Harald K. Wimmer; Allen D. Ziebur

Remarks on inertia theorems for matrices

Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 4, 556–561

Persistent URL: http://dml.cz/dmlcz/101351

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

REMARKS ON INERTIA THEOREMS FOR MATRICES

HARALD K. WIMMER, Graz, and ALLEN D. ZIEBUR, Binghamton

(Received March 26, 1974)

1. INTRODUCTION

In this note we give a unified treatment of two inertia results on the Ljapunov matrix equation

$$A^*H + HA = C\,, \quad C \geqq 0 \text{ (positive semidefinite)}\,, \quad H = H^*\,.$$

For a complex $n \times n$ matrix A the inertia, In A, of A is defined as the triple

$$\operatorname{In} A = (\pi(A), \nu(A), \delta(A))$$

where $\pi(A)$, $\nu(A)$ and $\delta(A)$ are respectively the numbers of eigenvalues of A with positive, negative and vanishing real part. If $\{\lambda_j \mid j = 1, 2, \ldots, k\}$ is the set of distinct eigenvalues of A, then A can be written in the form (see e.g. [11])

(1) $$A = \sum_{j=1}^{k} (\lambda_j P_j + N_j)$$

where $\{N_j\}$ is a set of nilpotent matrices and $\{P_j\}$ is a set of projection matrices such that

$$\sum_{j=1}^{k} P_j = I\,, \quad P_i P_j = P_j P_i = \delta_{ij} P_i\,, \quad P_i N_j = N_j P_i = \delta_{ij} N_i\,.$$

Equation (1) is easily derived from the Jordan form of A. We define

$$P_+ = \sum_{\mathrm{Re}\lambda_j > 0} P_j \quad \text{and} \quad P_- = \sum_{\mathrm{Re}\lambda_j < 0} P_j\,.$$

In the case $\delta(A) = 0$ we have $P_+ + P_- = I$. – H shall always denote a hermitian $n \times n$ matrix. $\delta(H) = 0$, then means H is nonsingular.

Our main tool will be the following theorem.

Theorem 1. *If A has no eigenvalues on the imaginary axis and*

$$A^*H + HA = C \tag{2}$$

holds, then

$$P_+^*H - HP_- = \frac{1}{2\pi}\int_{-\infty}^{\infty} [(A - iyI)^{-1}]^*C(A - iyI)^{-1}\,\mathrm{d}y\,. \tag{3}$$

Starting from (3) we will prove the following inertia theorems.

Theorem 2 [1, p. 432]. *Let A be a matrix with $\delta(A) = 0$. If*

$$A^*H + HA = C\,, \quad C \geqq 0\,,$$

then

$$\pi(H) \leqq \pi(A) \quad \text{and} \quad \nu(H) \leqq \nu(A)\,. \tag{4}$$

Theorem 3 [2], [9]. *If $A^*H + HA = C$, $C \geqq 0$ and*

$$\operatorname{rank}[C, A^*C, A^{*2}C, \ldots, A^{*^{n-1}}C] = n\,, \tag{5}$$

then $\operatorname{In} A = \operatorname{In} H$ *and* $\delta(A) = \delta(H) = 0$.

There are applications of Theorem 3 to continued fractions [10] and to the linear vibration equation [9].

2. TWO LEMMAS

For the proof of Theorem 1 we need the following lemma.

Lemma 1. *Let P_1 and P_2 be two $n \times n$ matrices with*

$$\operatorname{rank} P_1 + \operatorname{rank} P_2 \geqq n\,. \tag{6}$$

If H satisfies

$$P_1^*HP_1 \geqq 0 \quad \text{and} \quad P_2^*HP_2 \leqq 0\,, \tag{7}$$

then

$$\pi(H) \leqq \operatorname{rank} P_1 \quad \text{and} \quad \nu(H) \leqq \operatorname{rank} P_2\,. \tag{8}$$

Proof. Let H have the spectral decomposition

$$H = \sum_{r=1}^{h} \mu_r Q_r$$

where μ_r are the eigenvalues of H and the Q_r's are hermitian projection matrices with $Q_r Q_s = \delta_{rs} Q_r$. We put

$$Q_+ = \sum_{\mu_r > 0} Q_r \quad \text{and} \quad Q_- = \sum_{\mu_r < 0} Q_r .$$

We show that

$$(9) \qquad Q_+ \mathbf{C}^n \cap P_2 \mathbf{C}^n = \{0\} .$$

Suppose that $Q_+ u = P_2 v$, then

$$(10) \qquad (HQ_+ u, Q_+ u) = \sum_{\mu_r > 0} \mu_r (Q_r u, Q_r u) \geqq 0 .$$

On the other hand

$$(HQ_+ u, Q_+ u) = (HP_2 v, P_2 v) = (P_2^* H P_2 v, v) \leqq 0 .$$

Thus $(HQ_+ u, Q_+ u) = 0$ and by (10) $(Q_r u, Q_r u) = 0$ for each r with $\mu_r > 0$ and therefore $Q_+ u = 0$. (9) implies $\operatorname{rank} Q_+ + \operatorname{rank} P_2 \leqq n$. Similarly $\operatorname{rank} Q_- + \operatorname{rank} P_1 \leqq n$. The inequalities (8) are now immediate consequences of (6).

