Jarolím Bureš

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FREE CONSTRUCTIONS OF 2-STRUCTURES

JAROLÍM BUREŠ, Praha

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Among the geometric structures generalizing affine and projective planes there are also 2-structures introduced by H. KARZEL in [1]. A detailed classification of these structures was presented by V. HAVEL in the mimeographed text [2]. Simultaneously, he rose the question if the 2-structures of all the types mentioned in [2] exist. This paper contains construction of 2-structures of all such types using free extensions of incidence structures.

My thanks go to V. Havel for his valuable advice and remarks.

1. PRELIMINARIES

Definition 1 ([1] p. 192). A regular incidence structure $(\mathscr{P}, \mathscr{L}, \in)$ is called a 2-*structure* if there is a decomposition $\mathscr{L} = \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2$ of the set of lines $\mathscr{L}$ into disjoint non-empty sets, such that the following conditions are satisfied:

(1) $\forall A, B \in \mathscr{P},\ A \neq B\ \exists!\, g \in \mathscr{L}\ (A \in g \wedge B \in g)$;

(2) a) $\forall i \in \{1, 2\}\ \forall g, h \in \mathscr{L}_i\ (g = h \vee g \cap h = \emptyset)$;

b) $\forall A \in \mathscr{P},\ \forall i \in \{1, 2\}\ \exists!\, g_i \in \mathscr{L}_i\ (A \in g_i)$;

(3) $\forall_i \in \{1, 2\}\ \forall g \in \mathscr{L}_i,\ h \in \mathscr{L} \setminus \mathscr{L}_i\ (g \cap h \neq \emptyset)$.

Every affine plane is a 2-structure if we take for $\mathscr{L}_1$ and $\mathscr{L}_2$ two distinct classes of parallel lines and set $\mathscr{L}_0 = \mathscr{L} \setminus (\mathscr{L}_1 \cup \mathscr{L}_2)$ Hence we conclude.

Theorem 1 ([1] p. 193). *Every finite* 2-*structure is an affine plane.*

Definition 2. A 2-structure $(\mathscr{P}, \mathscr{L}, \in)$ is called a *weak affine plane* if there is an equivalence relation $\parallel$ on $\mathscr{L}$ called "*parallelism*" so that

(4) $g \parallel h \Rightarrow g = h \vee g \cap h = \emptyset$;

(5) $\forall P \in \mathscr{P}\ \forall g \in \mathscr{L}\ \exists!\, h \in \mathscr{L}\ (P \in h \wedge h \parallel g)$.

Definition 3 ([1] p. 198). Let M be a set, $\# M \geqq 2$, G a subset of the group S_M of all permutations of M with the properties

(6) $\forall a, a', b, b' \in M,\ a \neq b,\ a' \neq b'\ \exists!\, \gamma \in G\ (a^\gamma = a' \wedge b^\gamma = b')$;

(7) $\mathrm{Id} \in G$.

The pair (M, G) is called *a sharply doubly transitive set of permutations* (with identity).

Denote a 2-structure $(\mathscr{P}, \mathscr{L}, \in)$ with the base line e by $(\mathscr{P}, \mathscr{L}, \in, e)$. Further, denote the line from $\mathscr{L}_i$ going through the point A by $(A \to i)$ for $i = 1, 2$.

The relation between 2-structures with the base line and sharply doubly transitive sets of permutations is expressed in the following theorem.

Theorem 2 ([1] p. 198). a) *If $(\mathscr{P}, \mathscr{L}, \in, e)$ is a 2-structure with the base line e, put $M = \mathscr{L}_2$ and $G = \{\tilde{g} \mid g \in \mathscr{L}_0\}$ the set of all maps of M onto M defined by*

$$x^{\tilde{g}} = ((((x \cap g) \to 1) \cap e) \to 2)\,.$$

Then (M, G) is a sharply doubly transitive set of permutations.

