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Limits and colimits in generalized algebraic categories


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LIMITS AND COLIMITS IN GENERALIZED ALGEBRAIC CATEGORIES

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The paper presents a complete discussion of the existence of limits and colimits in a certain class of categories, which generalize categories of universal algebras. Given functors $F, G$ from sets to sets (with arbitrary variances) the generalized algebraic category $A(F, G)$ has objects $(X, \omega)$ where $X$ is a set and $\omega : FX \to GX$ is a mapping; morphisms from $(X, \omega)$ to $(X', \omega')$ are mappings $f : X \to X'$ for which the diagram, consisting of $Ff, Gf, \omega$ and $\omega'$ commutes. An example of such a category is presented by every category of universal algebras of a given type; and there are many others.

The generalized algebraic categories were introduced by V. TRNKOVA and P. GORALČÍK. A considerable amount of papers investigate limits and colimits in $A(F, G)$ with a common variance of $F$ and $G$. The present paper is devoted to the case that the variances differ. The diagram mentioned above is

\[
\begin{array}{ccc}
FX & \xrightarrow{\omega} & GX \\
\downarrow Ff & & \downarrow Gf \\
FX' & \xrightarrow{\omega'} & GX'
\end{array}
\]

$F$ covariant, $G$ contravariant $\quad F$ contravariant, $G$ covariant

We investigate the existence of limits and colimits in the categories $A(F, G)$ and $A(G, F)$ where $F$ is an arbitrary contravariant set functor and $G$ a covariant one. We omit the case that some of the functors is constant for the sake of brevity. The table below, where $+$ means that the limits exist and $-$ means the contrary, summarizes the results.

The results concerning limits in $A(F, G)$ are analogous to those concerning colimits in $A(G, F)$. We denote the analogous theorems by $\ast$ and, if the proofs are also quite similar, we omit them. The same holds for colimits in $A(F, G)$ and limits in $A(G, F)$. Notice that the results are independent of the choice of the functors, depending
<table>
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<td>( A(F, G) )</td>
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<td>+ IF (</td>
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<td>( A(G, F) )</td>
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<td>+ IF ( |F_0| = 0 ) or ( |G_0| = 1 )</td>
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*) + holds for an arbitrary connected diagram scheme.

only on their variances (the only exception is singleton and cosingleton). We prove, moreover, that whenever limits or colimits exist then, in the underlying sets, they coincide with the limits and colimits in \( \text{Set} \), the category of sets. More precisely, they are preserved by the natural forgetful functor \( \Box : A(F, G) \to \text{Set} \) (\( \Box(X, \omega) = X \) and \( \Box f = f \) for a morphism \( f \)).

The contents of the paper:

I. Preliminaries
II. Limits in \( A(F, G) \) (colimits in \( A(G, F) \))
III. Colimits in \( A(F, G) \) (limits in \( A(G, F) \))

I. PRELIMINARIES

A. Given a category \( \mathcal{D} \), \( \mathcal{D}^a \) denotes the class of objects and \( \mathcal{D}(a, b) \) the set of morphisms from \( a \) to \( b \).

B. The category with no objects will be denoted by \( \emptyset \). \( 1 \) will denote both the standard one-point set \( 1 = \{0\} \) and its cardinal. The power of a set \( X \) is denoted by \( \|X\| \).

C. A diagram in a category \( K \) is, as usual, a functor from a small category, called the scheme, into \( K \). \( K \) is said to have limits over a scheme \( \mathcal{D} \) if each (covariant) diagram over this scheme in \( K \) has a limit; analogously for colimits.

D. A small category \( \mathcal{D} \) is said to be connected if \( \mathcal{D} \neq \emptyset \) and whenever \( \mathcal{D} \) is a sum \( \mathcal{D} = \mathcal{D}_1 \lor \mathcal{D}_2 \) then either \( \mathcal{D}_1 = \emptyset \) or \( \mathcal{D}_2 = \emptyset \). (In particular, \( \emptyset \) is disconnected.) Clearly, a category \( \mathcal{D} \neq \emptyset \) is connected if and only if for each pair of its objects \( d, d' \) there exist objects \( d = d_0, d_1, \ldots, d_k = d' \) such that for every \( i = 1, \ldots, k \) either \( \mathcal{D}(d_{i-1}, d_i) \) or \( \mathcal{D}(d_i, d_{i-1}) \) is non-void.
E. The limit of the void diagram (the diagram over 0) is called the singleton. It is an object \( s \) such that from any object \( o \) there leads just one morphism to \( s \); analogously for cosingleton. In \( \text{Set} \), any one-point set is a singleton and the void set is a cosingleton.

