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SEVERAL NEW CHARACTERIZATIONS OF THE SPHERE

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1. Preliminaries. Let M be an ovaloid of class C^4 in the 3-dimensional Euclidean space E^3 . Let $\{U_\alpha\}$ be its covering by domains such that in each U_α there is a field of orthonormal frames $(M; v_1, v_2, v_3)$ with $v_1, v_2 \in T(M)$. The orientation of v_3 be chosen in such a way that the principal curvatures are positive. Then

$$(1) \quad \begin{aligned} dM &= \omega^1 v_1 + \omega^2 v_2, & dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3, \\ dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3, & dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2; \\ d\omega^i &= \omega^j \wedge \omega_j^i, & d\omega_i^j &= \omega_i^k \wedge \omega_k^j; & \omega_i^i + \omega_j^j &= 0, & \omega^3 &= 0. \end{aligned}$$

Using the well known prolongation procedure, we get the existence of functions (in each U_α) $a, b, c; \alpha, \dots, \delta; A, \dots, E$ such that

$$(2) \quad \omega_1^3 = a\omega^1 + b\omega^2, \quad \omega_2^3 = b\omega^1 + c\omega^2;$$

$$(3) \quad \begin{aligned} da - 2b\omega_1^2 &= \alpha\omega^1 + \beta\omega^2, \\ db + (a - c)\omega_1^2 &= \beta\omega^1 + \gamma\omega^2, \\ dc + 2b\omega_1^2 &= \gamma\omega^1 + \delta\omega^2; \end{aligned}$$

$$(4) \quad \begin{aligned} d\alpha - 3\beta\omega_1^2 &= A\omega^1 + (B - bK)\omega^2, \\ d\beta + (\alpha - 2\gamma)\omega_1^2 &= (B + bK)\omega^1 + (C + aK)\omega^2, \\ d\gamma + (2\beta - \delta)\omega_1^2 &= (C + cK)\omega^1 + (D + bK)\omega^2, \\ d\delta + 3\gamma\omega_1^2 &= (D - bK)\omega^1 + E\omega^2. \end{aligned}$$

As always,

$$(5) \quad K = ac - b^2, \quad 2H = a + c$$

denote the Gauss and mean curvature respectively.

Let $f : M \rightarrow \mathcal{R}$ be a function. The derivatives f_i, f_{ij} of its restriction to U_α with respect to the frames $(M; v_1, v_2, v_3)$ are introduced by means of the formulas

$$(6) \quad \begin{aligned} df &= f_1\omega^1 + f_2\omega^2; \\ df_1 - f_2\omega_1^2 &= f_{11}\omega^1 + f_{12}\omega^2, \quad df_2 + f_1\omega_1^2 = f_{12}\omega^1 + f_{22}\omega^2. \end{aligned}$$

It is easy to see that the functions

$$(7) \quad f_1^2 + f_2^2, \quad (a - c)(f_{11} - f_{22}) + 4bf_{12}$$

do not depend on the choice of the moving frames. The domain U_α be covered by coordinates (u, v) such that

$$(8) \quad \omega^1 = r(u, v) du, \quad \omega^2 = s(u, v) dv; \quad rs \neq 0;$$

such coordinates do exist. We get

$$(9) \quad \omega_1^2 = -s^{-1}r_v du + r^{-1}s_u dv$$

and

$$(10) \quad \begin{aligned} f_1 &= r^{-1}f_u, \quad f_2 = s^{-1}f_v; \\ f_{11} &= r^2f_{uu} - r^{-3}r_u f_u + r^{-1}s^{-2}r_v f_v, \\ f_{12} &= r^{-1}s^{-1}f_{uv} - r^{-2}s^{-1}r_v f_u - r^{-1}s^{-2}s_u f_v, \\ f_{22} &= s^{-2}f_{vv} + r^{-2}s^{-1}s_u f_u - s^{-3}s_v f_v. \end{aligned}$$

Thus we have the following version of the (weak) maximum principle: *Let f satisfy, in U_α , the equation*

$$(11) \quad a_{11}f_{11} + 2a_{12}f_{12} + a_{22}f_{22} + a_1f_1 + a_2f_2 + a_0f = \varphi$$

with $a_{ij}x^i x^j$ positive, $a_0 \leq 0$ and $\varphi \geq 0$, then f cannot have a positive maximum inside of U_α without being constant; see [1], p. 109.