b) *If (M, G) is a sharply doubly transitive set of permutations, put $\mathscr{P} = M \times M$, $\hat{\gamma} = \{(x, x^\gamma) \mid x \in M$,*

$$\langle a \rangle_1 = \{(a, y) \mid y \in M\}\,, \quad \langle a \rangle_2 = \{(x, a) \mid x \in M\}\,,$$

$$\mathscr{L}_0 = \{\hat{\gamma} \mid \gamma \in G\}\,, \quad \mathscr{L}_1 = \{\langle a \rangle_1 \mid a \in M\}\,, \quad \mathscr{L}_2 = \{\langle a \rangle_2 \mid a \in M\}\,.$$

Then $(\mathscr{P}, \mathscr{L}, \in, \widehat{\mathrm{Id}})$ is a 2-structure with the base line $\widehat{\mathrm{Id}}$ (see Fig. 1).

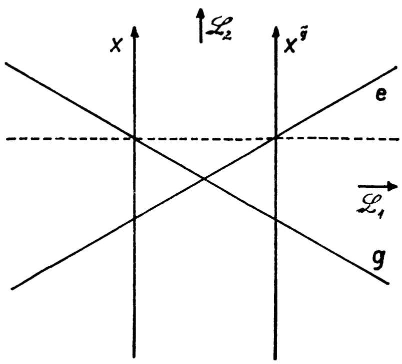

Figure 1.

Theorem 3 ([3] p. 292). *Let (M, G) be a sharply doubly transitive set of permutations. The corresponding 2-structure is an affine plane if and only if*

(8) $\forall a, b \in M\ \forall \gamma \in G\ a^\gamma \neq b\ \exists!\, \delta \in G\ [a^\delta = b \wedge (\forall x \in M x^\delta \neq x^\gamma)]$.

Definition 4. A permutation π of M si called *dispersive* if it satisfies

$$\forall x \in M \ (x^\pi \neq x) .$$

Denote by $\mathscr{R}_M$ the set of all dispersive permutations fro mS_M and for $G \subseteq S_M$ put $G' = G \cap \mathscr{R}_M$.

If M is finite, then (6) $\Rightarrow$ (8) and Theorem 3 implies Theorem 1. For the proof see e.g. [2]. Generally (in the infinite case) the implication is not true.

A counter-example (by M. Hall). Denote by N the set of all non-negative integers, and put $N_k = \{0, 1, \ldots, k\}$ for $k \in N$. A partial permutation of N is defined either as a permutation of N or as a bijection π from a finite subset $D_\pi \subseteq N$ onto $R_\pi \subseteq N$. We say that a partial permutation π has a height k if

$$N_k \subseteq D_\pi \cap R_\pi .$$

A finite set of partial permutations $S = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ is admissible of order k if

(9) there exists at most one $\gamma \in S$ such that $a^\gamma = c$, $b^\gamma = d$ for every $a, b, c, d \in N$, $a \neq b$, $c \neq d$.

(10) $\forall a, b, c, d \in N$, $a \neq b$, $c \neq d$, $a + b + c + d \leqq k$ $\exists! \gamma \in S$ $(a^\gamma = c \wedge \wedge b^\gamma = d)$.

(11) Every $\gamma \in S$ has the height k.

It is obvious that for fixed k we can construct many admissible sets of partial permutations of order k.

To every admissible set $S = \{\gamma_1, \ldots, \gamma_l\}$ of order k we can construct an admissible set $\tilde{S}$ of order $k + 1$ in two steps:

I. Denote by T the set of all elements $(a, b, c, d) \in N \times N \times N \times N$ with the properties

(i) $a \neq b$, $c \neq d$, $a + b + c + d = k + 1$.

(ii) There exists no $\gamma \in S$ such that $a^\gamma = c$ and $b^\gamma = d$. Arrange the elements of T into the finite sequence:

$$(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), \ldots, (a_m, b_m, c_m, d_m)$$

and define partial permutations $\gamma_{l+1}, \ldots, \gamma_{l+m}$ by

$$D_{\gamma_{l+i}} = \{a_i, b_i\} , \quad R_{\gamma_{l+i}} = \{c_i, d_i\} , \quad a_i^{\gamma_{l+i}} = c_i , \quad b_i^{\gamma_{l+i}} = d_i .$$

We obtain the set of partial permutations $S' = \{\gamma_1, \ldots, \gamma_l, \gamma_{l+1}, \ldots, \gamma_{l+m}\}$ which satisfies the conditions (9) and (10) but not (11).

