

Bohdan Zelinka

Tolerances and congruences on tree algebras

Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 4, 634–637

Persistent URL: <http://dml.cz/dmlcz/101358>

Terms of use:

© Institute of Mathematics AS CR, 1975

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

TOLERANCES AND CONGRUENCES ON TREE ALGEBRAS

BOHDAN ZELINKA, Liberec

(Received September 30, 1974)

The concept of tolerance was introduced by E. C. ZEEMAN [3] and studied on various types of algebras in [4], [5], [6], [7].

A tolerance is a reflexive and symmetric binary relation on a set.

Let $\mathfrak{A} = \langle A, \mathcal{F} \rangle$ be an algebra; A is its set of elements, \mathcal{F} is the set of operations on it. Let ξ be a tolerance on A . We say that ξ is compatible with \mathfrak{A} , if and only if the following condition is satisfied: If $f \in \mathcal{F}$ is an n -ary operation, where n is a positive integer, and $x_1, \dots, x_n, y_1, \dots, y_n$ are elements of A such that $(x_i, y_i) \in \xi$ for $i = 1, \dots, n$, then $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \xi$.

Tree algebras were introduced by L. NEBESKÝ [1]. A tree algebra (M, P) is an algebra with a non-empty finite set M of elements and with a ternary operation P satisfying the following axioms:

- I. $P(u, u, v) = u$;
- II. $P(u, v, w) = P(v, u, w) = P(u, w, v)$;
- III. $P(P(u, v, w), v, x) = P(u, v, P(w, v, x))$;
- IV. if $P(u, v, x) \neq P(v, w, x) \neq P(u, w, x)$, then $P(u, v, x) = P(u, w, x)$.

L. Nebeský has proved that there exists a one-to-one correspondence between tree algebras and trees; to a tree algebra (M, P) a tree T corresponds whose vertex set is M and $x = P(u, v, w)$ if and only if the vertex x of T is the common vertex of the path connecting u and v , the path connecting u and w and the path connecting v and w .

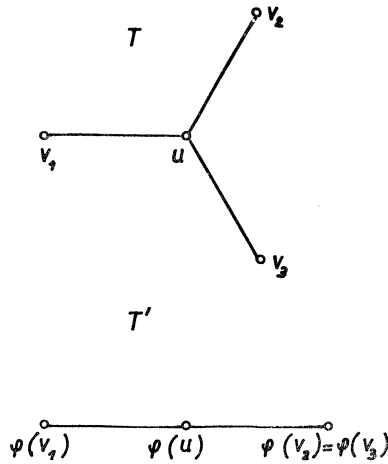
If (M, P) is a tree algebra and ξ a tolerance on M , then ξ is compatible with (M, P) if and only if the following assertion holds: If $x_1, x_2, y_1, y_2, z_1, z_2$ are elements of M , $(x_1, x_2) \in \xi$, $(y_1, y_2) \in \xi$, $(z_1, z_2) \in \xi$, then $(P(x_1, y_1, z_1), P(x_2, y_2, z_2)) \in \xi$.

If ξ is a tolerance compatible with (M, P) and moreover ξ is transitive, then ξ is a congruence on (M, P) .

We shall prove two theorems concerning tolerances and congruences on tree algebras.

Theorem 1. Let T be a tree, let (M, P) be the tree algebra corresponding to T . Let ξ be a tolerance on M . Then the following two assertions are equivalent:

- (1) ξ is compatible with (M, P) .
- (2) If $u \in M, v \in M, (u, v) \in \xi$, then $(x, y) \in \xi$ for any two vertices x, y of the path connecting u and v in T .



Proof. (1) \Rightarrow (2). Let x, y be two vertices of the path connecting u and v in T . Without loss of generality let x lie between u and y . Then $P(u, x, y) = x, P(v, x, y) = y$. As $(u, v) \in \xi, (x, x) \in \xi, (y, y) \in \xi$, we must have $(P(u, x, y), P(v, x, y)) = (x, y) \in \xi$.

(2) \Rightarrow (1). Let $x_1, x_2, y_1, y_2, z_1, z_2$ be elements of M (vertices of T), let $(x_1, x_2) \in \xi, (y_1, y_2) \in \xi, (z_1, z_2) \in \xi$. Let X (or Y , or Z) be the path connecting the vertices x_1, x_2 (or y_1, y_2 , or z_1, z_2 respectively) in T . Let $u_1 = P(x_1, y_1, z_1), u_2 = P(x_2, y_2, z_2)$. First suppose that x_1, y_1, z_1 are all distinct from u_1 . Let B_x (or B_y , or B_z) be the branch of T outgoing from u_1 and containing x_1 (or y_1 , or z_1 respectively). The branches B_x, B_y, B_z are pairwise distinct. If $u_1 = u_2$, then $(u_1, u_2) \in \xi$ and the assertion holds. Thus suppose that $u_1 \neq u_2$. This means that at least two of the vertices x_2, y_2, z_2 lie in the same branch outgoing from u_1 . Without loss of generality let x_2, y_2 lie in the same branch B outgoing from u_1 ; the branch B may coincide with some of the branches B_x, B_y, B_z , but obviously at most with one of them. Without loss of generality let $B \neq B_x$. Then X goes through u_1 . If z_2 does not belong to B , then the

