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LIMITS AND COLIMITS IN GENERALIZED ALGEBRAIC CATEGORIES

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The paper presents a complete discussion of the existence of limits and colimits in a certain class of categories, which generalize categories of universal algebras. Given functors F, G from sets to sets (with arbitrary variances) the *generalized algebraic category* $A(F, G)$ has objects (X, ω) where X is a set and $\omega : FX \to GX$ is a mapping; morphisms from (X, ω) to (X', ω') are mappings $f : X \to X'$ for which the diagram, consisting of Ff, Gf, ω and ω' commutes. An example of such a category is presented by every category of universal algebras of a given type; and there are many others.

The generalized algebraic categories were introduced by V. TRNKOVÁ and P. GORALČÍK. A considerable amount of papers investgate limits and colimits in $A(F, G)$ with a common variance of F and G. The present paper is devoted to the case that the *variances differ*. The diagram mentioned above is

$$\begin{tikzcd} FX \arrow[r, "\omega"] \arrow[d, "Ff"'] & GX \\ FX' \arrow[r, "\omega'"'] & GX' \arrow[u, "Gf"'] \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[r, "\omega"] & GX \arrow[d, "Gf"] \\ FX' \arrow[u, "Ff"] \arrow[r, "\omega'"'] & GX' \end{tikzcd}$$

F covariant, G contravariant $\qquad$ F contravariant, G covariant

We investigate the existence of limits and colimits in the categories $A(F, G)$ and $A(G, F)$ where F is an arbitrary contravariant set functor and G a covariant one. We omit the case that some of the functors is constant for the sake of brevity. The table below, where $+$ means that the limits exist and $-$ means the contrary, summarizes the results.

The results concerning limits in $A(F, G)$ are analogous to those concerning colimits in $A(G, F)$. We denote the analogous theorems by $*$ and, if the proofs are also quite similar, we omit them. The same holds for colimits in $A(F, G)$ and limits in $A(G, F)$. Notice that the results are independent of the choice of the functors, depending

	Finite Products	Equalizers or Pullbacks	Single-ton	Finite Sums	Coequal. or Pushouts	Cosin-gleton
$A(F, G)$	$-$	$-$	$+$ IFF $\lvert G1\rvert = 1$	$-$	$+$ *)	$-$
$A(G, F)$	$-$	$+$ *)	$-$	$-$	$-$	$+$ IFF $F\emptyset = \emptyset$ or $\lvert G\emptyset\rvert = 1$

*) $+$ holds for an arbitrary connected diagram scheme.

only on their variances (the only exception is singleton and cosingleton). We prove, moreover, that whenever limits or colimits exist then, in the underlying sets, they coincide with the limits and colimits in Set, the category of sets. More precisely, they are preserved by the natural forgetful functor $\square : A(F, G) \to \mathbf{Set}$ ($\square(X, \omega) = X$ and $\square f = f$ for a morphism f).

The contents of the paper:

I. Preliminaries
II. Limits in $A(F, G)$ (colimits in $A(G, F)$)
III. Colimits in $A(F, G)$ (limits in $A(G, F)$)

I. PRELIMINARIES

A. Given a category $\mathscr{D}$, $\mathscr{D}^\sigma$ denotes the class of objects and $\mathscr{D}(a, b)$ the set of morphisms from a to b.

B. The category with no objects will be denoted by $\emptyset$. 1 wil denote both the standard one-point set $1 = \{0\}$ and its cardinal. The power of a set X is denoted by $|X|$.

C. A diagram in a category K is, as usual, a functor from a small category, called the scheme, into K. K is said to have limits over a scheme $\mathscr{D}$ if each (covariant) diagram over this scheme in K has a limit; analogously for colimits.

