Alois Švec

On a global version of the Gauss-Bonnet theorem

Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 4, 638–644

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

ON A GLOBAL VERSION OF THE GAUSS-BONNET THEOREM

ALOIS ŠVEC, Praha

(Received November 18, 1974)

Let $M^m \subset E^{m+1}$ be a hypersurface; the induced fundamental tensors be a_{ij}, b_{ij}. On M^m, consider a tensor b'_{ij} such that the couple (a_{ij}, b'_{ij}) satisfies the Gauss and Codazzi equations. Is there a hypersurface $M' \subset E^{m+1}$ such that its induced fundamental tensors are exactly a_{ij}, b'_{ij}? In what follows, I give a partial answer to this question. It is usefull to consult my paper [1].

1. A cohomology theory. Let M be an orientable Riemannian manifold with the metric tensor a. Let us restrict ourselves to its coordinate neighborhood $U \subset M$ possessing the coordinates $(x^1, \ldots, x^n)$; as always, define the Christoffel symbols and the operator of covariant derivation by means of

$$\Gamma^k_{ij} = \tfrac{1}{2} a^{rk}(\partial_i a_{jr} + \partial_j a_{ir} - \partial_r a_{ij}), \tag{1}$$

$$\nabla_k T^{j_1 \ldots j_\beta}_{i_1 \ldots i_\alpha} = \partial_k T^{j_1 \ldots j_\beta}_{i_1 \ldots i_\alpha} - \sum_{\varrho=1}^{\alpha} \Gamma^r_{k i_\varrho} T^{j_1 \ldots j_\beta}_{i_1 \ldots i_{\varrho-1} r i_{\varrho+1} \ldots i_\alpha} + \sum_{\sigma=1}^{\beta} \Gamma^{j_\sigma}_{kr} T^{j_1 \ldots j_{\sigma-1} r_{\sigma+1} \ldots j_\beta}_{i_1 \ldots i_\alpha} \tag{2}$$

with $\partial_i = \partial/\partial x^i$, the summation convention being used throughout. Recall the formula

$$(\nabla_l \nabla_k - \nabla_k \nabla_l)\, T^{j_1 \ldots j_\beta}_{i_1 \ldots i_\alpha} = \sum_{\sigma=1}^{\beta} R^{j_\sigma}_{rkl} T^{j_1 \ldots j_{\sigma-1} r j_{\sigma+1} \ldots j_\beta}_{i_1 \ldots i_\alpha} - \sum_{\varrho=1}^{\alpha} R^r_{i_\varrho kl} T^{j_1 \ldots j_\beta}_{i_1 \ldots i_{\varrho-1} r i_{\varrho+1} \ldots i_\alpha}, \tag{3}$$

$$R^l_{ijk} = \partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^r_{ij} \Gamma^l_{kr} - \Gamma^r_{ik} \Gamma^l_{jr} \tag{4}$$

being the curvature tensor.

Definition. An *abstract hypersurface* is an orientable Riemannian manifold endowed with a symmetric 2-covariant tensor b satisfying

(5) $$\nabla_j b_{ik} = \nabla_i b_{jk} ,$$

(6) $$R^l_{ijk} = a^{rl}(b_{ij}b_{kr} - b_{ik}b_{jr}) .$$

Definition. Let M be an abstract hypersurface. For each domain $U \subset M$, let E^α_U; $\alpha = 0, \ldots, \dim M$; be the set of couples (φ, ψ), φ and ψ being an exterior α-form and an exterior vector α-form over U respectively; let E^α be the associated sheaf. The operator

(7) $$D \equiv D^\alpha : E^\alpha \to E^{\alpha+1}$$

be defined as follows. Let U be a coordinate neighbourhood,

(8) $$\varphi = R_{i_1 \ldots i_\alpha} \, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_\alpha} , \qquad R_{i_1 \ldots i_\alpha} \text{ skew-symmetric} ,$$

$$\psi = S^j_{i_1 \ldots i_\alpha} \frac{\partial}{\partial x^j} \otimes \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_\alpha} , \quad S^j_{i_1 \ldots i_\alpha} \quad \text{skew-symmetric} ,$$

then

(9) $$D^\alpha(\varphi, \psi) = (D^\alpha \varphi, D^\alpha \psi)$$

with

(10) $$D^\alpha \varphi = (\nabla_i R_{i_1 \ldots i_\alpha} + b_{ir} S^r_{i_1 \ldots i_\alpha}) \, \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_\alpha} ,$$

$$D^\alpha \psi = (\nabla_i S^j_{i_1 \ldots i_\alpha} - a^{rj} b_{ri} R_{i_1 \ldots i_\alpha}) \frac{\partial}{\partial x^j} \otimes \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_\alpha} .$$

The following two propositions are proved just on the level $\alpha = 0$, this being sufficient for our use.