Lemma 2. *Let P_1 and P_2 be two $n \times n$ matrices with $P_1 + P_2 = I$ and $P_i P_j = \delta_{ij} P_i$, $i, j = 1, 2$. If H satisfies*

$$(11) \qquad P_1^* H - H P_2 > 0 \text{ (positive definite)},$$

then In H *is given by*

$$\delta(H) = 0 , \quad \pi(H) = \operatorname{rank} P_1 , \quad \nu(H) = \operatorname{rank} P_2 .$$

Proof. Suppose $Hv = 0$, then $(v(P_1^* H - HP_2), v) = 0$ and because of (11) $v = 0$. This means $\delta(H) = 0$ and $\pi(H) + \nu(H) = n$, so that in (8) the equality signs hold.

3. PROOFS

Proof of Theorem 1. Let Γ be a positively-orientated simple closed curve that consists of a segment of the imaginary axis and of a left semi-circle of radius R around the origin. If R is greater than the spectral radius of A, then

$$(12) \qquad \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} \, dz = P_- \quad \text{and} \quad \frac{1}{2\pi i} \int_\Gamma (zI + A)^{-1} \, dz = P_+ .$$

The integrals in (12) exist and the formulas follow easily from (see e.g. [11])

$$(zI - A)^{-1} = \sum_{j=1}^{k} [(z - \lambda_j)^{-1} P_j + M_j] .$$

For $z \in \Gamma$ we write (2) as

$$(zI + A^*)^{-1} H + H(A - zI)^{-1} = (A^* + zI)^{-1} C(A - zI)^{-1} ,$$

divide both sides of this equation by $2\pi i$ and integrate around Γ. The integrals on the left-hand side are evaluated with (12) and since the right-hand side is $O(z^{-2})$ at infinity, we obtain (3). — Let us remark that (3) is a generalisation of a result of SMITH [5, p. 425] which was stated for the case of a stable matrix A, i.e. $P_+ = 0$, $P_- = I$.

Proof of Theorem 2. For $C \geqq 0$ the matrix

$$M = \int_{-\infty}^{\infty} [(A - iyI)^{-1}]^* C(A - iyI)^{-1} \, \mathrm{d}y \tag{13}$$

is also positive semidefinite, so $P_+^* H - HP_- \geqq 0$. If we put $P_1 = P_+$ and $P_2 = P_-$ and observe that rank $P_+ = \pi(A)$ and rank $P_- = \nu(A)$, then the inequalities (4) follow from Lemma 1.

Proof of Theorem 3. We first show that $\delta(A) = 0$. Assume the contrary, then there is a u, $u \neq 0$, and a real α such that $Au = i\alpha u$. Let r be a nonnegative integer, then $A^*(A^{*r}HA^r) + (A^{*r}HA^r) A = A^{*r}CA^r$. Hence

$$(A^{*r}CA^r u, u) = (-i\alpha + i\alpha)(A^{*r}HA^r u, u) = 0 .$$

$C \geqq 0$ implies $u^*A^*C = 0$ for $r = 0, 1, \dots, n - 1$. Thus rank $(C, A^*C, \dots$ $\dots, A^{*n-1}C) < n$, which contradicts to (5). Now that we know that $\delta(A) = 0$, we can write equation (3). We next show that $M > 0$ where M is given by (13). Suppose u is a vector such that $(Mu, u) = 0$. Then $((A^* + iyI)^{-1} C(A - iyI)^{-1} u, u) =$ $= 0$ or $C(A - iyI)^{-1} u = 0$ for all real y. Therefore

$$C(zI - A)^{-1} u = 0 \tag{14}$$

holds for all complex z which are not eigenvalues of A. Multiplying (14) by z^r and integrating around a curve which surrounds the eigenvalues of A we find that $CA^r u = 0$, $r = 0, 1, \dots, n - 1$. (5) implies $u = 0$ which means $M > 0$. Theorem 3 now follows directly from Lemma 2.