D. A small category $\mathscr{D}$ is said to be *connected* if $\mathscr{D} \neq \emptyset$ and whenever $\mathscr{D}$ is a sum $\mathscr{D} = \mathscr{D}_1 \vee \mathscr{D}_2$ then either $\mathscr{D}_1 = \emptyset$ or $\mathscr{D}_2 = \emptyset$. (In particular, $\emptyset$ is disconnected.) Clearly, a category $\mathscr{D} \neq \emptyset$ is connected if and only if for each pair of its objects d, d' there exist objects $d = d_0, d_1, \ldots, d_k = d'$ such that for every $i = 1, \ldots, k$ either $\mathscr{D}(d_{i-1}, d_i)$ or $\mathscr{D}(d_i, d_{i-1})$ is non-void.

E. The limit of the void diagram (the diagram over $\emptyset$) is called the *singleton*. It is an object s such that from any object o there leads just one morphism to s; analogously for *cosingleton*. In $\mathbf{Set}$, any one-point set is a singleton and the void set is a cosingleton.

F. Given a mapping $f : A \to B$, denote by $\operatorname{im} f$ the set of all $f(a)$ where $a \in A$. If f is the constant mapping to $b \in B$ we shall write $f = \text{const } b$.

G. Let H be a covariant set functor. An element t of $H1$ is said to be a distinguished point of H if, given arbitrary mappings $f, g : 1 \to X$, we have $Hf(t) = Hg(t) = t_X$. Clearly, for an arbitrary mapping $k : A \to B$ we have $Hk(t_A) = t_B$.

H. All monomorphisms in $\mathbf{Set}$ are coretractions, if the domain is not $\emptyset$. Thus, if f is a one-to-one mapping with a non-void domain then for any set functor H, if H is covariant then Hf is one-to-one and if H is contravariant then Hf is onto. Analogously for mappings onto. If H is an arbitrary non-constant set functor then $HX \neq \emptyset$ for any set $X \neq \emptyset$. We call, for short, a set functor constant if its restriction to the category of non-void sets is a constant functor.

I. A non-void collection $\{f_i\}_{i \in I}$ of mappings with a common domain $X \neq \emptyset$ is said to be a collective monomorphism if, given arbitrary distinct $a, b \in X$, there exists $i \in I$ such that $f_i(a) \neq f_i(b)$. We say that a covariant set functor H *preserves collective monomorphisms* (or, respectively, finite collective monomorphisms) if, given a collective monomorphism $\{f_i\}_{i \in I}$ (where I is finite), then also $\{Hf_i\}_{i \in I}$ is a collective monomorphism.

J. Throughout the following text, F and G denote respectively an arbitrary non-constant contravariant or covariant set functor.

K. Let H be an arbitrary non-constant set-functor (covariant or contravariant). It is proved in [6, 7] that for every cardinal number there exists a set X such that the power of HX is bigger than the cardinal number.

II. LIMITS IN $A(F, G)$ (COLIMITS IN $A(G, F)$)

Lemma 2,1. *$A(F, G)$ has a singleton if and only if $|G1| = 1$. The singleton has then a one-point underlying set (i.e., it is preserved by $\square$).*

Proof. If $|G1| = 1$ then there is just one object in $A(F, G)$ with the underlying set 1 and it is easy to verify that it is a singleton. If $|G1| \neq 1$ then $|G1| > 1$ ($G1 \neq \emptyset$ since G is non-constant). Assume that $A(F, G)$ has nevertheless a singleton (B, ω).

Choose $a, b \in G1$, $a \neq b$. There exists a unique t_a: $(1, \text{const}\ a) \to (B, \omega)$ and a unique t_b: $(1, \text{const}\ b) \to (B, \omega)$. Then clearly $\omega = \text{const}\,(G\,t_a(a)) = \text{const}\,(G\,t_b(b))$, in

$$\begin{tikzcd}
F1 \arrow[r, "\text{const}\ a"] & G1 \arrow[d] \\
FB \arrow[u] \arrow[d] \arrow[r, "\omega"] & GB \\
F1 \arrow[r, "\text{const}\ b"] & G1 \arrow[u]
\end{tikzcd}$$

particular $G\,t_a(a) = G\,t_b(b)$. Let $d : B \to 1$, then we have $\mathrm{d}t_a = \mathrm{d}t_b = \mathrm{id}_1$ and so $a = b$, a contradiction.