Proposition 1. *We have*

(11) $$D^2 = 0 \quad \textit{i.e.,} \quad D^{\alpha+1} D^\alpha = 0 \quad \textit{for} \quad \alpha = 0, \ldots, \dim M - 1 .$$

Proof. Suppose $\alpha = 0$, $(\varphi, \psi) \in E^0_U$,

(12) $$\varphi = R , \quad \psi = S^j \frac{\partial}{\partial x^j} .$$

Then

$$(13)\qquad D^0\varphi = (\nabla_i R + b_{ir}S^r)\,dx^i\,,\quad D^0\psi = (\nabla_i S^j - a^{rj}b_{ri}R)\,\frac{\partial}{\partial x^j}\otimes dx^i\,;$$

$$D^1D^0\varphi = U_{ij}\,dx^i\wedge dx^j \equiv \{\nabla_i(\nabla_j R + b_{jr}S^r) + b_{ir}(\nabla_j S^r - a^{sr}b_{sj}R)\}\,dx^i\wedge dx^j\,,$$

$$D^1D^0\psi = V^k_{ij}\,\frac{\partial}{\partial x^k}\otimes dx^i\wedge dx^j \equiv$$

$$\equiv \{\nabla_i(\nabla_j S^k - a^{rk}b_{rj}R) - a^{rk}b_{ri}(\nabla_j R + b_{js}S^s)\}\,\frac{\partial}{\partial x^k}\otimes dx^i\wedge dx^j\,.$$

We are going to show that $U_{ij} = U_{ji}$, $V^k_{ij} = V^k_{ji}$. Indeed,

$$U_{ij} - U_{ji} = (\nabla_i\nabla_j - \nabla_j\nabla_i)\,R + (\nabla_i b_{jr} - \nabla_j b_{ir})\,S^r = 0\,,$$

$$V^k_{ij} - V^k_{ji} = (\nabla_i\nabla_j - \nabla_j\nabla_i)\,S^k - a^{rk}(\nabla_i b_{rj} - \nabla_j b_{ri})\,R -$$

$$-a^{rk}(b_{ri}b_{js} - b_{rj}b_{is})\,S^s = 0$$

because of (3), (5) and (6).

Proposition 2. (Poincaré lemma) *Let* $(\Phi, \Psi)\in E^{\alpha+1}_U$, *and suppose* $D^{\alpha+1}(\Phi, \Psi) = (0, 0)$; *be given a point* $m\in U$. *Then there is a neighbourhood* $U_1\subset U$ *of* m *and a* $(\varphi, \psi)\in E^{\alpha}_{U_1}$ *such that* $(\Phi, \Psi) = D^{\alpha}(\varphi, \psi)$.

Proof. Suppose $\alpha = 0$. For

$$\Phi = M_i\,dx^i\,,\quad \Psi^i = N^j_i\,\frac{\partial}{\partial x^j}\otimes dx^i\,,$$

we have

$$D^1\Phi = (\nabla_i M_j + b_{ir}N^r_j)\,dx^i\wedge dx^j\,,$$

$$D^1\Psi = (\nabla_i N^k_j - a^{rk}b_{ri}M_j)\,\frac{\partial}{\partial x^k}\otimes dx^i\wedge dx^j\,.$$

From $D^1(\Phi, \Psi) = (0, 0)$,

$$(14)\qquad \nabla_i M_j - \nabla_j M_i = b_{jr}N^r_i - b_{ir}N^r_j\,,$$

$$\nabla_i N^k_j - \nabla_j N^k_i = a^{rk}(b_{ri}M_j - b_{rj}M_i)\,.$$

The element (φ, ψ) being (12), we have – according to (13) – to prove the local existence of R, S^j such that

(15) $$\nabla_i R = M_i - b_{ir} S^r , \quad \nabla_i S^j = N_i^j + a^{rj} b_{ri} R .$$

The integrability conditions of (15) being exactly (14), we are done.

