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Jarolím Bureš

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SOME REMARKS ON SURFACES IN THE 4-DIMENSIONAL EUCLIDEAN SPACE

JAROLÍM BUREŠ, Praha

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In the present paper isometric immersions of the 2-dimensional connected oriented Riemannian manifold into the 4-dimensional Euclidean space E^4 by means of invariants of the second order (e.g. Gaussian and mean curvature) are studied. A characterization of surfaces contained in a hyperplane, compact surfaces with constant mean curvature and non-negative Gaussian curvature and surfaces in the 3-dimensional sphere S^3 in E^4 is given.

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1. PRELIMINARIES

Let M^2 be a 2-dimensional connected oriented Riemannian C^∞ – manifold with an isometric immersion

$$x : M^2 \to E^4$$

of M^2 into the 4-dimensional Euclidean space E^4. Let $\mathcal{F}(M^2)$ and $\mathcal{F}(E^4)$ be the bundles of orthonormal frames of M^2 and E^4, respectively. Let $\mathcal{B}$ be the set of elements $b = (p, e_1, e_2, e_3, e_4)$ such that $(p, e_1, e_2) \in \mathcal{F}(M^2)$ and $(x(p), e_1, e_2, e_3, e_4) \in \mathcal{F}(E^4)$, whose orientations are coherent with the canonical one of E^4 with the identification $e_i \equiv \mathrm{d}x(e_i)$, $i = 1, 2$.

$\mathcal{B} \to M^2$ may be considered a principal bundle with the fiber $O(2) \times SO(2)$. Let

$$\tilde{x} : \mathcal{B} \to \mathcal{F}(E^4)$$

be the mapping defined naturally by $\tilde{x}(b) = (x(p), e_1, e_2, e_3, e_4)$.

By means of the immersion x we get on $\mathcal{B}$ the differential forms $\omega^1, \omega^2, \omega_1^2, \omega_1^3, \omega_2^3, \omega_1^4, \omega_2^4, \omega_3^4$ induced from the basic forms and the connection forms on $\mathcal{F}(E^4)$.

On $\mathscr{B}$ we have

$$
\begin{aligned}
(1)\qquad dx &= \omega^1 e_1 + \omega^2 e_2 \,, \\
de_A &= \omega_A^B e_B \,, \quad A, B = 1, 2, 3, 4 \,, \\
\omega_B^A &= -\omega_B^A \,.
\end{aligned}
$$

The system (1) being completely integrable, we have

$$
\begin{aligned}
(2)\qquad d\omega^1 &= \omega_1^2 \wedge \omega^2 \,, \quad d\omega^2 = -\omega_1^2 \wedge \omega^1 \,, \\
d\omega_1^2 &= -\omega_1^3 \wedge \omega_2^3 - \omega_1^4 \wedge \omega_2^4 \,, \\
d\omega_3^4 &= -\omega_1^3 \wedge \omega_1^4 - \omega_2^3 \wedge \omega_2^4 \,, \\
d\omega_i^r &= \omega_i^j \wedge \omega_j^r + \omega_i^t \wedge \omega_t^r \,, \quad i, j = 1, 2 \,, \quad i \neq j \,, \quad r, t = 3, 4 \,, \quad r \neq t \,, \\
\omega^1 \wedge \omega_1^3 &+ \omega^2 \wedge \omega_2^3 = 0 \,, \quad \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0 \,.
\end{aligned}
$$

From the last two equations of (2) and from Cartan's lemma (ω^1 and ω^2 are independent forms on M^2) we get

$$
\begin{aligned}
(3)\qquad \omega_1^3 &= a_1\omega^1 + a_2\omega^2 \,, \quad \omega_1^4 = b_1\omega^1 + b_2\omega^2 \,, \\
\omega_2^3 &= a_2\omega^1 + a_3\omega^2 \,, \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2 \,.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
(4)\qquad da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 &= \alpha_1\omega^1 + \alpha_2\omega^2 \,, \\
da_2 + (a_1 - a_3)\,\omega_1^2 - b_2\omega_3^4 &= \alpha_2\omega^1 + \alpha_3\omega^2 \,, \\
da_3 + 2a_2\omega_1^2 - b_3\omega_3^4 &= \alpha_3\omega^1 + \alpha_4\omega^2 \,, \\
db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 &= \beta_1\omega^1 + \beta_2\omega^2 \,, \\
db_2 + (b_1 - b_3)\,\omega_1^2 + a_2\omega_3^4 &= \beta_2\omega^1 + \beta_3\omega^2 \,, \\
db_3 + 2b_2\omega_1^2 + a_3\omega_3^4 &= \beta_3\omega^1 + \beta_4\omega^2
\end{aligned}
$$