Lemma 2,1*. *$A(G, F)$ has a cosingleton if and only if either $G\emptyset = \emptyset$ or $|F\emptyset| = 1$. The cosingleton has then the void underlying set (i.e., it is preserved by $\square$).*

Theorem 2,2. *The forgetful functor $\square$ preserves all limits which exist in $A(F, G)$.*

Proof. Let $D : \mathscr{D} \to A(F, G)$ be an arbitrary diagram. Due to Lemma 2,1 we may assume that $\mathscr{D}$ is non-void. For $d \in \mathscr{D}^\sigma$ denote $Dd = (X_d, \omega_d)$. Let $\langle X, \{f_d\}_{d \in \mathscr{D}^\sigma} \rangle$ be the limit of $\square \cdot D$ (in Set).

Assume that D has a limit in $A(F, G)$, $\langle (Z, \omega), \{r_d\}_{d \in \mathscr{D}^\sigma} \rangle$. Then $\langle Z, \{r_d\} \rangle$ is a bound of $\square \cdot D$ and so there exists a unique mapping $r : Z \to X$ with $r_d = f_d \,.\, r$ for all d. We shall prove that then also $\langle (X, Gr \,.\, \omega \,.\, Fr), \{f_d\} \rangle$ is a limit of D. Let $\langle (Z', \omega'), \{r'_d\} \rangle$ be an arbitrary bound of D. There exists a unique mapping $r' : Z' \to X$ with $r'_d = f_d \,.\, r'$. It clearly suffices to show that r' is a morphism, i.e. that $Gr \,.\, \omega \,.\, Fr = Gr' \,.\, \omega' \,.\, Fr'$. But this is a simple consequence of the fact that (Z, ω) is a limit of D.

Theorem 2,2*. *The forgetful functor $\square$ preserves all colimits which exist in $A(F, G)$.*

Definition. We say that a category K is *directed* if for each pair of its objects d_1, d_2 there exists an object d with $K(d_1, d) \neq \emptyset \neq K(d_2, d)$. The dual notion: *dual directed.*

Definition. A contravariant set functor H is said to *spread limits* over a scheme $\mathscr{D}$ if there exists a diagram $D : \mathscr{D} \to$ Set with a limit $\langle Z, \{f_d\}_{d \in \mathscr{D}^\sigma} \rangle$ such that $HZ \neq \bigcup \operatorname{im} Hf_d$.

Proposition 2,3. *Every non-constant contravariant set functor spreads limits over any scheme which is not dual directed.*

Proof. Let F be a non-constant contravariant set functor.

I. F spreads products of pairs.

Let $\{f, g\}$ be a collective monomorphism, $f : M \to X$, $g : M \to Y$. If $FM \neq \text{im } Ff \cup \text{im } Fg$ then F spreads the product of X, Y denoted $\langle X \times Y, \{\pi_X, \pi_Y\}\rangle$: let $h : M \to X \times Y$ be the unique mapping with $f = \pi_X h$, $g = \pi_Y h$. Then h is one-to-one and so Fh is onto and we have $\text{im } F\pi_X \cup \text{im } F\pi_Y = \text{im } F(\pi_X h) \cup \text{im } F(\pi_Y h) \neq F(X \times Y)$. Now, assume that for an arbitrary collective monomorphism $\{f, g\}$ we have $FM = \text{im } Ff \cup \text{im } Fg$. Then denote $P^- = \text{Hom}(-, 2)$; clearly $\{P^- \circ Ff, P^- \circ Fg\}$ is a collective monomorphism, thus $P^- \circ F$ preserves finite collective monomorphisms. It was proved in [10] that, whenever a covariant set functor preserves finite collective monomorphisms then it can be expressed as a sum $K_1 \vee K_2$ where K_1 has no distinguished point and K_2 is constant. Such a functor has clearly the following property: given an arbitrary distinguished point t of K and an arbitrary mapping $h : A \to B$ we have $Kh^{-1}(t_B) = \{t_A\}$. (This is, in fact, a reformulation of the above condition.) The proof of I will be completed if we show that $P^- \circ F$ does not fulfil this condition. As F is non-constant there clearly exists a mapping $f : A \to B$ such that Ff is not onto. Let $p \in FA - \text{im } Ff$ and let t be a distinguished point of $P^- \circ F : t = \emptyset$ (as an element of $\exp F1$); for every set X we have $t_X = \emptyset$. As $P^- \circ Ff(\{p\}) = \emptyset$, clearly $(P^- \circ Ff)^{-1}(t_B) \supsetneqq \{t_A\}$.