Theorem 1. *Let $\mathscr{V} \subset E^0$ be the sheaf of solutions $(\varphi, \psi) \in E^0$ of the system $D^0(\varphi, \psi) = (0, 0)$. Then*

(16) $$0 \to \mathscr{V} \to E^0 \xrightarrow{D^0} E^1 \xrightarrow{D^1} \dots$$

is a resolution of the sheaf $\mathscr{V}$.

Definition. Let $\mathscr{E}^\alpha$ be the additive group (over reals) of global sections of E^α. For $\alpha = 0, \dots, \dim M - 1$, let $\mathscr{Z}^\alpha \subset \mathscr{E}^\alpha$ be the subgroup of the global sections (Φ, Ψ) satisfying $D^\alpha(\Phi, \Psi) = (0, 0)$; for $\alpha = 1, \dots, \dim M$, let $\mathscr{B}^\alpha \subset \mathscr{E}^\alpha$ be the subgroup of global sections of the form $D^{\alpha-1}(\varphi, \psi)$ with $(\varphi, \psi) \in \mathscr{E}^{\alpha-1}$. The *cohomology groups* of our abstract hypersurface are then given by

(17) $$\mathscr{H}^\alpha = \mathscr{Z}^\alpha / \mathscr{B}^\alpha ; \quad \alpha = 1, \dots, \dim M - 1 .$$

Let M be an abstract hypersurface. Over M, consider the bundle V of vector euclidean spaces such that $V(m) \supset T_m(M)$, $\dim V(m) = \dim M + 1$ and both the scalar products coincide on $T_m(M)$. In V, consider the euclidean connection Γ given by

(18) $$\partial_j v_i = \Gamma_{ij}^k v_k + b_{ij} n , \quad \partial_i n = -a^{rk} b_{ri} v_k$$

with $v_i = \partial / \partial x^i$, $n(m) \perp T_m(M)$, $\langle n, n \rangle = 1$. This connection is integrable because of (5) and (6). Let

(19) $$w = A^i v_i + A n$$

be a Γ-parallel vector field in V. Because of

(20) $$\nabla_i w = (\nabla_i A^j - a^{rj} b_{ri} A) \, v_j + (\nabla_i A + b_{ir} A^r) \, n = 0 ,$$

we see that

(21) $$D^0 \left(A, A^j \frac{\partial}{\partial x^j} \right) = (0, 0) , \quad \text{i.e.,} \quad \left(A, A^j \frac{\partial}{\partial x^j} \right) \in \mathscr{V} .$$

Proposition 3. *$\mathscr{V}$ is the sheaf of Γ-parallel vector fields in V.*

2. Realization of abstract hypersurfaces. Let M be a hypersurface of the euclidean space E^{m+1}; its fundamental equations be

$$\partial_i M = v_i\,, \quad \partial_j v_i = \Gamma^k_{ij} v_k + b_{ij} n\,, \quad \partial_i n = -a^{rk} b_{ri} v_k\,. \tag{22}$$

Let $S \in E^{m+1}$ be a fixed point and $M = S + w$, w being given by (19). Then

$$(\nabla_i A^j - a^{rj} b_{ri} A)\, v_j + (\nabla_i A + b_{ir} A^r)\, n = v_i\,, \tag{23}$$

i.e.,

$$D^0 \left(A, A^j \frac{\partial}{\partial x^j}\right) = \left(0, \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i\right). \tag{24}$$

Let us remark that

$$D^1 \left(0, \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i\right) = (0, 0)\,, \quad \text{i.e.,} \quad \left(0, \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i\right) \in \mathscr{Z}^1\,. \tag{25}$$

Definition. Let M be an abstract hypersurface, its bundle V be constructed as above. The vector field (19) is called *Γ-central* if (24) holds true.

Obviously, we have the following assertion: Let M be an abstract hypersurface, $m \in M$ its fixed point. If, for each vector $w' \in V(m)$, there exists a global Γ-central vector field (19) with $w(m) = w'$, M is realizable as a hypersurface of the euclidean space.