and

$$
\begin{aligned}
(4')\qquad d\alpha_1 - 3\alpha_2\omega_1^2 - \beta_1\omega_3^4 &= A_1\omega^1 + (A_2 - a_2\mathscr{K})\,\omega^2 \,, \\
d\alpha_2 + (\alpha_1 - 2\alpha_3)\,\omega_1^2 - \beta_2\omega_3^4 &= \\
&= (A_2 + a_2\mathscr{K} + b_1 h)\,\omega^1 + (A_3 + a_1\mathscr{K} + b_2 h)\,\omega^2 \,, \\
d\alpha_3 + (2\alpha_2 - \alpha_4)\,\omega_1^2 - \beta_3\omega_3^4 &= (A_3 + a_3\mathscr{K})\,\omega^1 + (A_4 + a_2\mathscr{K})\,\omega^2 \,,
\end{aligned}
$$

$$\mathrm{d}\alpha_4 + 3\alpha_3\omega_1^2 - \beta_4\omega_3^4 = (A_4 - a_2\mathscr{K} + b_3h)\,\omega^1 + A_5\omega^2\,,$$

$$\mathrm{d}\beta_1 = 3\beta_2\omega_1^2 + \alpha_1\omega_3^4 = B_1\omega^1 + (B_2 - b_2\mathscr{K})\,\omega^2\,,$$

$$\mathrm{d}\beta_2 + (\beta_1 - 2\beta_3)\,\omega_1^2 + \alpha_2\omega_3^4 = $$
$$= (B_2 + b_2\mathscr{K} - a_1h)\,\omega^1 + (B_3 + b_1\mathscr{K} - a_2h)\,\omega^2\,,$$

$$\mathrm{d}\beta_3 + (2\beta_2 - \beta_4)\,\omega_1^2 + \alpha_3\omega_3^4 = (B_3 + b_3\mathscr{K})\,\omega^1 + (B_4 + b_2\mathscr{K})\,\omega^2\,,$$

$$\mathrm{d}\beta_4 + 3\beta_3\omega_1^2 + \alpha_4\omega_3^4 = (B_4 - b_2\mathscr{K} - a_3h)\,\omega^1 + B_5\omega^2\,.$$

If $(p, \tilde{e}_1, \tilde{e}_2, e_3, e_4)$ is another frame defined by

$$\tilde{e}_1 = \cos\varphi\,.\,e_1 + \sin\varphi\,.\,e_2\,, \tag{5}$$
$$\tilde{e}_2 = -\sin\varphi\,.\,e_1 + \cos\varphi\,.\,e_2$$

we have the transformation laws:

$$\tilde{a}_1 = a_1\cos^2\varphi + 2a_2\sin\varphi\cos\varphi + a_3\sin^2\varphi\,, \tag{6}$$
$$\tilde{a}_2 = a_2\cos 2\varphi + \tfrac{1}{2}(a_3 - a_1)\sin 2\varphi\,,$$
$$\tilde{a}_3 = a_1\sin^2\varphi - 2a_2\sin\varphi\cos\varphi + a_3\cos^2\varphi\,,$$
$$\tilde{b}_1 = b_1\cos^2\varphi + 2b_2\sin\varphi\cos\varphi + b_3\sin^2\varphi\,,$$
$$\tilde{b}_2 = b_2\cos 2\varphi + \tfrac{1}{2}(b_3 - b_1)\sin 2\varphi\,,$$
$$\tilde{b}_3 = b_1\sin^2\varphi - 2b_2\sin\varphi\cos\varphi + b_3\cos^2\varphi\,.$$

If we have $(p, e_1, e_2, \tilde{e}_3, \tilde{e}_4)$ with

$$\tilde{e}_3 = \cos\Theta\,.\,e_3 + \sin\Theta\,.\,e_4\,, \tag{7}$$
$$\tilde{e}_4 = -\sin\Theta\,.\,e_3 + \cos\Theta\,.\,e_4\,,$$

we obtain

$$\tilde{a}_1 = a_1\cos\Theta + b_1\sin\Theta\,, \tag{8}$$
$$\tilde{a}_2 = a_2\cos\Theta + b_2\sin\Theta\,,$$
$$\tilde{a}_3 = a_3\cos\Theta + b_3\sin\Theta\,,$$
$$\tilde{b}_1 = -a_1\sin\Theta + b_1\cos\Theta\,,$$
$$\tilde{b}_2 = -a_2\sin\Theta + b_2\cos\Theta\,,$$
$$\tilde{b}_3 = -a_3\sin\Theta + b_3\cos\Theta\,.$$