II. Assign every partial permutation $\gamma_i \in S'$ a permutation $\tilde{\gamma}_i \in \tilde{S}$ which extends γ_i. Now we construct successively the permutations $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{m+l}$.

Construction of $\tilde{\gamma}_1$: Let $(N \setminus D_{\gamma_1}) \cap N_{k+1} = \{t_1, \ldots, t_r\}$, $t_1 < t_2 < \ldots < t_r$ and $(N \setminus R_{\gamma_1}) \cap N_{k+1} = \{s_1, \ldots, s_u\}$, $s_1 < s_2 < \ldots < s_u$. We construct successively $t_1^{\tilde{\gamma}_1}, t_2^{\tilde{\gamma}_1}, \ldots, t_r^{\tilde{\gamma}_1}$ in such a way that at each step after the corresponding extension of γ_1 (9) is satisfied for the system S' with the extended γ_1 instead of the original γ_1.

Further, we choose successively $v_1, \ldots, v_u$ and set $v_p^{\tilde{\gamma}_1} = s_p$ in such a way that after each step the condition (9) is fulfilled. For $\tilde{\gamma}_1$ we have:

$$D_{\tilde{\gamma}_1} = D_{\gamma_1} \cup \{t_1, \ldots, t_r\} \cup \{v_1, \ldots, v_u\},$$

$$R_{\tilde{\gamma}_1} = R_{\gamma_1} \cup \{t_1^{\tilde{\gamma}_1}, \ldots, t_r^{\tilde{\gamma}_1}\} \cup \{s_1, \ldots, s_u\}, \quad \tilde{\gamma}_1 = \gamma_1 \quad \text{on} \quad D_{\gamma_1}.$$

This construction can be applied successively to the systems

$$S_2' = \{\gamma_2, \ldots, \gamma_{m+l}, \tilde{\gamma}_1\}, \quad \ldots, \quad S_{m+l}' = \{\gamma_{m+l}, \hat{\gamma}_1, \ldots, \gamma_{m+l-1}'\}$$

and after $m + l$ steps we obtain the desired $\tilde{S}$.

To every admissible set S_k of order k we can construct a sequence of admissible sets $\{S_{k+i}\}_{i=0}^{\infty}$ satisfying:

(i) order of $S_{k+i} \geqq k + i$.

(ii) There exists an injection $I_p^q : S_{k+p} \to S_{k+q}$, $p \in q$ such that $I_p^q(\gamma)$ is an extension of γ for every γ.

The system $\{S_{k+i}, I_p^q\}$ is an inductive system of sets and the inductive limit $\varinjlim S_{k+i} = S_\infty$ exists. An element from S_∞ is a permutation of N and the set S_∞ satisfies the conditions (9) and (10), i.e., S_∞ is a sharply doubly transitive set of permutations of N.

Special examples. A. If we take $S_2 = \{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_1 = \text{Id}$,

$$0^{\gamma_2} = 1, \quad 1^{\gamma_2} = 2, \quad 2^{\gamma_2} = 0 \quad (2r-1)^{\gamma_2} = 2r, \quad (2r)^{\gamma_2} = 2r - 1,$$

$$(2r+1)^{\gamma_3} = 2r, \quad (2r)^{\gamma_3} = 2r + 1$$

so we have

$$\gamma_1, \gamma_2, \gamma_3 \in S_\infty, \quad 0^{\gamma_2} = 1, \quad 0^{\gamma_3} = 1, \quad \gamma_2\gamma_1^{-1} \in \mathcal{R}, \quad \gamma_3\gamma_1^{-1} \in \mathcal{R}.$$

B. We take $S = \{\gamma_1, \gamma_2\}$, $\gamma_1 = \text{Id.}$,

$$D_{\gamma_2} = \{0, 1, 2\}, \quad 0^{\gamma_2} = 1, \quad 1^{\gamma_2} = 0, \quad 2^{\gamma_2} = 2$$

and complete the construction in the following way: To every permutation γ we add one fixed element, i.e. $x \in N$ with $x^\gamma = x$ if there is no one left. There exists no permutation $\gamma \in S_\infty$ satisfying $\gamma\gamma_1^{-1} \in \mathcal{R}$.