II. F spreads limits over any scheme $\mathscr{D}$ which is not dual directed.

Let X, Y be sets with the product $\langle X \times Y, \{\pi_X, \pi_Y\}\rangle$ such that $F(X \times Y) \neq (\text{im } F\pi_X \cup \text{im } F\pi_Y)$. Further, let d_1, d_2 be objects of $\mathscr{D}$ such that for an arbitrary object d either $\mathscr{D}(d, d_1) = \emptyset$ or $\mathscr{D}(d, d_2) = \emptyset$. Define a diagram $D : \mathscr{D} \to \mathbf{Set}$:

$$Dd = X \quad \text{if} \quad \mathscr{D}(d, d_1) \neq \emptyset\,,$$

$$Dd = Y \quad \text{if} \quad \mathscr{D}(d, d_2) \neq \emptyset\,,$$

$$Dd = 1 \quad \text{if} \quad \mathscr{D}(d, d_i) = \emptyset\,, \quad i = 1, 2\,;$$

let δ be a morphism from d to d'

$$D\delta = id_X \quad \text{if} \quad Dd' = X\,,$$

$$D\delta = id_Y \quad \text{if} \quad Dd' = Y\,,$$

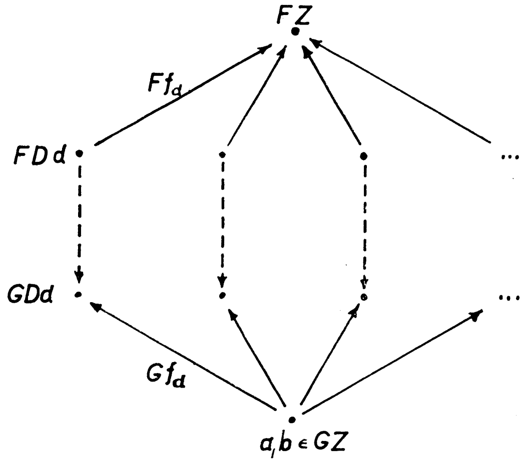
$$D\delta = \text{const} \quad \text{if} \quad Dd' = 1\,.$$

Clearly, $\langle X \times Y, \{f_d\}\rangle$, where f_d is either π_X or π_Y or const, is a limit of D in $\mathbf{Set}$ and $F(X \times Y) \neq \bigcup \text{im } Ff_d$. This completes the proof.

Theorem 2,4. *If F spreads limits over a scheme $\mathscr{D}$ then $A(F, G)$ has not limits over $\mathscr{D}$.*

Proof. Let $D'_0 : \mathscr{D} \to \mathrm{Set}$ be a diagram whose limit $\langle Z', \{f'_d\}\rangle$ is spread by F. Let M be a set with $|GM| > 1$ (it exists as G is non-constant). Let D_0 be the sum of the diagram D'_0 and of the constant diagram to M (then, for each object d, $D_0 d = D'_0 d \vee M$; denote by g_d the inclusion of $D'_0 d$ into $D_0 d$). If $\langle Z, \{f_d\}\rangle$ is a limit of D_0 then there exists a unique $g : Z' \to Z$ with $f_d g = g_d f'_d$; g is clearly one-to-one. Then F spreads the limit of D_0, too: if $FZ = \bigcup \operatorname{im} Ff_d$ then $FZ' = \bigcup \operatorname{im} Fg\, Ff'_d$ (as Fg is onto) but $FZ' \supsetneqq \bigcup \operatorname{im} Ff'_d \supset \bigcup \operatorname{im} Ff'_d F g_d = \bigcup \operatorname{im} Fg\, Ff'_d$.