We obtain the following functions on M^2 depending only on the immersion $x : M^2 \to E^4$:

$$\mathscr{H} = (a_1 + a_3)^2 + (b_1 + b_3)^2 \qquad \text{(mean curvature)}, \tag{9}$$
$$\mathscr{K} = a_1 a_3 - a_2^2 + b_1 b_3 - b_2^2 \qquad \text{(Gauss curvature)},$$
$$h = a_1 b_2 - a_2 b_1 + a_2 b_3 - a_3 b_2 \quad \text{(torsion)},$$
$$k = (a_1 b_2 - a_2 b_1)(a_2 b_3 - a_3 b_2) - \tfrac{1}{4}(a_1 b_3 - a_3 b_1)^2 .$$

The Riemannian metric is given by

$$I = (\omega^1)^2 + (\omega^2)^2 .$$

For $\mathscr{K}$ and h we have the relations

$$d\omega_1^2 = -\mathscr{K}\omega^1 \wedge \omega^2 , \quad d\omega_3^4 = -h\omega^1 \wedge \omega^2 . \tag{10}$$

The functions h and k are connected with the invariant form

$$\Phi = (a_1 b_2 - a_2 b_1)(\omega^1)^2 + (a_1 b_3 - a_3 b_1)\,\omega^1\omega^2 + (a_2 b_3 - a_3 b_2)(\omega^2)^2 \tag{11}$$

it is easy to see that $\Phi = 0$ yields the conjugate net of $x(M^2)$.

The mean curvature vector is given by

$$\xi = (a_1 + a_3)\, e_3 + (b_1 + b_3)\, e_4 \tag{12}$$

with $\|\xi\|^2 = \mathscr{H}$.

If $\mathscr{H} \neq 0$ on M^2 we can choose (locally) moving frames (the mean curvature frame) (e_1, e_2, e_3, e_4) so that

$$e_3 = \frac{\xi}{\|\xi\|} .$$

In this case we have $b_1 + b_3 = 0$ and $\mathscr{H} = (a_1 + a_3)^2$.

Example 1. Standard sphere S^2 in E^4. S^2 can be represented by

$$x_1 = a \sin u \cos v , \quad x_2 = a \sin u \sin v , \quad x_3 = a \cos u , \tag{13}$$
$$x_4 = 0 , \quad 0 \leqq u \leqq \pi , \quad 0 \leqq v \leqq 2\pi .$$

Putting

$$e_1 = (\cos u \cos v, \ \cos u \sin v, \ -\sin u, \ 0) , \tag{14}$$
$$e_2 = (\sin v, \ \cos v, \ 0, 0) ,$$
$$e_3 = (\sin u \cos v, \ \sin u \sin v, \ \cos u, \ 0) ,$$
$$e_4 = (0, 0, 0, 1)$$

we have

$$(15)\qquad \begin{aligned} dx &= a\,du\,e_1 + a\sin u\,dv\,e_2\,,\\ \omega^1 &= a\,du\,,\quad \omega^1 = a\sin u\,dv\,,\\ \omega_1^3 &= -\frac{1}{a}\omega^1\,,\quad \omega_2^3 = -\frac{1}{a}\omega^2\,,\quad \omega_1^2 = \frac{1}{a}\operatorname{cotg} u\omega^2\,,\\ \omega_1^4 &= \omega_2^4 = \omega_3^4 = 0\,,\quad a_1 = a_3 = -\frac{1}{a}\,,\quad a_2 = 0\,,\quad b_1 = b_2 = b_3 = 0\,. \end{aligned}$$

Hence

$$\mathscr{K} = \frac{1}{a^2}\,,\quad \mathscr{H} = \frac{4}{a^2}\,,\quad h = 0\,,\quad k = 0\,,\quad \mathscr{H} = 4\mathscr{K}\,.$$