The examples A and B determine 2-structures which are not affine planes.

Remark. As we extend only finite partial permutations and we have only a finite system of permutations we can always choose the elements of N in the step II so that (9) and (10) are satisfied.

2. CONSTRUCTION OF 2-STRUCTURES

Definition 5. Let (M, G) be a sharply doubly transitive set of permutations. The triple $(a, b, \gamma) \in M \times M \times G$ is called *admissible* ([2] p. 7) if $a^\gamma \neq b$.

Denote by $\mathfrak{M}$ the set of all admissible triples of (M, G). For $(a, b, \gamma) \in \mathfrak{M}$ define the multiplicity $N(a, b, \gamma)$:

$N(a, b, \gamma) = 0$ if there is no $\delta \in G$ such that $a^\delta = b$, $\gamma\delta^{-1} \in \mathscr{R}_M$;

$N(a, b, \gamma) = 1$ if there is only one $\delta \in G$ such that $a^\delta = b$, $\gamma\delta^{-1} \in \mathscr{R}_M$;

$N(a, b, \gamma) = 2$ if there exist $\delta_1, \delta_2 \in G$, such that $\delta_1 \neq \delta$, $a^{\delta_i} = b$, $\gamma\delta_i^{-1} \in \mathscr{R}_M$, $i = 1, 2$.

We say that: (M, G) has the type (j) for $j \in \{0, 1, 2\}$ if every $(a, b, \gamma) \in \mathfrak{M}$ has the multiplicity (j);

(M, G) has the type (i, j) if there exist $(a, b, \gamma) \in \mathfrak{M}$ with the multiplicity i, $(a', b', \gamma') \in \mathfrak{M}$ with the multiplicity j and one with the multiplicity k, $\{i, j, k\} = \{0, 1, 2\}$;

(M, G) has the type $(0, 1, 2)$ if there exist triples with all multiplicities.

The notions of Definition 5 have a geometric meaning for the corresponding 2-structure.

If $(\mathscr{P}, \mathscr{L}, \in)$ is the corresponding 2-structure, then the condition $\gamma\delta^{-1} \in \mathscr{R}_M$ means that the corresponding lines $\tilde{\gamma}$ and $\tilde{S}$ have $\tilde{S}$ have no intersection point.

Theorem 1 and Theorem 3 imply

Theorem 4. a) *(M, G) has the type (1) if and only if the corresponding 2-structure is an affine plane.*

b) *If M is a finite set, then (M, G) has always the type (1).*

Similarly, for a 2-structure $(\mathscr{P}, \mathscr{L}, \in) = \mathscr{I}$ we define a pair $(P, l) \in \mathscr{P} \times \mathscr{L}$ to be admissible if $P \notin l$.

We say that an admissible pair (P, l) has the multiplicity j (writing $\tilde{N}(P, l) = j$) for $j \in \{0, 1, 2\}$ in the following cases:

$$\tilde{N}(P, l) = 0 \Leftrightarrow^{\text{Def}} \forall l_1 \in \mathscr{L}(P \in l_1 \Rightarrow l \cap l_1 = \emptyset),$$

$$\tilde{N}(P, l) = 1 \Leftrightarrow^{\text{Def}} \exists! \, l_1 \in \mathscr{L} \ (P \in l_1 \wedge l \cap l_1 = \emptyset),$$

$$\tilde{N}(P, l) = 2 \Leftrightarrow^{\text{Def}} \exists! \, l_1, l_2 \in \mathscr{L}_1 \ l_1 \neq l_2 \ \forall i \in \{1, 2\} \ (P \in l_i \wedge l \cap l_1 = \emptyset).$$

A 2-structure $\mathscr{I}$ has the type (j) if every admissible pair has the multiplicity j, type

(j, k) if there exists an admissible pair of the multiplicity j and an admissible pair of the multiplicity k while no admissible pair has the multiplicity m. (Here $\{j, k, m\} = \{0, 1, 2\}$.)