The important special case of Theorem 3 where C is a positive definite matrix is due to TAUSSKY [7] and OSTROWSKI and SCHNEIDER [4].

4. STEIN'S EQUATION

Theorems corresponding to those on Ljapunov's equation (2) can be derived for Stein's equation

$$(15) \qquad A^*HA - H = C\,.$$

If A is given in the form (1), we define

$$P_c = \sum_{|\lambda_j|<1} P_j \quad \text{and} \quad P_x = \sum_{|\lambda_j|>1} P_j\,.$$

Let Δ be the positively orientated unit circle. Suppose A has no eigenvalue of modulus 1. Then

$$(16) \qquad P_c = \frac{1}{2\pi i}\int_\Delta (zI - A)^{-1}\,\mathrm{d}z \quad \text{and} \quad P_x = \frac{1}{2\pi i}\int_\Delta A(zA - I)^{-1}\,\mathrm{d}z\,.$$

For $z \in \Delta$ write (15) as

$$HA(zA - I)^{-1} + (A^* - zI)^{-1}\,H = (A^* - zI)^{-1}\,C(zA - I)^{-1}\,.$$

Using (16) we obtain

$$(17) \qquad HP_x - P_c^*H = \frac{1}{2\pi i}\int_\Delta (A^* - zI)^{-1}\,C(zA - I)^{-1}\,\mathrm{d}z =$$

$$= \frac{1}{2\pi}\int_\Delta [(A - e^{-i\theta}I)^{-1}]^*\,C(A - e^{-i\theta}I)^{-1}\,\mathrm{d}\theta\,.$$

Equation (17) is a generalisation of another result of Smith [6, p. 214]. There it was assumed that $P_c = I$ and $P_x = 0$. – By the same method we used for Theorem 3 we can refine a theorem which is mentioned in [8].

Theorem 4. *If* $A^*HA - H = C$, $C \geqq 0$ *and* $\operatorname{rank}(C, A^*C, \ldots, A^{*n-1}C) = n$, *then* A *has no eigenvalues of modulus* 1. *The number of eigenvalues of* A *with modulus less [greater] than* 1 *is equal to the number of negative [positive] eigenvalues of* H.

The results derived in this note for the equations (2) and (15) can not be extended to the more general matrix equation

$$\sum_{\varrho,\sigma=0}^{m} c_{\varrho\sigma} A^{*\varrho} H A^{\sigma} = C, \quad c_{\varrho\sigma} = \overline{c_{\sigma\varrho}}\,,$$

as the following example shows. Take

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$A^T H_1 A + H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} > 0, \quad A^T H_2 A + H_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} > 0,$$

but $\operatorname{In} H_1 \neq \operatorname{In} H_2$. — Generalisations of inertia theorems of a different type are contained in [3].

References

[1] *D. Carlson* and *H. Schneider:* Inertia theorems for matrices: The semidefinite case, J. Math. Anal. Appl. *6* (1963), 430—446.

[2] *Ch.-T. Chen:* A generalisation of the inertia theorem, SIAM J. Appl. Math. *25* (1973), 158—161.

[3] *R. D. Hill:* Inertia theory for simultaneously triangulable complex matrices, Linear Algebra Appl. *2* (1969), 131—142.

[4] *A. Ostrowski* and *H. Schneider:* Some theorems on the inertia of general matrices, J. Math. Anal. Appl. *4* (1962), 72—84.

[5] *R. A. Smith:* Bounds for quadratic Lyapunov functions, J. Math. Anal. Appl. *12* (1965), 425—435.

[6] *R. A. Smith:* Matrix calculations for Lyapunov quadratic forms, J. Diff. Equations *2* (1966), 208—217.

[7] *O. Taussky:* A generalization of a theorem by Lyapunov, J. Soc. Ind. Appl. Math. *9* (1961), 640—643.

[8] *O. Taussky:* Matrices C with $C^n \to 0$, J. Algebra *1* (1964), 5—10.

[9] *H. K. Wimmer:* Inertia theorems for matrices, controllability and linear vibrations, Linear Algebra Appl. *8* (1974), 337—343.

[10] *H. K. Wimmer:* An inertia theorem for tridiagonal matrices and a criterion of Wall on continued fractions, Linear Algebra Appl. *9* (1974), 41—44.

[11] *A. D. Ziebur:* On determining the structure of A by analysing e^{At}, SIAM Review *12* (1970), 98—102.

Authors' addresses: H. K. Wimmer, Technische Hochschule Graz, 2 Math. Institut, A-8010 Graz, Kopernikusgasse 24, Österreich; A. D. Ziebur, State University of New York at Binghamton, Department of Mathematics, Binghamton, N. Y. 13901, U.S.A.