Example 2. The standard flat torus T^2 in E^4. We have

$$(16)\qquad \begin{aligned} &x_1 = a\cos u\,,\quad x_2 = a\sin u\,,\quad x_3 = b\cos v\,,\quad x_4 = b\sin v\,,\\ &a, b > 0\,,\quad 0 \leqq u \leqq 2\pi\,,\quad 0 \leqq v \leqq 2\pi\,. \end{aligned}$$

We take the following frames over T^2

$$(17)\qquad \begin{aligned} e_1 &= (-\sin u, \cos u, 0, 0)\,,\\ e_2 &= (0, 0, -\sin v, \cos v)\,,\\ e_3 &= \frac{1}{\sqrt{(a^2 + b^2)}}(a\cos u,\ a\sin u,\ b\cos v,\ b\sin v)\,,\\ e_4 &= \frac{1}{\sqrt{(a^2 + b^2)}}(b\cos u,\ b\sin u,\ -a\cos v,\ -a\sin v)\,, \end{aligned}$$

obtaining thus

$$(18)\qquad \begin{aligned} &\omega^1 = a\,du\,,\quad \omega^2 = b\,dv\,,\quad \omega_1^2 = \omega_3^4 = 0\,,\\ &\omega_1^3 = \frac{b}{\sqrt{(a^2 + b^2)}}\omega^1\,,\quad \omega_1^4 = \frac{1}{\sqrt{(a^2 + b^2)}}\omega^1\,,\\ &\omega_2^3 = -\frac{a}{\sqrt{(a^2 + b^2)}}\omega^2\,,\quad \omega_2^4 = \frac{1}{\sqrt{(a^2 + b^2)}}\omega^2\,. \end{aligned}$$

Hence

$$\mathscr{K} = 0\,,\quad \mathscr{H} = \frac{1}{a^2} + \frac{1}{b^2}\,,\quad h = 0\,,\quad k = -\frac{1}{4a^2b^2}$$

on T^2.

The geometrical meaning of the functions h, k is expressed by the following

Theorem 1. *Let $x : M^2 \to E^4$ be an isometric imbedding of a connected oriented Riemannian 2-dimensional manifold M^2 into the Euclidean space E^4. If there is a hyperplane E of E^4 such that $x(M^2) \subseteq E$ then $h \equiv k \equiv 0$ on M^2.*

If $h \equiv 0$, $k \equiv 0$ and $\mathscr{K} > 0$ (or $\mathscr{K} < 0$) on M^2, there is a hyperplane E of E^4 such that $x(M^2) \subseteq E$.

Proof. a) If $x(M^2) \subseteq E$, the surface M^2 can be covered by domains $\{U_\alpha\}$ in such a way that, in each U_α, we can choose moving frames (x, e_1, e_2, e_3, e_4) such that e_4 is the constant unit vector field vertical to E. Thus we have $de_4 = 0$ on U_α and $\omega_1^4 = \omega_2^4 = \omega_3^4 = 0$, i.e. $b_1 = b_2 = b_3 = 0$ and $k \equiv 0$, $h \equiv 0$ on U_α.

b) Let $h \equiv k \equiv 0$ on M^2, let us have a covering of M^2 by domains $\{U_\alpha\}$ and in each U_α, a moving frame (x, e_1, e_2, e_3, e_4) so that (1)–(4) holds.

From $h \equiv k \equiv 0$ it follows

$$(19) \qquad a_1b_2 - a_2b_1 + a_2b_3 - a_3b_2 = 0\,,$$
$$(a_1b_2 - a_2b_1)(a_2b_3 - a_3b_2) - \tfrac{1}{4}(a_1b_3 - a_3b_1)^2 = 0$$

that is

$$(20) \qquad a_1b_2 - a_2b_1 = a_3b_2 - a_2b_3\,,$$
$$(a_1b_2 - a_2b_1)^2 + \tfrac{1}{4}(a_1b_3 - a_3b_1)^2 = 0\,.$$

This implies

$$(21) \qquad a_1b_2 = a_2b_1\,, \quad a_1b_3 = a_3b_1\,, \quad a_2b_3 = a_3b_2\,.$$

We can prove:

(I) There exists a normal frame $(\tilde{e}_3, \tilde{e}_4)$ so that for every tangent frame $(\tilde{e}_1, \tilde{e}_2)$ it holds $\tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = 0$ with respect to the frame $(x, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)$.