We say it has the type $(0, 1, 2)$ if there exist admissible pairs with all multiplicities.

Theorem 5. a) (M, G) *has the type* α *if and only if the corresponding* 2-*structure has the same type.*

$$(\alpha \in \{(0), (1), (2), (0, 1), (0, 2), (1, 2), (0, 1, 2)\}.)$$

b) *If* $(\mathscr{P}, \mathscr{L}, \in)$ *is a* 2-*structure of the type* 0 *then every two distinct lines from* $\mathscr{L}_0$ *have an intersection point.*

The proof is obvious.

The main result of this paper is

Theorem 6. *There exist* 2-*structures of all types.*

We shall construct 2-structures of all types except the type (1) (this type is that of every affine plane). All 2-structures constructed in the sequel are not affine planes. For better description of the construction we take two singular points $\alpha \notin \mathscr{P}$, $\beta \notin \mathscr{P}$ and suppose that α lies exactly on all lines of $\mathscr{L}_1$, β lies exactly on the lines of $\mathscr{L}_2$ and there is no line containing both α and β.

Let K be an arbitrary set, $\# K \geqq 2$. Put

$$\mathscr{P}^0 = K \times K \cup \{\alpha, \beta\}, \quad \mathscr{L}^0 = \mathscr{L}_1^0 \cup \mathscr{L}_2^0 \cup \mathscr{L}_0^0, \quad \mathscr{I}^0 = (\mathscr{P}^0, \mathscr{L}^0, \in),$$

$\mathscr{L}_1^0$ is formed by all subsets of $\mathscr{P}^0$ of the form

$$\{(k, x) \mid x \in K\} \cup \{\alpha\} \quad \forall k \in K,$$

$\mathscr{L}_2^0$ is formed by all subsets of $\mathscr{P}^0$ of the form

$$\{(x, k) \mid x \in K\} \cup \{\beta\}$$

and $\mathscr{L}_0^0$ will be defined in each case separately.

We proceed by induction constructing step by step the incidence structures $\mathscr{I}^0 = (\mathscr{P}^i, \mathscr{L}^i, \in)$ in the following ways.

Case I. $\mathscr{L}_0^0 = \emptyset$ a) $\mathscr{L}^i$ is obtained from $\mathscr{L}^{i-1}$ by adding all lines joining the points which have not been joined so far (i.e. new lines $\{A, B\}$, $A, B \in \mathscr{P}^{i-1}$ such that there exists no line from $\mathscr{L}^{i-1}$ going through A and B) with the exception of the line joining α and β.

b) $\mathscr{P}^i$ is obtained from $\mathscr{P}^{i-1}$ by adding all intersection points of the pairs of lines which have no intersection point in $\mathscr{P}^{i-1}$ (i.e. new distinct points $\{l, m\}$, $l, m \in \mathscr{L}^i$ with $l \cap m = \emptyset$ in $\mathscr{P}^{i-1}$, we set $\{l, m\} \in l, m$.)

Case II. $\mathscr{L}_0^0 = \emptyset$, $\#K \geqq 3$

a) $\mathscr{L}^i$ is obtained as in Ia.

b) $\mathscr{P}^i$ is obtained by adding to $\mathscr{P}^{i-1}$ all intersection points of lines going through α or β with other lines.

Case III. For $\#K \geqq 3$, we fix an arbitrary point $A \in K \times K$.

a) $\mathscr{L}^i$ is obtained as in Ia.

b) $\mathscr{P}^i$ is obtained by adding to $\mathscr{P}^{i-1}$ all intersection points of all lines going through α or β with other lines, and all intersections of lines going through A with other lines.