We shall now find a diagram $D : \mathscr{D} \to A(F, G)$ which has no limit in $A(F, G)$. As $|GM| > 1$, clearly also $|GZ| > 1$; let a, b be distinct elements of GZ. The diagram D is defined as follows: for each object d, $Dd = (D_0 d, \text{const}\, G f_d(a))$ and,



moreover, $\square \,.\, D = D_0$. Assume that D has a limit in $A(F, G)$. Then, due to Theorem 2,2, without loss of generality the limit is $\langle (Z, \omega), \{f_d\}\rangle$ for a suitable $\omega : FZ \to GZ$. We have clearly two bounds of $D : \langle (Z, \text{const}\, a), \{f_d\}\rangle$ and $\langle (Z, \omega'), \{f_d\}\rangle$ where $\omega' = \text{const}\, a$ on $\bigcup \operatorname{im} Ff_d$ and $\omega = \text{const}\, b$ on $FZ - \bigcup \operatorname{im} Ff_d$. Therefore there exists a unique $r : (Z, \text{const}\, a) \to (Z, \omega)$ such that for all $d, f_d r = f_d$ (which implies that $r = id_Z$ and so $\omega = \text{const a}$); furthermore there exists a unique $r' : (Z \,.\, \omega') \to (Z, \omega)$ such that $f_d r = f_d$ and so $\omega' = \omega = \text{const}\, a$, a contradiction. Therefore D has no limit in $A(F, G)$.

Corollary 2,5. *$A(F, G)$ has not limits over any scheme which is not dual directed. In particular, it has not finite products and pullbacks.*

Theorem 2,5*. *$A(G, F)$ has not colimits over any scheme which is not directed. In particular, it has not finite sums and pushouts.*

Proof.

I. $A(G, F)$ has not finite sums.

As both functors G, F are non-constant there clearly exist sets U, V with the sum $\langle U \vee V, \{i_U, i_V\}\rangle$ such that $|\mathrm{im}\, Gi_U - \mathrm{im}\, Gi_V| > |G1|$ and that Fi_U is not a bijection. Then Fi_U is not one-to-one and we may choose distinct $t, u \in F(U \vee V)$ with $Fi_U(t) = Fi_U(u)$; moreover, we choose an arbitrary $a \in \mathrm{im}\, Gi_U - \mathrm{im}\, Gi_V$. Then

$$\begin{tikzcd}
 & G(U \vee V) & \\
GU \arrow[ur, "Gi_U"] \arrow[d, "\mathrm{const}\, Fi_U(t)"'] & & GV \arrow[ul, "Gi_V"'] \arrow[d, "\mathrm{const}\, Fi_V(t)"] \\
FU & & FV \\
 & F(U \vee V) \arrow[ul, "Fi_U"] \arrow[ur, "Fi_V"'] &
\end{tikzcd}$$

$A(G, F)$ has no sum of $(U, \mathrm{const}\, Fi_U(t))$ and $(V, \mathrm{const}\, Fi_V(t))$. Assume the contrary. Then due to Theorem 2,2 the sum would have a form $\langle (U \vee V, \omega), \{i_U, i_V\}\rangle$ for a suitable ω. Clearly, $\langle (U \vee V, \mathrm{const}\, t), \{i_U, i_V\}\rangle$ is a cobound and so there exists a unique $r : (U \vee V, \omega) \to (U \vee V, \mathrm{const}\, t)$ such that $r \,.\, i_U = i_U$ and $r \,.\, i_V = i_V$ – then, of course, $r = id_{U \vee V}$, which proves that $\omega = \mathrm{const}\, t$. On the other hand, as $a \notin \mathrm{im}\, Gi_V$, we have another cobound $\langle (U \vee V, \omega'), \{i_U, i_V\}\rangle$ where $\omega'(a) = u$, else $\omega' = \mathrm{const}\, t$. Then, reasoning as above, we get $\omega' = \omega$ – a contradiction.