If (x, e_1, e_2, e_3, e_4) is an arbitrary frame satisfying ($b_1 \neq 0$ similarly for $b_2 \neq 0$ or $b_3 \neq 0$) the equations (21) imply

$$a_3 = \frac{b_3}{b_1}a_1\,, \quad a_2 = \frac{b_2}{b_1}a_1\,.$$

If $a_1 = 0$ we set $\tilde{e}_3 = e_4$, $\tilde{e}_4 = e_3$. Assume that $a_1 \neq 0$. For $e_3' = \cos\Theta \cdot e_3 + \sin\Theta \cdot e_4$, $e_4' = -\sin\Theta \cdot e_3 + \cos\Theta \cdot e_4$ we have

$$b_1' = -a_1\sin\Theta + b_1\cos\Theta\,, \quad b_2' = \frac{b_2}{b_1}(-a_1\sin\Theta + b_1\cos\Theta)\,,$$
$$b_3' = \frac{b_3}{b_1}(-a_1\sin\Theta + b_1\cos\Theta)$$

and taking Θ such that $-a_1\sin\Theta + b_1\cos\Theta = 0$ we obtain the desired result (I).

Let $(x, e_1, e_2, \tilde{e}_3, \tilde{e}_4)$ be a moving frame on U_α such that

$$b_1 = b_2 = b_3 = 0 .$$

The equations (4) imply

$$(22) \qquad a_1\omega_3^4 = \beta_1\omega^1 + \beta_2\omega^2 , \quad a_3\omega_3^4 = \beta_3\omega^1 + \beta_4\omega^2 ,$$
$$a_2\omega_3^4 = \beta_2\omega^1 + \beta_3\omega^2 .$$

There is a frame $(x, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)$ on U_α with $a_2 = 0$. Hence we have for this frame $\beta_1 = \beta_3 = 0$ and

$$a_1\omega_3^4 = \beta_1\omega^1 , \quad a_3\omega_3^4 = \beta_4\omega^2 .$$

If $\mathscr{K} = a_1a_3 \neq 0$ on U_α then $\omega_3^4 = 0$, $de_4 = 0$ i.e. e_4 is a constant vector.

Remark. If $h \equiv k \equiv 0$ on M^2 we can choose a covering of M^2 by domains $\{U_\alpha\}$ in such a way that, in each U_α, we can choose moving frames satisfying $\omega_3^4 = 0$ or $\mathscr{K} = 0$ at each point $p \in U_\alpha$.

Theorem 2. *Let M^2 be an oriented 2-dimensional connected Riemannian manifold, $x : M^2 \to E^4$ an isometric immersion of M^2 into the Euclidean space E^4 and $\mathscr{H}$, $\mathscr{K}$, h, k functions on M^2 defined by (5). Then we have:*

(i) $\mathscr{H} \geqq 4\mathscr{K}$, $h^2 \geqq 2k$.

(ii) *If $\mathscr{H} \neq 0$ and $\mathscr{H} = 4\mathscr{K}$ then $x(M^2)$ is contained in a 2-dimensional sphere $S^2 \subset E^4$.*

(iv) *if $h^2 = 2k$ then $h = k = 0$.*

(iii) *If $\mathscr{H} = (4 - \varepsilon^2)\mathscr{K}$, $\varepsilon \neq 0$ then $\mathscr{H} = \mathscr{K} = 0$ and $x(M^2)$ is contained in a plane $F^2 \subset E^4$.*

(v) *If $\mathscr{H} = 0$ then $\mathscr{K} \leqq 0$.*

Proof. (i) It is

$$\mathscr{H} - 4\mathscr{K} = (a_1 - a_3)^2 + 4a_2^2 + (b_1 - b_3)^2 + 4b_2^2 \geqq 0 ,$$
$$h^2 - 2k = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + \tfrac{1}{2}(a_1b_3 - a_3b_1)^2 \geqq 0 .$$

(ii) If $\mathscr{H} \neq 0$, $\mathscr{H} = 4\mathscr{K}$ then we have $a_1 = a_3$, $a_2 = 0$, $b_1 = b_2 = b_3 = 0$ from (i).

(iii) From $\mathscr{H} = (4 - \varepsilon^2)\mathscr{K}$ it follows that $a_1 = a_2 = a_3 = 0$, $b_1 = b_2 = b_3 = 0$, i.e. $\mathscr{H} = \mathscr{K} = 0$ and $x(M^2)$ is a submanifold of a plane from E^4.

(iv) and (v) follows immediately from (i).