Case IV. For $\#K \geqq 3$. Let $\Delta = \{(k, k) \mid k \in K\}$ and let Δ_1 be a subset of $K \times K$ satisfying

(i) $\forall k \in K \ \exists!\ x(k, x) \in \Delta_1$,

(ii) $\forall k \in K \ \exists!\ y(y, k) \in \Delta_1$,

(iii) $\Delta \cap \Delta_1 = \emptyset$.

Fix $A \in \Delta_1$ and put $\mathscr{L}_0^0 \in \{\Delta, \Delta_1\}$.

a) $\mathscr{L}_i$ is obtained as in Ia.

b) $\mathscr{P}^i$ is obtained from $\mathscr{P}^{i-1}$ by adding the intersection points of all lines from $\mathscr{L}^i$ with the lines going through α or β and the intersection points of all lines, with the exception of Δ_1, going through A with the line Δ.

Case V. For $\#K \geqq 3$. Choose Δ, Δ_1 and set $\mathscr{L}_0^0 = \{\Delta, \Delta_1\}$ as in IV.

a) $\mathscr{L}^i$ is obtained as in Ia.

b) $\mathscr{P}^i$ is obtained from $\mathscr{P}^{i-1}$ by adding the intersection points of all lines, with the only exception of the intersection point of Δ with Δ_1.

Case VI. For $\#K \geqq 3$. Choose Δ as in IV and take $A \notin \Delta$. Set $\mathscr{L}_0^0 = \{\Delta\}$.

a) $\mathscr{L}^i$ is obtained as in Ia.

b) $\mathscr{P}^i$ is obtained from $\mathscr{P}^{i-1}$ by adding the intersection points of all lines from $\mathscr{L}^i \setminus \{\Delta\}$ and further by adding the intersection points of Δ with the lines going through α or β.

A modification of VI is obtained from VI if we substitute b) by b)':

b)' $\mathscr{P}^i$ is obtained from $\mathscr{P}^{i-1}$ by adding the following intersection points: Intersection points of any line with the lines going through α or β, and intersection points of the lines going through A with all lines from $\mathscr{L}^i \setminus \{\Delta\}$.

In all these cases we add intersection points only if the corresponding lines do not intersect in $\mathscr{P}^{i-1}$, and we set

$$\mathscr{P} = \bigcup_{i=0}^{\infty} \mathscr{P}^i \setminus \{\alpha, \beta\}, \quad \mathscr{L} = \bigcup_{i=1}^{\infty} \mathscr{L}^i, \quad \mathscr{L}_1 = \{g \in \mathscr{L} \mid \alpha \in g\}, \quad \mathscr{L}_2 = \{g \in \mathscr{L} \mid \beta \in g\}.$$

Theorem 7. *The incidence structures constructed in* I–VI *are 2-structures which are not affine planes.*

They are of the following types: In the case I of the type (0), *in the case* II *of the type* (2), *in the case* III *of the type* (0, 2), *in the case* IV *of the type* (0, 1), *in the case* V *of the type* (1, 2) *and in the case* VI *of the type* (0, 1, 2)

Proof. From the part (a) of the constructions it follows that any two points can be joined by a unique line and to each point of $\mathscr{P}$ there exists a unique line from $\mathscr{L}_1$ and a unique line from $\mathscr{L}_2$ going through the point. As a consequence of the part b) we get that every line from $\mathscr{L}_i$ intersects every line from $\mathscr{L} \setminus \mathscr{L}_i$ $(i = 1, 2)$.

Distinct lines from $\mathscr{L}_1$ or $\mathscr{L}_2$ have no intersection point. Moreover, in the individual cases we have:

I. Every pair of lines from $\mathscr{L}_0$ has an intersection point, every admissible pair has the multiplicity 0 and the 2-structure is of the type (0).

II. Let $l, m \in \mathscr{L}_0$ be arbitrary lines. Then there exists an index k so that $l, m \in \mathscr{L}^k$). If l does not intersect m at a point from $\mathscr{P}^k$, then l does not intersect m.