II. If $\mathscr{D}$ is not directed then $A(G, F)$ has not limits over $\mathscr{D}$.

Let d_1, d_2 be such objects of $\mathscr{D}$ that for each object d either $\mathscr{D}(d_1, d) = \emptyset$ or $\mathscr{D}(d_2, d) = \emptyset$. Let $W_1 = (U, \mathrm{const}\, Fi_U(t))$ and $W_2 = (V, \mathrm{const}\, Fi_V(t))$; we proved above that W_1 and W_2 have no sum. Denote by p_U and p_V the void mappings to the set U and V, respectively. Then $i_U \,.\, p_U = i_V \,.\, p_V$ and we put $x = F(i_U \,.\, p_U)(t)$. Finally, put $W_3 = (\emptyset, \mathrm{const}\, x)$.

To construct a diagram $D : \mathscr{D} \to A(G, F)$ with no colimit we proceed in the same way as in part II of the proof of Proposition 2,3, using W_1, W_2, W_3 instead of $X, Y, 1$ (in particular, $D\delta$ is the void mapping if the domain is W_3).

Theorem 2,6. *$A(F, G)$ has not equalizers.*

Proof. As F and G are non-constant functors there clearly exist mappings $f, g : X \to Y$ such that (a) $FX \neq \mathrm{im}\, Ff \cup \mathrm{im}\, Fg$ (i.e., there exists $a \in FX - (\mathrm{im}\, Ff \cup \mathrm{im}\, Fg)$) and (b) the equalizer of Gf, Gg in Set is neither GX nor $\emptyset$ (i.e., there exist

$t, u \in GX$ such that $G f(u) \neq G g(u)$ and $G f(t) = G g(t) = t_1$). Put $\omega : FX \to GX$, $\omega(a) = u$, else $\omega = \text{const}\, t$. Then clearly $f, g : (X, \omega) \to (Y, \text{const}\, t_1)$ and we shall

$$\begin{tikzcd}
FY \arrow[r, "\text{const}\, t_1"] \arrow[d, shift right=2, "Ff"'] \arrow[d, shift left=2, "Fg"] & GY \\
FX \arrow[r, "\omega"] \arrow[d, "Fi"'] & GX \arrow[u, shift right=2, "Gf"] \arrow[u, shift left=2, "Gg"'] \\
FZ \arrow[r, "\tilde{\omega}"] & GZ \arrow[u, "Gi"']
\end{tikzcd}$$

prove that f, g have no equalizer in $A(F, G)$: if $i : (Z, \tilde{\omega}) \to (X, \omega)$ fulfils $fi = gi$ then $Gi \,.\, \tilde{\omega} \,.\, F\, i(a) = \omega(a) = u$ and we get a contradiction as $Gf \,.\, G\, i(x) \neq Gg \,.\, G\, i(x)$ where $x = \tilde{\omega} F\, i(a)$.

Theorem 2.6*. *$A(G, F)$ has not coequalizers.*

III. COLIMITS IN $A(F, G)$ (LIMITS IN $A(G, F)$)

Lemma 3,1. *$A(F, G)$ has not finite sums.*

Proof. Let A_1, A_2 be sets with the product $\langle A_1 \times A_2, \{\pi_1, \pi_2\}\rangle$ such that $F(A_1 \times A_2) \neq \operatorname{im} F\pi_1 \cup \operatorname{im} F\pi_2$ (see Proposition 2,3). The sets can be certainly chosen so that $|G(A_1 \times A_2)| > 1$. Choose distinct $x, y \in G(A_1 \times A_2)$ and let $t \in F(A_1 \times A_2)$ fulfil $t \notin \operatorname{im} F\pi_i$, $i = 1, 2$.