F. Given a mapping \( f : A \rightarrow B \), denote by \( \text{im} f \) the set of all \( f(a) \) where \( a \in A \). If \( f \) is the constant mapping to \( b \in B \) we shall write \( f = \text{const} b \).

G. Let \( H \) be a covariant set functor. An element \( t \) of \( H1 \) is said to be a distinguished point of \( H \) if, given arbitrary mappings \( f, g : 1 \rightarrow X \), we have \( Hf(t) = Hg(t) = t_X \). Clearly, for an arbitrary mapping \( k : A \rightarrow B \) we have \( Hk(t_A) = t_B \).

H. All monomorphisms in \( \text{Set} \) are coretractions, if the domain is not 0. Thus, if \( f \) is a one-to-one mapping with a non-void domain then for any set functor \( H \), if \( H \) is covariant then \( Hf \) is one-to-one and if \( H \) is contravariant then \( Hf \) is onto. Analogously for mappings onto. If \( H \) is an arbitrary non-constant set functor then \( HX \neq 0 \) for any set \( X \neq 0 \). We call, for short, a set functor constant if its restriction to the category of non-void sets is a constant functor.

I. A non-void collection \( \{f_i\}_{i \in I} \) of mappings with a common domain \( X \neq 0 \) is said to be a collective monomorphism if, given arbitrary distinct \( a, b \in X \), there exists \( i \in I \) such that \( f_i(a) \neq f_i(b) \). We say that a covariant set functor \( H \) preserves collective monomorphisms (or, respectively, finite collective monomorphisms) if, given a collective monomorphism \( \{f_i\}_{i \in I} \) (where \( I \) is finite), then also \( \{Hf_i\}_{i \in I} \) is a collective monomorphism.

J. Throughout the following text, \( F \) and \( G \) denote respectively an arbitrary non-constant contravariant or covariant set functor.

K. Let \( H \) be an arbitrary non-constant set-functor (covariant or contravariant). It is proved in [6, 7] that for every cardinal number there exists a set \( X \) such that the power of \( HX \) is bigger than the cardinal number.

II. LIMITS IN \( A(F, G) \) (COLIMITS IN \( A(G, F) \))

Lemma 2.1. \( A(F, G) \) has a singleton if and only if \( |G1| = 1 \). The singleton has then a one-point underlying set (i.e., it is preserved by \( \square \)).

Proof. If \( |G1| = 1 \) then there is just one object in \( A(F, G) \) with the underlying set 1 and it is easy to verify that it is a singleton. If \( |G1| \neq 1 \) then \( |G1| > 1 \) \( (G1 \neq 0 \) since \( G \) is non-constant). Assume that \( A(F, G) \) has nevertheless a singleton \( (B, \omega) \).
Choose \( a, b \in G_1, a \neq b \). There exists a unique \( t_a^* : (1, \text{const } a) \to (B, \omega) \) and a unique \( t_b^* : (1, \text{const } b) \to (B, \omega) \). Then clearly \( \omega = \text{const } (G t_a(a)) = \text{const } (G t_b(b)) \), in particular \( G t_a(a) = G t_b(b) \). Let \( d : B \to 1 \), then we have \( dt_a = dt_b = \text{id}_1 \) and so \( a = b \), a contradiction.

**Lemma 2.1**. \( A(G, F) \) has a cosingleton if and only if either \( G\emptyset = \emptyset \) or \( |F\emptyset| = 1 \). The cosingleton has then the void underlying set (i.e., it is preserved by \( \square \)).

**Theorem 2.2**. The forgetful functor \( \square \) preserves all limits which exist in \( A(F, G) \).

Proof. Let \( D : \mathcal{D} \to A(F, G) \) be an arbitrary diagram. Due to Lemma 2.1 we may assume that \( \mathcal{D} \) is non-void. For \( d \in \mathcal{D} \) denote \( Dd = (X_d, \omega_d) \). Let \( \langle X, \{f_d\}_{d \in \mathcal{D}} \rangle \) be the limit of \( \square \cdot D \) (in \( \text{Set} \)).