If (A, l) is an admissible pair, then there exists k such that $A \in \mathscr{P}^k$, $l \in \mathscr{L}^k$. Let us choose points $A_1, A_2 \in \mathscr{P}^{k+1}$, A_1, A_2, A non-collinear, and $A_1, A_2 \notin l$ (e.g., newly added intersection points). Then AA_1 and AA_2 do not inersect l. The 2-structure is of the type (2).

III. If $B \in \mathscr{P}$, $B \neq A$ and $l \in \mathscr{L}_0$ is a line not going through A, (B, l) is an admissible pair, then (B, l) has the multiplicity 2 by virtue of II. If $m \in \mathscr{L}_0$ is a line going through A and $B \notin m$ an arbitrary point, then every line going through B intersects m. Similarly if (A, t) is an admissible pair then every line going through A_0 intersects t. The pairs $\{(A, l) \mid A \notin l\}$, $\{B, m) \mid B \notin m,\ A \in m\}$ are of the multiplicity 0, the 2-structure is thus of the type (0, 2).

IV. If $B \in \mathscr{P}$, $B \neq A$ and (B, l) is an admissible pair then there exist at least two lines going through B and not intersecting l. The pair (B, l) has the multiplicity 2. Similarly, the pair (A, l) for $l \neq \Delta$ has the multiplicity 2. The pair (A, Δ) has the multiplicity 1. The 2-structure is of the type (1, 2).

V. Through every point $A \in \Delta$ there goes a unique line non-intersecting Δ_1, namely Δ. Similarly, through every point of Δ_1 there goes a unique line non-intersecting Δ, namely Δ_1. The admissible pairs $\{(A, \Delta_1) \mid A \in \Delta\}$, $\{(B, \Delta), B \in \Delta_1\}$ have the multiplicity 1. The others have the multiplicity 0 (see I). The 2-structure has the type (0, 1).

VI. Let (B, Δ) be an admissible pair and $B \in \mathscr{P}^k$. If $A_1, A_2 \in \mathscr{P}^k$ are such that B, A_1, A_2 are non-collinear and BA_1, BA_2 do not intersect Δ in $\mathscr{P}^k$ then the lines BA_1, BA_2 do not intersect Δ even in $\mathscr{P}$, and the pair (B, Δ) has the multiplicity 2. Choose now $X \in \Delta$ and a line m non-intersecting Δ. (Such a line surely exists; it is e.g. the line joining points not lying on Δ in $\mathscr{L}^i$). Then there is a unique line going through X and non-intersecting m, namely Δ. The pairs (X, Δ) have the multiplicity 1. If l is a line intersecting Δ, then every line going through X intersects l and thus (X, l) has the multiplicity 0. The 2-structure is of the type $(0, 1, 2)$.

VI′. Here we can proceed similarly as in VI. Admissible pairs (A, l), $l \neq \Delta$ has the multiplicity 0, the admissible pair (A, Δ) has the multiplicity 2. The admissible pairs of the type (X, l) for $X \in \Delta$, $A \in l$, and $l \cap \Delta = \emptyset$ have the multiplicity 1, the admissible pairs (B, l), $A \in l$, $B \notin \Delta$ have the multiplicity 0, the other admissible pairs have the multiplicity 2. The 2-structure has the type $(0, 1, 2)$.

Definition 6. A regular incidence structure $(\mathscr{P}, \mathscr{L}, \in)$ provided with a decomposition $\mathscr{L} = \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2$ satisfying (2), (3) from Definition 1 is called a *partial* 2-*structure.*

Remark. After a modification, the constructions I–VI can be applied to an arbitrary partial 2-structure.

We arrive at various 2-structures also when studying some well-known algebraic structures. This questions will be discussed in the last part of this paper.