We shall prove that $A(F, G)$ has no sum of (X, ω_X) and (Y, ω_Y), where $X = Y = A_1 \times A_2$; $\omega_X = \text{const}\, x$, $\omega_Y(t) = y$, else $\omega_Y = \text{const}\, x$. To do this, it clearly suffices to find distinct cobounds with underlying mappings π_1 and π_2. It follows immediately from the properties of A_1, A_2 and of x, y, t that such cobounds are $\langle (A_1, \text{const}\, G\, \pi_1(x)), \{\pi_1, \pi_2\}\rangle$ and $\langle (A_2, \text{const}\, G\, \pi_2(x)), \{\pi_1, \pi_2\}\rangle$.

Lemma 3,1*. *$A(G, F)$ has not finite products.*

Proof. For an arbitrary set M denote by r_M the mapping from M to 1. As both F and G is non-constant, there clearly exists a set X such that $|GX| > |G1|$ and $|FX| > |F1|$. Put $A = X \times \{1, 2\}$, $b = A \cup 1$. Define $f, g : A \to B$:

$$\begin{aligned}
\text{for each} \quad x \in X\,, \quad & g(\langle x, 1\rangle) = 0 \quad (\text{recall that } 1 = \{0\}) \\
& g(\langle x, 2\rangle) = \langle x, 2\rangle\,, \\
& f(\langle x, i\rangle) = \langle x, 1\rangle\,, \quad i = 1, 2\,.
\end{aligned}$$

Choose $h : B \to X$ so that hf is onto while hg is constant. As $\operatorname{im} f \cap \operatorname{im} g = \emptyset$ and $|G(\operatorname{im} g)| > |G1|$, clearly there exists $u \in \operatorname{im} Gg - \operatorname{im} Gf$; choose $t \in GA$ with $G\, g(t) = u$.

Let us prove that there exist $q \in F1$ and $a \in FB$ such that $F\, f(a) \neq F\, r_A(q) = F\, g(a)$. As hg is constant there exists $j : 1 \to X$ with $jr_A = hg$. As $|FX| > |F1|$ there exists $q \in F1$ with $|(Fj)^{-1}(q)| > 1$. Choose $b \in FX$, $b \neq F\, r_X(q)$ with $F\, j(b) = q$. Put $a = F\, h(b)$. As hf is onto, $F(hf)$ is one-to-one and as $r_A = r_X hf$ we have $F\, f(a) \neq F\, r_A(q)$. Furthermore, $r_A = r_X hf$ and so $F\, g(a) = F(jr_A)(b) = F\, r_A(q)$.

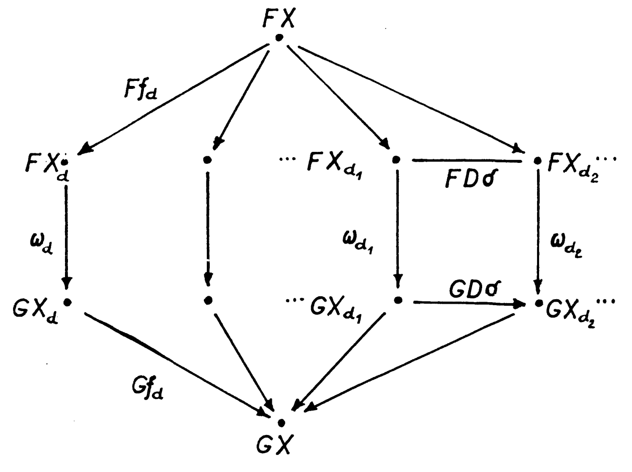
We shall show that $A(G, F)$ has no product of (B, ω_1) and (B, ω_2) where $\omega_1 = \operatorname{const} F\, r_B(q)$, $\omega_2(u) = a$ else $\omega_2 = \operatorname{const} F\, r_B(q)$. Let, to the contrary, $\langle (Z, \omega), \{\varphi_1, \varphi_2\}\rangle$ be their product. As $u \notin \operatorname{im} Gf$ we have a bound $\langle (A, \operatorname{const} F\, r_A(q), \{f, f\}\rangle$ and as $F\, g(a) = F\, r_A(q)$ we have a bound $\langle (A, \operatorname{const} F\, r_A(q)), \{g, g\}\rangle$. Therefore there exist mappings h, k such that $f = \varphi_1 h = \varphi_2 h$ and $g = \varphi_1 k = \varphi_2 k$. Consequently $F\, \varphi_2(a) = F\, r_Z(q)$; hence $\omega(G\, k(t)) = F\varphi_1 \,.\, \omega_1 \,.\, G\varphi_1(G\, k(t)) = F\, r_Z(q)$ (because $r_Z = r_B \,.\, \varphi_1$) and $\omega(G\, k(t)) = F\varphi_2 \,.\, \omega_2 \,.\, G\varphi_2(G\, k(t)) = F\varphi_2 \,.\, \omega_2 \,.\, (u) = F\, \varphi_2(a)$. This leads to a contradiction as $Fh(F\, \varphi_2(a)) = F\, f(a) \neq F\, r_A(q) = Fh(F\, r_Z(q)) = Fh(F\, \varphi_2(a))$.