Assume that \( D \) has a limit in \( A(F, G) \), \( \langle (Z, \omega), \{r_d\}_{d \in \mathcal{D}} \rangle \). Then \( \langle Z, \{r_d\} \rangle \) is a bound of \( \square \cdot D \) and so there exists a unique mapping \( r : Z \to X \) with \( r_d = f_d \cdot r \) for all \( d \).

We shall prove that then also \( \langle (X, G r \cdot \omega \cdot Fr), \{f_d\} \rangle \) is a limit of \( D \). Let \( \langle (Z', \omega'), \{r'_d\} \rangle \) be an arbitrary bound of \( D \). There exists a unique mapping \( r' : Z' \to X \) with \( r'_d = f_d \cdot r' \). It clearly suffices to show that \( r' \) is a morphism, i.e. that \( Gr \cdot \omega \cdot Fr = Fr' \cdot \omega' \cdot Fr' \). But this is a simple consequence of the fact that \( (Z, \omega) \) is a limit of \( D \).

**Theorem 2.2**. The forgetful functor \( \square \) preserves all colimits which exist in \( A(F, G) \).

**Definition.** We say that a category \( K \) is directed if for each pair of its objects \( d_1, d_2 \) there exists an object \( d \) with \( K(d_1, d) \neq \emptyset \neq K(d_2, d) \). The dual notion: dual directed.

**Definition.** A contravariant set functor \( H \) is said to spread limits over a scheme \( \mathcal{D} \) if there exists a diagram \( D : \mathcal{D} \to \text{Set} \) with a limit \( \langle Z, \{f_d\}_{d \in \mathcal{D}} \rangle \) such that \( HZ \neq \emptyset \neq \bigcup \text{im } Hf_d \).
Proposition 2,3. Every non-constant contravariant set functor spreads limits over any scheme which is not dual directed.

Proof. Let $F$ be a non-constant contravariant set functor.

I. $F$ spreads products of pairs.

Let $\{f, g\}$ be a collective monomorphism, $f : M \to X$, $g : M \to Y$. If $FM \neq \pm \mathrm{im}\, Ff \cup \mathrm{im}\, Fg$ then $F$ spreads the product of $X$, $Y$ denoted $\langle X \times Y, \{\pi_X, \pi_Y\}\rangle$: let $h : M \to X \times Y$ be the unique mapping with $f = \pi_X h$, $g = \pi_Y h$. Then $h$ is one-to-one and so $Fh$ is onto and we have $\mathrm{im}\, F\pi_X \cup \mathrm{im}\, F\pi_Y = \mathrm{im}\, F(\pi_X h) \cup \mathrm{im}\, F(\pi_Y h) \neq \pm F(X \times Y)$. Now, assume that for an arbitrary collective monomorphism $\{f, g\}$ we have $FM = \mathrm{im}\, Ff \cup \mathrm{im}\, Fg$. Then denote $P^- = \mathrm{Hom}(\cdot, 2)$; clearly $\{P^- \circ Ff, P^- \circ Fg\}$ is a collective monomorphism, thus $P^- \circ F$ preserves finite collective monomorphisms. It was proved in [10] that, whenever a covariant set functor preserves finite collective monomorphisms then it can be expressed as a sum $K_1 \lor K_2$ where $K_1$ has no distinguished point and $K_2$ is constant. Such a functor has clearly the following property: given an arbitrary distinguished point $t$ of $K$ and an arbitrary mapping $h : A \to B$ we have $Kh^{-1}(t_B) = \{t_A\}$. (This is, in fact, a reformulation of the above condition.) The proof of I will be completed if we show that $P^- \circ F$ does not fulfil this condition. As $F$ is non-constant there clearly exists a mapping $f : A \to B$ such that $Ff$ is not onto. Let $p \in FA = Ff$ and let $t$ be a distinguished point of $P^- \circ F : t = \emptyset$ (as an element of $\exp\, F1$); for every set $X$ we have $t_X = \emptyset$. As $P^- \circ F(t) = 0$, clearly $(P^- \circ F)^{-1}(t_B) \not\subseteq \{t_A\}$.