Definition. Let M be an abstract hypersurface. Its I-*deformation* $M(t)$ is given by a tensor

$$\beta_{ij}(t) = b^{(0)}_{ij} + t b^{(1)}_{ij} + t^2 b^{(2)}_{ij} + \ldots\,; \quad b^{(0)}_{ij} = b_{ij}\,, \quad b^{(\alpha)}_{ij} = b^{(\alpha)}_{ji}\,; \tag{26}$$

such that the manifold M together with a_{ij}, $\beta_{ij}(t)$ is an abstract hypersurface for each t. I am going to restrict myself to the formal theory, the series (26) being a formal one and the tensors $b^{(\alpha)}_{ij}$ satisfying ($\alpha = 1, 2, \ldots$)

$$\nabla_j b^{(\alpha)}_{ik} = \nabla_i b^{(\alpha)}_{jk}\,, \tag{27}$$

$$\sum_{\beta=0}^{\alpha} (b^{(\beta)}_{ij} b^{(\alpha-\beta)}_{kl} - b^{(\beta)}_{ik} b^{(\alpha-\beta)}_{jl}) = 0\,. \tag{28}$$

The connection $\Gamma(t)$ on V associated to the abstract hypersurface $M(t)$ is given by the equations

$$\partial_j v_i = \Gamma^k_{ij} v_k + \beta_{ij}(t)\, n\,, \quad \partial_i n = -a^{rk} \beta_{ri}(t)\, v_k\,. \tag{29}$$

Now, my question is if there exists, in V, a global $\Gamma(t)$-central formal vector field

(30) $w(t) = (A^i_{(0)} + tA^i_{(1)} + t^2A^i_{(2)} + \ldots)\, v_i + (A_{(0)} + tA_{(1)} + t^2A_{(2)} + \ldots)\, n$

for each t and for each global Γ-central vector field $A^i_{(0)}v_i + A_{(0)}n$ of M. Let $A^i_{(0)}v_i + A_{(0)}n$ be a Γ-central vector field and (30) a $\Gamma(t)$-central vector field. Then ($\alpha = 1, 2, \ldots$)

(31)
$$\nabla_i A^j_{(\alpha)} - a^{rj}b_{ri}A_{(\alpha)} = \sum_{\beta=0}^{\alpha-1} a^{rj}b^{(\alpha-\beta)}_{ri}A_{(\beta)}\,,$$

$$\nabla_i A_{(\alpha)} + b_{ir}A^r_{(\alpha)} = -\sum_{\beta=0}^{\alpha-1} b^{(\alpha-\beta)}_{ir}A^r_{(\beta)}\,,$$

i.e.,

(32)
$$D^0\left(A_{(\alpha)}, A^j_{(\alpha)}\frac{\partial}{\partial x^j}\right) = (\varphi_{(\alpha)}, \psi_{(\alpha)}) \quad \text{with}$$

$$\varphi_{(\alpha)} = -\sum_{\beta=0}^{\alpha-1} b^{(\alpha-\beta)}_{ir}A^r_{(\beta)}\,\mathrm{d}x^i\,, \quad \psi_{(\alpha)} = \sum_{\beta=0}^{\alpha-1} a^{rj}b^{(\alpha-\beta)}_{ri}A_{(\beta)}\frac{\partial}{\partial x^j}\otimes \mathrm{d}x^i\,.$$

The following assertion holds true: Let $A^i_{(0)}v_i + A_{(0)}n$ be a Γ-central vector field and let the vector fields $A^i_{(\alpha)}v_i + A_{(\alpha)}n$ ($\alpha = 1, \ldots, \gamma - 1$) satisfy the equations (31) for $\alpha = 1, \ldots, \gamma - 1$; then