2. Two generalized H -theorems. On M , consider the function

$$(12) \quad f = 2(H^2 - K) = \frac{1}{2}(a - c)^2 + 2b^2;$$

of course, $f \geq 0$, and we have $f = 0$ in the umbilical points. We get

$$(13) \quad f_1 = (a - c)(\alpha - \gamma) + 4b\beta, \quad f_2 = (a - c)(\beta - \delta) + 4b\gamma;$$

$$\begin{aligned}
 (14) \quad f_{11} &= (c^2 - ac + 4b^2)K + (\alpha - \gamma)^2 + 4\beta^2 + (a - c)(A - C) + 4bB, \\
 f_{12} &= 2b(a + c)K + (\alpha - \gamma)(\beta - \delta) + 4b\gamma + (a - c)(B - D) + 4bC, \\
 f_{22} &= (a^2 - ac + 4b^2)K + (\beta - \delta)^2 + 4\gamma^2 + (a - c)(C - E) + 4bD.
 \end{aligned}$$

The covariant derivatives of the mean curvature H are given by

$$\begin{aligned}
 (15) \quad 2H_1 &= \alpha + \gamma, \quad 2H_2 = \beta + \delta; \\
 (16) \quad 2H_{11} &= cK + A + C, \quad 2H_{12} = B + D, \quad 2H_{22} = aK + C + E.
 \end{aligned}$$

Multiplying the equations (14), (16) successively by 1, 0, 1, $c - a$, $-4b$, $a - c$ and adding them, we get

$$\begin{aligned}
 (17) \quad f_{11} + f_{22} - 4Kf &= 2\{(a - c)(H_{11} - H_{22}) + 4bH_{12} + H_1^2 + H_2^2\} + \\
 &+ 2\{(2\gamma - H_1)^2 + (2\beta - H_2)^2\}.
 \end{aligned}$$

Thus we obtain

Theorem 1. *Let $M \subset E^3$ be an ovaloid, and let its mean curvature satisfy*

$$(18) \quad (a - c)(H_{11} - H_{22}) + 4bH_{12} + H_1^2 + H_2^2 \geq 0.$$

Then M is a sphere.

Next, we are going to prove the following

Theorem 2. *Let $M \subset E^3$ be a part of an ovaloid such that: (i) M has a net of lines of curvature, (ii) ∂M consists of umbilical points, (iii) v_1 and v_2 being the unit tangent vector fields of the lines of curvature and $S : M \rightarrow \mathcal{R}$ a function satisfying*

$$(19) \quad S^2 \leq 57 - 40\sqrt{2},$$

we have

$$(20) \quad v_1v_1H - v_2v_2H + S[v_1v_2]H = 0.$$

Then M is a part of a sphere.

The lines of curvature being given by

$$(21) \quad b(\omega^1)^2 + (c - a)\omega^1\omega^2 - b(\omega^2)^2 = 0,$$

we have to suppose

$$(22) \quad b = 0;$$

a and c are then the principal curvatures respectively. From (3) and (4),

$$(23) \quad v_1 a = \alpha, \quad v_2 a = \beta, \quad v_1 c = \gamma, \quad v_2 c = \delta;$$

$$(24) \quad \begin{aligned} (a - c) v_1 v_1 a &= 3\beta^2 + (a - c) A, \\ (a - c) v_1 v_2 a &= (2\gamma - \alpha) \beta + (a - c) B, \\ (a - c) v_2 v_1 a &= 3\beta\gamma + (a - c) B, \\ (a - c) v_2 v_2 a &= (2\gamma - \alpha) \gamma + (a - c) (C + aK), \\ (a - c) v_1 v_1 c &= (\delta - 2\beta) \beta + (a - c) (C + cK), \\ (a - c) v_1 v_2 c &= -3\beta\gamma + (a - c) D, \\ (a - c) v_2 v_1 c &= (\delta - 2\beta) \gamma + (a - c) D, \\ (a - c) v_2 v_2 c &= -3\gamma^2 + (a - c) E. \end{aligned}$$

The equation (20) implies

$$(25) \quad \begin{aligned} \beta^2 + \gamma^2 + \alpha\gamma + \beta\delta - (a - c)^2 K + \\ + (a - c) (A - E) - S(\alpha\beta + 2\beta\gamma + \gamma\delta) = 0. \end{aligned}$$

The elimination of A, C, E from (14_{1,3}) and (25) yields

$$(26) \quad \begin{aligned} f_{11} + f_{22} - 4Kf &= (\alpha - \frac{3}{2}\gamma + \frac{1}{2}S\beta)^2 + (\delta - \frac{3}{2}\beta + \frac{1}{2}S\gamma)^2 + \\ &+ \frac{1}{4}\{(7 - S^2)\beta^2 + 20S\beta\gamma + (7 - S^2)\gamma^2\}. \end{aligned}$$

The last term is non-negative for each β and γ because of (19). This concludes the proof of our theorem.