II. $F$ spreads limits over any scheme $\emptyset$ which is not dual directed.

Let $X$, $Y$ be sets with the product $\langle X \times Y, \{\pi_X, \pi_Y\}\rangle$ such that $F(X \times Y) \neq \pm (\mathrm{im}\, F\pi_X \cup \mathrm{im}\, F\pi_Y)$. Further, let $d_1$, $d_2$ be objects of $\emptyset$ such that for an arbitrary object $d$ either $\emptyset(d, d_1) = \emptyset$ or $\emptyset(d, d_2) = \emptyset$. Define a diagram $D : \emptyset \to \text{Set}$:

\begin{align*}
Dd &= X & \text{if} & \emptyset(d, d_1) = \emptyset, \\
Dd &= Y & \text{if} & \emptyset(d, d_2) = \emptyset, \\
Dd &= 1 & \text{if} & \emptyset(d, d_i) = \emptyset, & i = 1, 2; \\
\end{align*}

let $\delta$ be a morphism from $d$ to $d'$

\begin{align*}
D\delta &= id_X & \text{if} & Dd' = X, \\
D\delta &= id_Y & \text{if} & Dd' = Y, \\
D\delta &= \text{const} & \text{if} & Dd' = 1.
\end{align*}

Clearly, $\langle X \times Y, \{f_d\}\rangle$, where $f_d$ is either $\pi_X$ or $\pi_Y$ or const, is a limit of $D$ in $\text{Set}$ and $F(X \times Y) \neq \bigcup \mathrm{im}\, Ff_d$. This completes the proof.

Theorem 2,4. If $F$ spreads limits over a scheme $\emptyset$ then $A(F, G)$ has not limits over $\emptyset$.  

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Proof. Let $D_0 : \mathcal{D} \to \text{Set}$ be a diagram whose limit $\langle Z', \{f_d'\} \rangle$ is spread by $F$. Let $M$ be a set with $|GM| > 1$ (it exists as $G$ is non-constant). Let $D_0$ be the sum of the diagram $D_0$ and of the constant diagram to $M$ (then, for each object $d$, $D_0 d = D_0' d \vee M$; denote by $g_d$ the inclusion of $D_0' d$ into $D_0 d$). If $\langle Z, \{f_d\} \rangle$ is a limit of $D_0$ then there exists a unique $g : Z' \to Z$ with $f_d g = g_d f_d'$; $g$ is clearly one-to-one. Then $F$ spreads the limit of $D_0$, too: if $FZ = \bigcup \text{im } Ff_d$ then $FZ' = \bigcup \text{im } Fg Ff_d'$ (as $Fg$ is onto) but $FZ' \supseteq \bigcup \text{im } Ff_d' \supseteq \bigcup \text{im } Ff_d Fg_d = \bigcup \text{im } Fg Ff_d'$.

We shall now find a diagram $D : \mathcal{D} \to A(F, G)$ which has no limit in $A(F, G)$. As $|GM| > 1$, clearly also $|GZ| > 1$; let $a, b$ be distinct elements of $GZ$. The diagram $D$ is defined as follows: for each object $d$, $Dd = (D_0 d, \text{const } Gf_d(a))$ and,

\[
\begin{array}{c}
FZ \\
Ff_d \\
FDd \\
GZd \\
Gf_d \\
a, b \in GZ
\end{array}
\]

moreover, $\square : D = D_0$. Assume that $D$ has a limit in $A(F, G)$. Then, due to Theorem 2.2, without loss of generality the limit is $\langle (Z, \omega), \{f_d\} \rangle$ for a suitable $\omega : FZ \to GZ$. We have clearly two bounds of $D : \langle (Z, \text{const } a), \{f_d\} \rangle$ and $\langle (Z, \omega'), \{f_d\} \rangle$ where $\omega' = \text{const } a$ on $\bigcup \text{im } Ff_d$ and $\omega = \text{const } b$ on $FZ - \bigcup \text{im } Ff_d$. Therefore there exists a unique $r : (Z, \text{const } a) \to (Z, \omega)$ such that for all $d$, $f_d r = f_d$ (which implies that $r = id_Z$ and so $\omega = \text{const } a$); furthermore there exists a unique $r' : (Z, \omega') \to (Z, \omega)$ such that $f_d r' = f_d$ and so $\omega' = \omega = \text{const } a$, a contradiction. Therefore $D$ has no limit in $A(F, G)$.