path $R(x_2, z_2)$ goes through u_1 and has a common subpath $R'(u_1, x_2)$ with X ; this subpath $R'(u_1, x_2)$ connects u_1 and x_2 . The path $S(y_2, z_2)$ contains also u_1 . Then u_2 is the common vertex of $R(x_2, z_2)$ and $S(y_2, z_2)$ which is in B and whose distance from u_1 is maximal. As u_2 belongs to $R'(u_1, x_2)$ and $R'(u_1, x_2)$ is a subpath of X , the vertices u_1, u_2 belong both to X and $(u_1, u_2) \in \xi$. If z_2 belongs to B , then either $B = B_z$, or Z contains u_1 . If $B = B_z$, then Y contains u_1 . All vertices x_2, y_2, z_2 are in B . Suppose that Y contains u_1 . Let $Q(u_1, x_2), Q'(u_1, y_2)$ be the paths connecting u_1 with x_2 and y_2 respectively. Let w be the common vertex of $Q(u_1, x_2), Q'(u_1, y_2)$ whose distance from u_1 is maximal. The path connecting w and x_2 has only one common vertex w with the path connecting w and y_2 ; their union is the path connecting x_2 and y_2 (this path is unique, because T is a tree). Thus u_2 lies on this path. But $Q(u_1, x_2)$ is a subpath of X and $Q'(u_1, y_2)$ is a subpath of Y . This means that u_2 belongs either to X , or to Y . As u_1 belongs to both X and Y , we have $(u_1, u_2) \in \xi$. Analogously if Z contains u_1 . Thus the proof is complete for the case when x_1, y_1, z_1 are all distinct from u_1 . Now let u_1 coincide with one of the vertices x_1, y_1, z_1 . If $u_1 = x_1$, then the above proof is adapted so that B_x is not a branch, but the one-vertex subgraph of T consisting of u_1 ; analogously if $u_1 = y_1$ or $u_1 = z_1$.

Theorem 2. *Let T be a tree, let (M, P) be the tree algebra corresponding to T . Let ξ be an equivalence on M . Then the following two assertions are equivalent:*

- (1) ξ is a congruence on (M, P) .
- (2) Each equivalence class of ξ induces a subtree of T .

Proof. (1) \Rightarrow (2). As ξ is a congruence on (M, P) , it is a tolerance compatible with (M, P) . Thus all vertices of a path connecting two vertices of one equivalence class of ξ belong to this equivalence class and the subgraph of T induced by this class is connected. Any connected subgraph of a tree is its subtree.

(2) \Rightarrow (1). The assertion (2) from this theorem implies the assertion (2) from Theorem 1. According to Theorem 1, ξ is then a tolerance compatible with (M, P) . As ξ is transitive, it is a congruence on (M, P) .

In [2] a connected homomorphism of a graph G onto a graph G' is defined as a homomorphism φ of G onto G' such that for each vertex y of G' the set of all vertices x of G such that $\varphi(x) = y$ induces a connected subgraph of G .

Corollary. *Let T, T' be trees, let $(M, P), (M', P')$ be the tree algebras corresponding to them. Then each connected homomorphism of T onto T' is a homomorphism of (M, P) onto (M', P') and vice versa.*

There exist also homomorphisms of one tree onto another which are not connected. An example is in Fig. 1. This homomorphism φ is not a homomorphism of (M, P) onto (M', P') , because $P(v_1, v_2, v_3) = u$, $P'(\varphi(v_1), \varphi(v_2), \varphi(v_3)) = \varphi(v_2) \neq \varphi(u)$.

References:

- [1] *L. Nebeský*: Algebraic Properties of Trees. Acta Univ. Carol. Philologica — Monographia XXV, 1969.
- [2] *O. Ore*: Theory of Graphs. Providence 1962.
- [3] *E. C. Zeeman*: The Topology of the Brain and Visual Perception. In: The Topology of 3-Manifolds, Ed. by K. M. Fort, pp. 240—256.
- [4] *B. Zelinka*: Tolerance in Algebraic Structures. Czech. Math. J. 20 (1970), 179—183.
- [5] *B. Zelinka*: Tolerance in Algebraic Structures II. Czech. Math. J. 25 (1975), 175—178.
- [6] *I. Chajda* and *B. Zelinka*: Tolerance Relations on Lattices. Čas. pěstov. mat. 99 (1974), 334—339.
- [7] *B. Zelinka*: Tolerance Relations on Semilattices. Comment. Math. Univ. Carol. 16 (1975), 333—338.

Author's address: 461 17 Liberec 1, Komenského 2, ČSSR (Katedra matematiky Vysoké školy strojní a textilní).