Let us add the following remark. Let M satisfy (i) and (ii) of Theorem 2. On M , consider the second order operators of the form

$$(27) \quad P = S_1 v_1 v_1 + S_2 v_2 v_2 + S_3 v_1 v_2 + S_4 v_2 v_1,$$

$S_1, \dots, S_4 : M \rightarrow \mathcal{R}$ being functions. Now, our problem is to determine the class of operators (27) with the following property: The surface M satisfying $PH = 0$ (and knowing nothing more about it), we are able to prove by means of the maximum principle using the function $H^2 - K$ that it is a part of a sphere. It is not difficult to see that the just defined class of operators is given by (iii) of Theorem 2.

3. A generalization of the K -theorem. We are going to prove a somewhat modified version of Theorem 1.

Theorem 3. Let $M \subset E^3$ be an ovaloid satisfying, at each of its points, the following conditions: (i)

$$(28) \quad (a - c)(K_{11} - K_{22}) + 4bK_{12} \geq 0,$$

(ii) $k_1 \leq k_2$ being the principal curvatures, we have $k_2 \leq 3k_1$. Then M is a sphere.

We have

$$(29) \quad K_1 = a\gamma + c\alpha - 2b\beta, \quad K_2 = a\delta + c\beta - 2b\gamma;$$

$$(30) \quad K_{11} = (ac - 2b^2)K + 2\alpha\gamma - 2\beta^2 + cA - 2bB + aC,$$

$$K_{12} = -b(a + c)K + \alpha\delta - \beta\gamma + cB - 2bC + aD,$$

$$K_{22} = (ac - 2b^2)K + 2\beta\delta - 2\gamma^2 + cC - 2bD + aE.$$

Multiplying the equations (14), (30) successively by $c, -2b, a, c - a, -4b, a - c$ and adding them together, we get

$$(31) \quad cf_{11} - 2bf_{12} + af_{22} - 4HKf = (a - c)(K_{11} - K_{22}) + 4bK_{12} + T$$

with

$$(32) \quad T = a(3\beta^2 + \delta^2 + 2\gamma^2 - 2\alpha\gamma) - \\ - 2b(\alpha + \gamma)(\beta + \delta) + c(\alpha^2 + 3\gamma^2 + 2\beta^2 - 2\beta\delta).$$

We have to prove $T \geq 0$. The term T being an invariant of the surface (which is easy to see), it is sufficient to prove the inequality in a generic point $m \in M$ using a convenient field of moving frames around m . Let us choose this field in such a way that $b(m) = 0$. Then, at m ,

$$(33) \quad T = c^{-1}(c\alpha - a\gamma)^2 + a^{-1}(a\delta - c\beta)^2 + \\ + 2H\{a^{-1}(3a - c)\beta^2 + c^{-1}(3c - a)\gamma^2\};$$

$a(m)$ and $c(m)$ being the principal curvatures at m , we are done.

The classical K -theorem follows easily. Our starting point is the equation (30), we have to prove that $K = \text{const.}$ implies $T \geq 0$. Around a generic point $m \in M$, choose the moving frames as above. We have

$$K_1 = a\gamma + c\alpha = 0, \quad K_2 = a\delta + c\beta = 0 \quad \text{at } m,$$

and we get the existence of numbers ϱ, σ such that

$$\alpha = \varrho a, \quad \gamma = -\varrho c, \quad \beta = \sigma a, \quad \delta = -\sigma c \quad \text{at } m.$$

Then

$$T(m) = (3a^2 + 2ac + 3c^2)(a\sigma^2 + c\varrho^2) \geq 0.$$

4. Another generalization of the H -theorem. Consider the invariant 1-form

$$(34) \quad \tau = (a\beta + b\gamma - b\alpha - c\beta)\omega^1 + (a\gamma + b\delta - b\beta - c\gamma)\omega^2$$

on M ; it is easy to see that

$$(35) \quad d\tau = -2\{\beta^2 + \gamma^2 - \alpha\gamma - \beta\delta + 2(H^2 - K)K\}\omega^1 \wedge \omega^2.$$

The following assertion follows: Let M be a surface with ∂M umbilical, then

$$(36) \quad \int_M \{\beta^2 + \gamma^2 - \alpha\gamma - \beta\delta + 2(H^2 - K)K\}\omega^1 \wedge \omega^2 = 0.$$

Now, we are in the position to prove

Theorem 4. *Let M be a surface with ∂M umbilical, and let*

$$(37) \quad 4(H^2 - K)K \geq H_1^2 + H_2^2.$$

The point $m \in M$ being non-umbilical, there exists its neighbourhood $U \subset M$ such that $K = 0$ in U . (I do not suppose M to be an ovaloid!)