Corollary 2.5. $A(F, G)$ has not limits over any scheme which is not dual directed. In particular, it has not finite products and pullbacks.

Theorem 2.5*. $A(G, F)$ has not colimits over any scheme which is not directed. In particular, it has not finite sums and pushouts.

Proof.
I. $A(G, F)$ has not finite sums.

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As both functors $G, F$ are non-constant there clearly exist sets $U, V$ with the sum $(U \lor V, \{i_U, i_V\})$ such that $|\text{im } G_i_U - \text{im } G_i_V| > |G|$ and that $F_i_U$ is not a bijection. Then $F_i_U$ is not one-to-one and we may choose distinct $t, u \in F(U \lor V)$ with $F_i_U(t) = F_i_U(u)$; moreover, we choose an arbitrary $a \in \text{im } G_i_U - \text{im } G_i_V$. Then 

$A(G, F)$ has no sum of $(U, \text{const } F_i_U(t))$ and $(V, \text{const } F_i_V(t))$. Assume the contrary. Then due to Theorem 2.2 the sum would have a form $\langle(U \lor V, \omega), \{i_U, i_V\}\rangle$ for a suitable $\omega$. Clearly, $\langle(U \lor V, \text{const } t), \{i_U, i_V\}\rangle$ is a cobound and so there exists a unique $r : (U \lor V, \omega) \rightarrow (U \lor V, \text{const } t)$ such that $r \cdot i_U = i_U$ and $r \cdot i_V = i_V$ — then, of course, $r = \text{id}_{U \lor V}$, which proves that $\omega = \text{const } t$. On the other hand, as $a \notin \text{im } G_i_V$, we have another cobound $\langle(U \lor V, \omega'), \{i_U, i_V\}\rangle$ where $\omega'(a) = = u$, else $\omega' = \text{const } t$. Then, reasoning as above, we get $\omega' = \omega$ — a contradiction.

II. If $\mathcal{D}$ is not directed then $A(G, F)$ has not limits over $\mathcal{D}$.

Let $d_1, d_2$ be such objects of $\mathcal{D}$ that for each object $d$ either $\mathcal{D}(d_1, d) = \emptyset$ or $\mathcal{D}(d_2, d) = \emptyset$. Let $W_1 = (U, \text{const } F_i_U(t))$ and $W_2 = (V, \text{const } F_i_V(t))$; we proved above that $W_1$ and $W_2$ have no sum. Denote by $p_U$ and $p_V$ the void mappings to the set $U$ and $V$, respectively. Then $i_U \cdot p_U = i_V \cdot p_V$ and we put $x = F(i_U \cdot p_U)(t)$. Finally, put $W_3 = (\emptyset, \text{const } x)$.

To construct a diagram $D : \mathcal{D} \rightarrow A(G, F)$ with no colimit we proceed in the same way as in part II of the proof of Proposition 2.3, using $W_1$, $W_2$, $W_3$ instead of $X, Y, 1$ (in particular, $D\delta$ is the void mapping if the domain is $W_3$).

**Theorem 2.6.** $A(F, G)$ has not equalizers.

**Proof.** As $F$ and $G$ are non-constant functors there clearly exist mappings $f, g : X \rightarrow Y$ such that (a) $FX \neq \text{im } Ff \cup \text{im } Fg$ (i.e., there exists $a \in FX - (\text{im } Ff \cup \text{im } Fg)$) and (b) the equalizer of $Gf, Gg$ in $\text{Set}$ is neither $GX$ nor $\emptyset$ (i.e., there exist
\( t, u \in GX \) such that \( Gf(u) + Gg(u) \) and \( Gf(t) = Gg(t) = t_1 \). Put \( \omega : FX \to GX \), \( \omega(a) = u \), else \( \omega = \text{const} \). Then clearly \( f, g : (X, \omega) \to (Y, \text{const} t_1) \) and we shall prove that \( f, g \) have no equalizer in \( A(F, G) \): if \( i : (Z, \bar{\omega}) \to (X, \omega) \) fulfills \( fi = gi \) then \( Gi \cdot \bar{\omega} \cdot Fi(a) = \omega(a) = u \) and we get a contradiction as \( Gf \cdot Gfi(x) \neq Gg \cdot Gi(x) \) where \( x = \omega Fi(a) \).

