentiable boundary, and let $S = W^{k,p}(\Omega)$ be the Sobolev space. Let Γ_0 be a relatively open subset of the boundary $\partial\Omega$ with infinitely differentiable relative boundary; let $\Gamma_n \subset \partial\Omega$ be a sequence of relatively open sets such that

(i) $$\Gamma_0 = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \Gamma_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \Gamma_i ;$$

(ii) for any neighbourhood U of Γ_0 there exists n_0 such that for $n \geqq n_0$, $\Gamma_n \subset U$.

Let us put $H_n = \{u \in S \mid u = 0 \text{ on } \Gamma_n\}$, $n = 0, 1, 2, \ldots$.

Then $H_n \to H_0$ in the sense of Definition 1 (see [1]).

Theorem 1. *Let S be a reflexive Banach space. Let the sequence of functionals Φ_n over S, $n = 0, 1, 2, \ldots$ be given with the following properties*:
(i) (*differenciability*): *Φ_n have Gateaux differentials $T_n(x)$ at every point $x \in S$*;
(ii) (*Strong monotonicity*):

$$\forall x, y \in S : \langle x - y, T_n x - T_n y \rangle \geqq C\|x - y\|^p , \quad C > 0 , \quad p > 1 ;$$

(iii) (*boundedness*): *Operators T_n are bounded uniformly with respect to n, i.e., for any K_1 there exists K_2 such that $\|u\| \leqq K_1 \Rightarrow |||T_n(u)||| \leqq K_2$*;
(iv) *for any $u \in S$*

$$T_n u \to T_0 u .$$

Let H_n, $n = 0, 1, 2, \ldots$ be a sequence of closed subspaces of S; let $\omega_n \in S$, $f_n \in S'$ be given, $\omega_n \to \omega_0$, $f_n \to f_0$, $H_n \to H_0$.

Then the problem

(2) $$\forall v \in H_n \langle v, T_n(\omega_n + w) \rangle = \langle v, f_n \rangle , \quad w \in H_n$$

has a unique solution w_n for any n and

(3) $$w_n \to w_0 .$$

Proof. Conditions (i), (ii), (iii) quarantee coercivity of T_n, boundedness, continuity and convexity of Φ_n; of course, it is sufficient to consider the relation

(4) $$\Phi_n(x) - \Phi_n(y) = \int_0^1 \langle x - y, T_n(\tau x + (1 - \tau) y) \rangle \, d\tau ,$$

which holds for arbitrary $x, y \in S$. Hence the existence and unicity of solutions w_n of the problem (2) follows from the general theory, and

(5) $$w_n = v_n - \omega_n ,$$

where

(6) $$\alpha_n = \Phi_n(v_n) - \langle v_n, f_n \rangle = \min_{v \in \omega_n + H_n} (\Phi_n(v) - \langle v, f_n \rangle) ,$$

see [2] or [4].

We shall show that all the points v_n lie in some ball. Of course, putting $y = 0$ we obtain from (ii)

$$\frac{\langle x, T_n(x)\rangle}{\|x\|} \geqq C\|x\|^{p-1} - |||T_n(0)||| .$$

Putting, in the case of necessity, $\tilde{\Phi}_n = \Phi_n - \Phi_n(0)$, we can suppose $\Phi_n(0) = 0$ and using the equality (4) we obtain

$$\Phi_n(x) \geqq C_1\|x\|^p - C_2\|x\| .$$

We have $\omega_n \to \omega_0$, $f_n \to f_0$ and thus we obtain that there exist two constants M, R such that

$$|||f_n||| \leqq M , \quad |||T_n(0)||| \leqq M , \quad \|\omega_n\| \leqq R \quad \text{and} \quad (\|v\| \leqq R \Rightarrow |\Phi_n(v)| \leqq M) .$$

Thus we have

$$\Phi_n(\omega_n) - \langle\omega_n, f_n\rangle \leqq M(1 + R) ,$$

$$\Phi_n(x) - \langle x, f_n\rangle \geqq C_1\|x\|^p - (M + C_2)\|x\| .$$

It follows that for Q sufficiently large

$$\|x\| \geqq Q \Rightarrow \Phi_n(x) - \langle x, f_n\rangle \geqq \Phi_n(\omega_n) - \langle\omega_n, f_n\rangle$$

and hence $\|v_n\| \leqq Q$, $n = 0, 1, 2, \ldots$.

The theorem will be proved if we show that for arbitrary subsequence of $\{v_n\}$ we can choose a "subsubsequence", which tends weakly to v_0. For the sake of simplicity of notation, we shall consider the original sequence v_n instead of its subsequence.

We have $\|v_n\| \leqq Q$; thanks to the reflexivity of S we can choose a subsequence v_{k_n}, which tends weakly to an element $\bar{v}_0 \in S$. It follows from (1, i) that $\bar{v}_0 \in H_0$. Moreover, α_n form a bounded sequence $(|\alpha_n| \leqq \max_{n=1,2} (|\Phi_n(\omega_n)| + MR))$ and so we can suppose that our choice is such that $\alpha_{k_n} \to \bar{\alpha}$.