Theorem 3. *Let $x : M^2 \to E^4$ be an isometric immersion of a compact connected oriented 2-dimensional Riemannian manifold into the Euclidean space E^4 such that:*

(i) $\mathscr{K} > 0$ *and* $\mathscr{H} = \text{const.}$ *on* M^2,

(ii) *there exists a covering of* M^2 *by domains* $\{U_\alpha\}$ *such that, in each* U_α, *it holds* $\omega_3^4 = 0$ *with respect to the mean curvature frame (i.e. the torsion form of* x *is zero).*

Then $x(M^2)$ *is a 2-dimensional sphere in* E^4.

Proof. From the inequality $\mathscr{K} > 0$ on M^2 it follows immediately that $\mathscr{H} > 0$ on M^2 and, for each U_α, we may consider the mean curvature frame (x, e_1, e_2, e_3, e_4). In virtue of (i) it is $\mathrm{d}\mathscr{H} = 0$, i.e.

$$(23) \qquad \begin{aligned} (a_1 + a_3)(\alpha_1 + \alpha_3) &= 0\,, \\ (a_1 + a_3)(\alpha_2 + \alpha_4) &= 0 \end{aligned}$$

and $a_1 + a_3 \neq 0$ implies $\alpha_1 + \alpha_3 = 0$, $\alpha_2 + \alpha_4 = 0$. From (4) we have

$$(a_1 + a_3)\,\omega_3^4 = (\beta_1 + \beta_3)\,\omega^1 + (\beta_2 + \beta_4)\,\omega^2$$

and by virtue of $\omega_3^4 = 0$ we get

$$\beta_1 + \beta_3 = 0\,, \quad \beta_2 + \beta_4 = 0\,.$$

Using the equations (4′) for this case, we obtain

$$(24) \qquad \begin{aligned} A_1 + A_3 &= -a_3\mathscr{K}\,, \quad A_2 + A_4 = 0\,, \quad A_3 + A_5 = -a_1\mathscr{K}\,, \\ B_1 + B_3 &= -b_3\mathscr{K}\,, \quad B_2 + B_4 = 0\,, \quad B_3 + B_5 = -b_1\mathscr{K}\,. \end{aligned}$$

Let τ be the 1-form on M^2 defined by

$$\tau = - * \mathrm{d}\mathscr{K}$$

Then

$$\mathrm{d}\tau = (\mathscr{K}(\mathscr{H} - 4\mathscr{K}) + 4(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2))\,\mathrm{d}V\,.$$

Stokes' theorem implies

$$\int_{M^2} \mathrm{d}\tau = 0$$

i.e.

$$\int_{M^2} [\mathscr{K}(\mathscr{H} - 4\mathscr{K}) + 4(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2)]\,\mathrm{d}V = 0$$

and from $\mathscr{K} > 0$, $\mathscr{H} \geqq 4\mathscr{K}$ it follows $\mathscr{H} = 4\mathscr{K}$ and $x(M^2)$ is a 2-dimensional sphere in E^4.

Remark. If the condition (i) of Theorem 3 is replaced by

(i′) $$\mathscr{K} \geqq 0 \quad \text{and} \quad \mathscr{H} = \text{const} > 0$$

while the condition (ii) remains unchanged, $x(M^2)$ is etiher a sphere or $\mathscr{K} = 0$ holds on M^2.

2. SURFACES IN S^3

Let $x(M^2)$ be a submanifold of the 3-dimensional sphere with the center at the point S and with diameter $1/r$. If $\{U_\alpha\}$ is a covering of M^2 by domains U_α and (x, e_1, e_2, e_3, e_4) are orthogonal frame fields on each U_α with $x \in U_\alpha$ and

$$x = S - \frac{1}{r} e_4$$

then the equations (1)–(4) are satisfied.

Especially

$$\mathrm{d}x = -\frac{1}{r} \mathrm{d}e_4 ,$$

$$\omega^1 e_1 + \omega^2 e_2 = \frac{-1}{r} (-\omega_1^4 e_1 - \omega_2^4 e_2 - \omega_3^4 e_3) .$$

Hence

(25) $$\omega_1^4 = r\omega^1 , \quad \omega_2^4 = r\omega^2 , \quad \omega_3^4 = 0 \quad (b_1 = b_3 = r,\ b_2 = 0)$$

and from (2) we get

$$\omega_1^3 = a_1\omega^1 + a_2\omega^2 , \quad \omega_2^3 = a_2\omega^1 + a_3\omega^2 .$$