**Theorem 2.6*.** \( A(G, F) \) has not coequalizers.

### III. COLIMITS IN \( A(F, G) \) (LIMITS IN \( A(G, F) \))

**Lemma 3.1.** \( A(F, G) \) has not finite sums.

**Proof.** Let \( A_1, A_2 \) be sets with the product \( \langle A_1 \times A_2, \{\pi_1, \pi_2\} \rangle \) such that \( F(A_1 \times A_2) \neq \text{im } F\pi_i \cup \text{im } F\pi_j \) (see Proposition 2.3). The sets can be certainly chosen so that \( |G(A_1 \times A_2)| > 1 \). Choose distinct \( x, y \in G(A_1 \times A_2) \) and let \( t \in F(A_1 \times A_2) \) fulfill \( t \notin \text{im } F\pi_i, i = 1, 2 \).

We shall prove that \( A(F, G) \) has no sum of \( (X, \omega_X) \) and \( (Y, \omega_Y) \), where \( X = Y = A_1 \times A_2 ; \omega_X = \text{const } x, \omega_Y(t) = y \), else \( \omega_Y = \text{const } x \). To do this, it clearly suffices to find distinct cobounds with underlying mappings \( \pi_1 \) and \( \pi_2 \). It follows immediately from the properties of \( A_1, A_2 \) and of \( x, y, t \) that such cobounds are \( \langle (A_1, \text{const } G\pi_1(x)), \{\pi_1, \pi_2\} \rangle \) and \( \langle (A_2, \text{const } G\pi_2(x)), \{\pi_1, \pi_2\} \rangle \).

**Lemma 3.1*.** \( A(G, F) \) has not finite products.

**Proof.** For an arbitrary set \( M \) denote by \( r_m \) the mapping from \( M \) to \( 1 \). As both \( F \) and \( G \) is non-constant, there clearly exists a set \( X \) such that \( |GX| > |G1| \) and \( |FX| > |F1| \). Put \( A = X \times \{1, 2\}, b = A \cup 1 \). Define \( f, g : A \to B \):

- for each \( x \in X \), \( g(\langle x, 1 \rangle) = 0 \) (recall that \( 1 = \{0\} \))
- \( g(\langle x, 2 \rangle) = \langle x, 2 \rangle \),
- \( f(\langle x, 1 \rangle) = \langle x, 1 \rangle, \quad i = 1, 2 \).
Choose $h : B \rightarrow X$ so that $hf$ is onto while $hg$ is constant. As $\text{im } f \cap \text{im } g = \emptyset$ and $|G(\text{im } g)| > |G1|$, clearly there exists $u \in \text{im } Gg - \text{im } Gf$; choose $t \in GA$ with $Gg(t) = u$.

Let us prove that there exist $q \in F1$ and $a \in FB$ such that $Ff(a) \neq Fr_A(q) = Fg(a)$. As $hg$ is constant there exists $j : 1 \rightarrow X$ with $jr_A = hg$. As $|FX| > |F1|$ there exists $q \in F1$ with $|(Fj)^{-1}(q)| > 1$. Choose $b \in FX$, $b \neq Fx(q)$ with $Fj(b) = q$. Put $a = Fh(b)$. As $hf$ is onto, $F(hf)$ is one-to-one and as $r_A = rxh$ we have $Ff(a) \neq Fr_A(q)$. Furthermore, $r_A = rxhf$ and so $Fg(a) = F(jr_A)(b) = Fr_A(q)$.

We shall show that $A(G, F)$ has no product of $(B, \omega_1)$ and $(B, \omega_2)$ where $\omega_1 = \text{const } Fr_B(q)$, $\omega_2(u) = a$ else $\omega_2 = \text{const } Fr_B(q)$. Let, to the contrary, $<(Z, \omega), \{\varphi_1, \varphi_2\}>$ be their product. As $u \notin \text{im } Gf$ we have a bound $<(A, \text{const } Fr_A(q), \{f, f\})$ and as $Fg(a) = Fr_A(q)$ we have a bound $<(A, \text{const } Fr_A(q)), \{g, g\}>$. Therefore there exist mappings $h, k$ such that $f = \varphi_1h = \varphi_2h$ and $g = \varphi_1k = \varphi_2k$. Consequently $F\varphi_2(a) = Fr_z(q)$; hence $\omega(Gk(t)) = F\varphi_1.\omega_1 \cdot G\varphi_1(Gk(t)) = Fr_d(q)$ (because $r_z = r_B \cdot \varphi_1$) and $\omega(Gk(t)) = F\varphi_2 \cdot \omega_2 \cdot G\varphi_2(Gk(t)) = Fr_2 \cdot \omega_2 \cdot (u) = = F\varphi_2(a)$. This leads to a contradiction as $Fh(F\varphi_2(a)) = Ff(a) \neq Fr_A(q) = = Fh(Fr_d(q)) = Fh(F\varphi_2(a))$.