Rewriting $T_n(\tau x + (1 - \tau) y) = T_n(y) + \{T_n(y + \tau(x - y)) - T_n(y)\}$ we obtain from (4) and (ii)

$$\Phi_n(x) \geqq \Phi_n(y) + \langle x - y, T_n(y)\rangle . \tag{7}$$

So we can write

$$\alpha_{k_n} = \Phi_{k_n}(\bar{v}_0) + \{\Phi_{k_n}(v_{k_n}) - \Phi_{k_n}(\bar{v}_0)\} - \langle v_{k_n}, f_{k_n}\rangle \geqq$$

$$\geqq \Phi_{k_n}(\bar{v}_0) + \langle v_{k_n} - \bar{v}_0, T_{k_n}(\bar{v})\rangle - \langle v_{k_n}, f_{k_n}\rangle .$$

It follows from (4), (iii) and (iv) that $\Phi_n(\bar{v}_0) \to \Phi_0(\bar{v}_0)$ and so $\bar{\alpha} = \liminf \alpha_{k_n} \geqq \geqq \Phi_0(\bar{v}_0) - \langle\bar{v}_0, f_0\rangle \geqq \alpha_0$. (Remember that we have supposed $\Phi_n(0) = 0$.)

Let us suppose $\bar{\alpha} > \alpha_0$. We obtain from (1, ii) that there exists a sequence $u_n \in H_{k_n}$, $u_n \to v_0 - \omega_0$ and hence $z_n = u_n + \omega_n \to v_0$. We have now

$$\Phi_{k_n}(z_n) = \Phi_{k_n}(v_0) + \Phi_{k_n}(z_n) - \Phi_{k_n}(v_0) ;$$

but from (4) and (iii) we have

$$|\Phi_{k_n}(z_n) - \Phi_{k_n}(v_0)| = \left| \int_0^1 \langle z_n - v_0, T_{k_n}(\tau z_n + (1 - \tau) v_0) \rangle \, d\tau \right| \leqq$$

$$\leqq \|z_n - v_0\| \int_0^1 |||T_{k_n}(\tau z_n + (1 - \tau) v_0)||| \, d\tau \to 0 ,$$

and so

$$\Phi_{k_n}(z_n) - \langle z_n, f_{k_n} \rangle \to \Phi_0(v_0) - \langle v_0, f_0 \rangle = \alpha_0 .$$

On the other hand, $\Phi_{k_n}(z_n) - \langle z_n, f_{k_n} \rangle \geqq \alpha_{k_n}$, $\lim (\Phi_{k_n}(z_n) - \langle z_n, f_{k_n} \rangle) \geqq \bar{\alpha} > \alpha_0$, which is a contradiction. Thus we have

$$\Phi_0(\bar{v}_0) - \langle \bar{v}_0, f_0 \rangle = \Phi_0(v_0) - \langle v_0, f_0 \rangle$$

and thanks to the unicity of the point in which the minimum is attained, $\bar{v}_0 = v_0$ which proves the theorem.

Theorem 2. *Let S be a reflexive Banach space and let $\Phi_n, f_n, \omega_n, H_n$, satisfy the assumptions of Theorem 1. Let us denote by w_n the solutions of the problem (2) for $n = 0, 1, 2, \ldots$.*

Then $w_n \to w_0$.

Proof. Because of strong convergence of ω_n it is sufficient to prove that $w_n + \omega_n = v_n \to v_0$. In virtue of Theorem 1, $v_n \to v_0 = w_0 + \omega_0$. On the other hand, we have

$$C\|v_n - v_0\|^p \leqq \langle v_n - v_0, T_n(v_n) - T_n(v_0) \rangle$$

and hence, if we show that $\langle v_n - v_0, T_n(v_n) - T_n(v_0) \rangle \to 0$, the theorem will be proved.

There exists a sequence $\{z_n\}$, $z_n \in H_n + \omega_n$, $z_n \to v_0$. We have

$$\langle v_n - v_0, T_n v_n - T_n v_0 \rangle = \langle v_n - z_n, T_n v_n \rangle + \langle z_n - v_0, T_n v_n \rangle +$$

$$+ \langle v_n - v_0, -T_n v_0 \rangle = \langle v_n - z_n, f_n \rangle + \langle z_n - v_0, T_n v_n \rangle + \langle v_n - v_0, -T_n v_0 \rangle \to 0$$

which proves the theorem.

Example 2. Let S, H_n be the Banach spaces defined in Example 1. Let the function $P = P(x, \xi, \eta)$ be given, defined for $x \in \bar{\Omega}$, $\xi = (\xi_0, \xi_1, \ldots, \xi_N) \in R_{N+1}$, $\eta \in R_1$, with the following properties:

(i) P has all derivatives up to the second order continuous and bounded in $\bar{\Omega} \times \times R_{N+1} \times R_1$.

(ii) $$\exists C > 0 \ \forall x \in \Omega \ \forall \zeta \in R_{N+1} \ \forall \eta \in R_1 \ \forall \xi \in R_{N+1}$$

$$\sum_{i=0}^{N} \frac{\partial^2 P(x, \zeta, \eta)}{\partial \zeta_i \, \partial \zeta_j} \, \xi_i \xi_j \geqq C \sum_{i=0}^{N} \xi_i^2 .$$

Let us define

$$\Phi_n(u) = \int_\Omega P(x, u, \nabla u, \alpha_n) \, dx ,$$

where $\alpha_n \in L_2(\Omega)$, $n = 0, 1, 2, \ldots, \alpha_n \to \alpha_0$.

Than the assumptions of Theorem 2 are fulfilled and

$$\langle v, T_n u \rangle = \int_\Omega v \frac{\partial P}{\partial \xi_0} (x, u, \nabla u, \alpha_n) \, dx + \sum_{i=1}^{\infty} \int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial P}{\partial \xi_i} (x, u, \nabla u, \alpha_n) \, dx .$$

Together with Example 1, we obtain a result concerning the dependence of solution of a boundary value problem not only on the boundary conditions but on the type of these conditions, too.

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