Thus

$$\mathscr{H} = (a_1 + a_3)^2 + 4r^2 , \quad \mathscr{K} = a_1 a_3 - a_2^2 + r^2 ,$$

$$h = 0 , \quad k = -r^2(a_2^2 + \tfrac{1}{4}(a_1 - a_3)^2) , \quad \beta_1 = \beta_2 = \beta_3 = 0 .$$

Lemma. *A compact surface $M^2 \subset S^3$ is a flat torus if and only if it holds, on M^2*

$$\mathscr{H} = \text{const} , \quad \mathscr{K} = 0 .$$

Proof. If $\mathscr{K} = 0$ on M^2 then

$$a_1 a_3 - a_2^2 = -r^2 < 0$$

There is a covering of M^2 by domains $\{U_\alpha\}$ such that, in each there is a field of tangent frames with $a_2 = 0$.

This implies $a_1 = \text{const}$, $a_3 = \text{const}$, $a_1 \neq a_3$, $(a_1 - a_3)\,\omega_1^2 = 0$ implies $\omega_1^2 = 0$, and M^2 is a flat torus.

Theorem 4. *Let $x : M^2 \to E^4$ be an isometric immersion of compact connected oriented 2-dimensional Riemannian manifold into E^4 with $x(M^2) \subset S^3$. Further suppose that $\mathscr{K} \geqq 0$ and $\mathscr{H} = \text{const}$ on M^2.*

Then either $\mathscr{K} = 0$ on M^2 and $x(M^2)$ is a flat torus or $\mathscr{H} = 4\mathscr{K}$ on M^2 and $x(M^2)$ is a 2-dimensional sphere.

Proof. There is a covering of M^2 by domains $\{U_\alpha\}$ with moving frames (x, e_1, e_2, e_3, e_4) in each U_α, such that (1)–(4) and (25) are satisfied.

Then

$$\omega_3^4 = 0\,, \quad \omega_1^4 = r\omega^1\,, \quad \omega_2^4 = r\omega^2\,, \quad \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0\,, \quad h = 0\,.$$

From $\mathscr{H} = \text{const}$ it follows

$$0 = \mathrm{d}(a_1 + a_3) = (\alpha_1 + \alpha_3)\,\omega^1 + (\alpha_2 + \alpha_4)\,\omega^2$$

i.e.

$$\alpha_1 + \alpha_3 = 0\,, \quad \alpha_2 + \alpha_4 = 0\,.$$

For the 1-form τ on M^2 defined by

$$\tau = -*\,\mathrm{d}\mathscr{K}$$

it is

$$\mathrm{d}\tau = \left(\mathscr{K}(\mathscr{H} - 4\mathscr{K}) + 4(\alpha_1^2 + \alpha_2^2)\right)\mathrm{d}V$$

and from Stokes' theorem

$$\int_{M^2} [\mathscr{K}(\mathscr{H} - 4\mathscr{K}) + 4(\alpha_1^2 + \alpha_2^2)]\,\mathrm{d}V = 0$$

we obtain either

(A) $$\alpha_1 = 0\,, \quad \alpha_2 = 0\,, \quad \mathscr{K} = 0$$

or

(B) $$\alpha_1 = 0\,, \quad \alpha_2 = 0\,, \quad \mathscr{H} - 4\mathscr{K} = 0\,.$$

By Lemma, M^2 is a flat torus in the case (A) while in the case (B) we have

$$a_1 = a_3 = a\,, \quad \mathscr{H} = 4(a^2 + r^2)\,, \quad \mathscr{K} = a^2 + r^2\,,$$

$$\omega_1^3 = a\,\omega^1\,, \quad \omega_2^3 = a\omega^2\,, \quad \omega_1^4 = r\omega^1\,, \quad \omega_2^4 = r\omega\,.$$

If $a \neq 0$ then $\mathrm{d}(x + a^{-1}e_3) = 0$, i.e. $x + a^{-1}e_3$ is the center of $M^2 \equiv S^2$ with the radius a.

For $a = 0$. M^2 is a great sphere in S^3.

Remark: If $x(M^2) \subset S^3$ with $k = 0$ on M^2 then $x(M^2)$ is a submanifold of a 2-dimensional sphere $S^2 \subset S^3$.

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Author's address: 118 00 Praha 1, Malostranské nám. 25, ČSSR (Matematicko-fyzikální fakulta UK).