**Lemma 3.2.** $A(F, G)$ has no cosingleton.

**Proof.** Let, to the contrary, $(B, \omega_0)$ be a cosingleton. As the functors $F, G$ are non-constant there exists a set $X$ with $|FX| > |GB|$ and $|GX| > |GB|$. Let $\omega : FX \rightarrow GX$ be an arbitrary mapping with $|\text{im } \omega| > GB$. There exists a unique morphism $f : (B, \omega_0) \rightarrow (X, \omega)$; then $\omega = Ff \cdot \omega_0 \cdot Gf$ which is a contradiction as $|\text{im } \omega_0| < |\text{im } \omega|$.

**Lemma 3.2*.** $A(G, F)$ has no singleton.

**Theorem 3.3.** $A(F, G)$ has colimits over a given scheme if and only if the scheme is connected. In this case the colimits are preserved by the forgetful functor $\square$. 

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**Diagram:**

- $FX$ with $Ff_d$ and $\omega_d$.
- $FX_d$ with $FD\sigma$ and $\omega_{d_1}$.
- $G_X$ with $GD\sigma$ and $\omega_{d_2}$.
- $\ldots$ with $\omega_{d_1}$, $\omega_{d_2}$, and $\omega_{d_3}$.
- $G_Xd$ with $\omega_{d_1}$, $\omega_{d_2}$, and $\omega_{d_3}$.
Proof. It follows from Lemmas 3.1 and 3.2 that \( A(F, G) \) has not colimits over any non-connected scheme \( D \): if \( D \neq 0 \) and \( D = D_1 \vee D_2 \), let \( A_1, A_2 \) be algebras in \( A(F, G) \) which have no sum and let \( D : D \to A(F, G) \) be the diagram which on \( D_1 \) is constant to \( A_1 \) and on \( D_2 \) is constant to \( A_2 \). Then \( D \) has no colimit in \( A(F, G) \).

To prove the theorem we shall show that if \( D \) is an arbitrary connected scheme then \( A(F, G) \) has colimits over \( D \) preserved by \( \Box \). Let \( D : D \to A(F, G) \), let \( Dd = (X_d, \omega_d) \) and let \( \langle X, \{f_d\}_{d \in D} \rangle \) be the colimit of \( D \) in \( \text{Set} \). We try to find \( \omega : FX \to GX \) such that \( \langle X, \omega \rangle, \{f_d\} \) is the colimit of \( D \). To this end verify that \( \omega = Gf_\delta \cdot \omega_\delta \cdot Ff_\delta \) is independent of \( \delta \). In fact, if there exists a morphism \( \delta : d \to d' \) then clearly \( Gf_\delta \cdot \omega_\delta \cdot Ff_\delta = Gf_\delta' \cdot \omega_\delta' \cdot Ff_\delta' \). Now apply the fact that \( D \) is connected (see I.D.).

Using the fact that \( \langle X, \{f_d\} \rangle \) is the colimit of \( D \) in \( \text{Set} \), it is now rather easy to prove that \( \omega \) is the desired mapping.

**Corollary 3.4.** \( A(F, G) \) has pushouts and coequalizers.

**Theorem 3.3*.** \( A(G, F) \) has limits over a given scheme if and only if the scheme is connected. In this case the limits are preserved by the forgetful functor \( \Box \).

**Corollary 3.4*.** \( A(G, F) \) has pullbacks and equalizers.

As a consequence of the results of both parts of the paper we obtain:

**Theorem 3.5.** Let \( H_1, H_2 \) be non-constant set functors with different variances. Whenever the category \( A(H_1, H_2) \) has limits or colimits over a given diagram scheme then they are preserved by the forgetful functor \( \Box \).

**References**


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