(33)
$$D^1(\varphi_{(\gamma)}, \psi_{(\gamma)}) = (0, 0)\,.$$

The proof of this assertion is nothing more than a tiresome exercise; let us therefore restrict ourselves to the case $\gamma = 1$. From (23), (27) and (28), we get

$$\nabla_i A^j_{(0)} = a^{rj}b_{ri}A_{(0)} + \delta^j_i\,, \quad \nabla_i A_{(0)} = -b_{ir}A^r_{(0)}\,,$$

$$\nabla_j b^{(1)}_{ik} = \nabla_i b^{(1)}_{jk}\,, \quad b_{ij}b^{(1)}_{kl} - b_{ik}b^{(1)}_{jl} + b^{(1)}_{ij}b_{kl} - b^{(1)}_{ik}b_{jl} = 0\,.$$

Now,

$$\varphi_{(1)} = -b^{(1)}_{ir}A^r_{(0)}\,\mathrm{d}x^i\,, \quad \psi_{(1)} = a^{rj}b^{(1)}_{ri}A_{(0)}\frac{\partial}{\partial x^j}\otimes \mathrm{d}x^i\,;$$

$$D^1\varphi_{(1)} = \{-\nabla_i(b^{(1)}_{kr}A^r_{(0)}) + b_{is}a^{rs}b^{(1)}_{rk}A_{(0)}\}\,\mathrm{d}x^i \wedge \mathrm{d}x^k =$$
$$= (-A^r_{(0)}\nabla_i b^{(1)}_{kr} - b^{(1)}_{ki})\,\mathrm{d}x^i \wedge \mathrm{d}x^k = 0\,,$$

$$D^1\psi_{(1)} = \{\nabla_i(a^{rj}b^{(1)}_{rk}A_{(0)}) + a^{rj}b_{ri}b^{(1)}_{ks}A^s_{(0)}\}\frac{\partial}{\partial x^j}\otimes \mathrm{d}x^i \wedge \mathrm{d}x^k =$$

$$= \{a^{rj}A_{(0)}\nabla_i b^{(1)}_{rk} + a^{rj}A^s_{(0)}(b_{ri}b^{(1)}_{ks} - b_{is}b^{(1)}_{rk})\}\frac{\partial}{\partial x^j}\otimes \mathrm{d}x^i \wedge \mathrm{d}x^k = 0\,.$$

Let $A^i_{(0)}v_i + A_{(0)}n$ be an arbitrary global Γ-central vector field. Suppose the existence of global vector fields $A^i_{(\alpha)}v_i + A_{(\alpha)}n$; $\alpha = 1, \ldots, \gamma - 1$; satisfying $(32)_{\alpha=1,\ldots,\gamma-1}$. We are looking for the existence of a global vector field $A^i_{(\gamma)}v_i + A_{(\gamma)}n$ satisfying

$$D^0\left(A_{(\gamma)}, A^j_{(\gamma)}\frac{\partial}{\partial x^j}\right) = \left(\varphi_{(\gamma)}, \psi_{(\gamma)}\right). \tag{34}$$

We know that $D^1(\varphi_{(\gamma)}, \psi_{(\gamma)}) = (0, 0)$, i.e., $(\varphi_{(\gamma)}, \psi_{(\gamma)}) \in \mathscr{Z}^1$. The existence of a global solution of (34) implies $(\varphi_{(\gamma)}, \psi_{(\gamma)}) \in \mathscr{B}^1$ and we get

Theorem 2. *Let E^{m+1} be the euclidean space, $V(E^{m+1})$ its vector space, $S \in E^{m+1}$ its fixed point, M^m a manifold and $w : M^m \to V(E^{m+1})$ a map for which $\mu(M^m)$, μ being the map $\mu(m) = S + w(m)$, is a hypersurface. On M^m, consider the induced structure of an abstract hypersurface. Let $M^m(t)$ be a formal* I*-deformation of M^m. If $\mathscr{H}^1 = 0$, there are maps $w_{(\alpha)} : M^m \to V(E^{m+1})$; $\alpha = 1, 2, \ldots$; such that the hypersurface $\mu_t(M^m)$,*

$$\mu_t(m) = S + w(m) + tw_{(1)}(m) + \ldots, \quad m \in M^m, \tag{35}$$

has the induced structure $a_{ij}, \beta_{ij}(t)$.

Bibliography

[1] *A. Švec:* On infinitesimal isometries of a hypersurface. Czech. Math. J., *24* (*99*) 1974, 150—163.

Author's address: 118 00 Praha 1, Malostranské nám. 25, ČSSR (Matematicko-fyzikální fakulta UK).