Because of (15), (36) may be rewritten as

$$(38) \quad \int_M \{(2\beta - H_2)^2 + (2\gamma - H_1)^2 - H_1^2 - H_2^2 + 4(H^2 - K)K\}\omega^1 \wedge \omega^2 = 0.$$

From (37), we obtain $2\beta = H_2$, $2\gamma = H_1$, i.e.,

$$(39) \quad \alpha = 3\gamma, \quad \delta = 3\beta.$$

In a suitable neighborhood U of the point m , we may choose the frames in such a way that $b = 0$, $a \neq c$; the equations (3) reduce to

$$(40) \quad da = 3\gamma\omega^1 + \beta\omega^2, \quad (a - c)\omega_1^2 = \beta\omega^1 + \gamma\omega^2, \quad dc = \gamma\omega^1 + 3\beta\omega^2.$$

By exterior differentiation, taking regard of (40₂),

$$(41) \quad \begin{aligned} 3(d\gamma - \beta\omega_1^2) \wedge \omega^1 + (d\beta + \gamma\omega_1^2) \wedge \omega^2 &= 0, \\ (d\beta + \gamma\omega_1^2) \wedge \omega^1 + (d\gamma - \beta\omega_1^2) \wedge \omega^2 &= (c - a)ac\omega^1 \wedge \omega^2, \\ (d\gamma - \beta\omega_1^2) \wedge \omega^1 + 3(d\beta + \gamma\omega_1^2) \wedge \omega^2 &= 0. \end{aligned}$$

Thus there is a function ϱ such that

$$(42) \quad d\beta + \gamma\omega_1^2 = (\varrho - ac^2)\omega^2, \quad d\gamma - \beta\omega_1^2 = (\varrho - a^2c)\omega^1.$$

By exterior differentiation,

$$(43) \quad (d\varrho - 3a^2\beta\omega^2) \wedge \omega^1 = 0, \quad (d\varrho - 3c^2\gamma\omega^1) \wedge \omega^2 = 0,$$

i.e.,

$$(44) \quad d\varrho = 3c^2\gamma\omega^1 + 3a^2\beta\omega^2.$$

A further exterior differentiation yields

$$(45) \quad (a - c)\beta\gamma = 0.$$

Suppose $\beta = 0 \neq \gamma$, the case $\beta \neq 0 = \gamma$ being symmetric. We have

$$(46) \quad da = 3\gamma\omega^1, \quad dc = \gamma\omega^1,$$

and the equation $H_1^2 + H_2^2 = 4(H^2 - K)K$ turns out to be

$$(47) \quad a^3c - 2a^2c^2 + ac^3 = 4\gamma^2.$$

We get (taking regard of $\gamma \neq 0$!)

$$(48) \quad \begin{aligned} a^3 + 13a^2c - 9ac^2 + 3c^3 &= 8\varrho, \\ 11a^2 + 30ac - 21c^2 &= 0, \quad 2a + c = 0, \quad 7\gamma = 0 \end{aligned}$$

by a series of successive differentiations; hence a contradiction. Thus we have $\beta = \gamma = 0$, i.e., $\omega_1^2 = 0$ because of (40₂) and $a \neq c$. But this means $ac\omega^1 \wedge \omega^2 = 0$, i.e., $K = 0$.

5. General Weingarten surfaces. We are going to prove

Theorem 5. *Let M be an ovaloid or a part of it bounded by umbilical points. Let $H, K : M \rightarrow \mathcal{R}$ be its mean and Gauss curvature respectively. On a domain \mathcal{D} of \mathcal{R}^2 containing $H(M) \times K(M)$, be given a function $f(x, y)$ satisfying*

$$(49) \quad F(x, y) := f_x^2 + 4xf_xf_y + 4yf_y^2 > 0$$

on \mathcal{D} . If $f(H, K) = 0$, M is a sphere or a part of it.

We have

$$(50) \quad \begin{aligned} 2 dH &= (\alpha + \gamma) \omega^1 + (\beta + \delta) \omega^2, \\ dK &= (a\gamma + c\alpha - 2b\beta) \omega^1 + (a\delta + c\beta - 2b\gamma) \omega^2. \end{aligned}$$

Let $m \in M$ be a generic point. Around m , choose the moving frames in such a way that $b(m) = 0$. The equation $f(H, K) = 0$ implies

$$(51) \quad \begin{aligned} (f_H + 2cf_K) \alpha + (f_H + 2af_K) \gamma &= 0, \\ (f_H + 2cf_K) \beta + (f_H + 2af_K) \delta &= 0 \quad \text{at } m. \end{aligned}$$

Because of

$$(f_H + 2cf_K)(f_H + 2af_K) = F(H, K) > 0,$$

α and γ as well as β and δ have opposite signs at m . Thus $\beta^2 + \gamma^2 - \alpha\gamma - \beta\delta \geq 0$ at m ; applying now (36), we are done.

Bibliography

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