Theorem 8. *Let $\mathscr{F} = (Q, +, \cdot)$ be a proper reduced quasifield (i.e. there exist $a, b, c \in Q$, $a \neq b$ such that the equation $-a \cdot x + b \cdot x = c$ has no solution). The corresponding incidence structure is a 2-structure (namely, a weak affine plane) which is not an affine plane.*

Further, if $\mathscr{F}$ satisfies the condition

$$(12) \qquad \exists b', c' \in Q \setminus \{0\} \; \forall m \in Q \setminus \{b'\} \; \exists x \in Q \quad (-m \cdot x + b' \cdot x = c')$$

then the 2-structure has the type $(1, 2)$. If (12) is not satisfied the 2-structures has the type (2).

Proof. The 2-structure $\mathscr{I} = (\mathscr{P}, \mathscr{L}, \in)$ which corresponds to $\mathscr{F}$ has the form

$$\mathscr{P} = Q \times Q\,, \quad \mathscr{L}_1 = \{\{(a, x) \mid x \in Q \mid a \in Q\}\}\,, \quad \mathscr{L}_2 = \{\{(x, a) \mid x \in Q\} \mid a \in Q\}\,,$$
$$\mathscr{L}_0 = \{\{(x, y) \mid y = m \cdot x + b \mid x \in Q\}\,, \quad m \neq 0\,, \quad m, b \in Q\,.$$

The pair (P, l), $P = (0, 0)$ and $l = \{(x, ax + c) \mid x \in Q\}$ has the multiplicity 2, because there exist two lines $l_1 = \{(x, ax) \mid x \in Q\}$ and $l_2 = \{(x, bx) \mid x \in Q\}$ non-intersecting l and going through P.

If the condition (12) is satisfied, then the pair (P, q), $P = (0, 0)$ and $q = \{(x, mx + c) \mid x \in Q\}$ has the multiplicity 1, because every line $q'_1 \neq q_1$, $q_1 = \{(x, mx) \mid x \in Q\}$ which goes through P intersects q. There is no admissible pair of the multiplicity 0.

If the condition (12) is not satisfied then no lines of the form

$$\{(x, bx + c_1) \mid x \in Q\}\,, \quad \{(x, mx + c_2) \mid x \in Q\}$$

have an intersection point. Every admissible pair has the multiplicity 2 and the 2-structure has the type (2).

If we define the relation $\parallel$ on $\mathscr{L}$ in such a way that the first class of equivalence is the set $\mathscr{L}_1$, the second is $\mathscr{L}_2$ and for $l_1, l_2 \in \mathscr{L}_0$ of the form

$$l_1 = \{(x, rx + t_1) \mid x \in Q\}, \quad l_2 = \{(x, sx + t_2) \mid x \in Q\}$$

we set $l_1 \parallel l_2$ if $r = s$, we obtain the relation which has all the properties from Definition 3.

Theorem 9. *For a proper nonplanar nearfield* $\mathscr{F} = (Q, +, \cdot)$ (*i.e., there exist* t, s, $t \neq 1$ *such that the equation* $t \,.\, x = x + s$ *has no solution*) *the condition* (12) *is not satisfied. The corresponding* 2*-structure has the type* 2.

Proof. Theorem 9 follows immediately from Theorem 8 and Theorem 3,2 of [6].

Let $\mathscr{F} = (Q, +, \cdot)$ be a nonplanar quasifield, $\mathscr{I}(\mathscr{F})$ the corresponding 2-structure. If we construct the sharply doubly transitive set of permutations (M, G) which corresponds to $\mathscr{I}(\mathscr{F})$ with the base line $\Delta = \{(x, x) \mid x \in Q\}$ we obtain the permutations from G in the form

$$\gamma \in G \Rightarrow x^{\gamma} = m \,.\, x + b \,.$$

Then G is a permutation group and $G^{\cdot}$ is a subgroup of G. $G^{\cdot}$ is the set of all elements $\bar{b}$ of the form

$$x^{\bar{b}} = x + b\,, \quad b \in Q\,.$$

Remark. There exists a non-planar nearfield (see e.g. [5, [6]) and a non-planar quasifield (see [4]).

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Author's address: 118 00 Praha, Malostranské nám. 25, ČSSR (Matematicko-fyzikální fakulta UK).