Lemma 3,2. *$A(F, G)$ has no cosingleton.*

Proof. Let, to the contrary, (B, ω_0) be a cosingleton. As the functors F, G are non-constant there exists a set X with $|FX| > |GB|$ and $|GX| > |GB|$. Let $\omega : FX \to GX$ be an arbitrary mapping with $|\operatorname{im} \omega| > GB$. There exists a unique morphism $f : (B, \omega_0) \to (X, \omega)$; then $\omega = Ff \,.\, \omega_0 \,.\, Gf$ which is a contradiction as $|\operatorname{im} \omega_0| < |\operatorname{im} \omega|$.

Lemma 3,2*. *$A(G, F)$ has no singleton.*

Theorem 3,3. *$A(F, G)$ has colimits over a given scheme if and only if the scheme is connected. In this case the colimits are preserved by the forgetful functor* □.



Proof. It follows from Lemmas 3,1 and 3,2 that $A(F, G)$ has not colimits over any non-connected scheme $\mathcal{D}$: if $\mathcal{D}_i \neq \emptyset$ and $\mathcal{D} = \mathcal{D}_1 \vee \mathcal{D}_2$, let $\mathcal{A}_1, \mathcal{A}_2$ be algebras in $A(F, G)$ which have no sum and let $D : \mathcal{D} \to A(F, G)$ be the diagram which on $\mathcal{D}_1$ is constant to $\mathcal{A}_1$ and on $\mathcal{D}_2$ is constant to $\mathcal{A}_2$. Then D has no colimit in $A(F, G)$.

To prove the theorem we shall show that if $\mathcal{D}$ is an arbitrary connected scheme then $A(F, G)$ has colimits over $\mathcal{D}$ preserved by $\square$. Let $D : \mathcal{D} \to A(F, G)$, let $Dd = (X_d, \omega_d)$ and let $\langle X, \{f_d\}_{d \in \mathcal{D}^\sigma} \rangle$ be the colimit of D in Set. We try to find $\omega : FX \to GX$ such that $\langle (X, \omega), \{f_d\} \rangle$ is the colimit of D. To this end verify that $\omega = Gf_d \, . \, \omega_d \, . \, Ff_d$ is independent of d. In fact, if there exists a morphism $\delta : d \to d'$ then clearly $Gf_d \, . \, \omega_d \, . \, Ff_d = Gf_{d'} \, . \, \omega_{d'} \, . \, Ff_d'$. Now apply the fact that $\mathcal{D}$ is connected (see I.D.).

Using the fact that $\langle X, \{f_d\} \rangle$ is the colimit of $\square \, . \, D$ in Set, it is now rather easy to prove that ω is the desired mapping.

Corollary 3,4. *$A(F, G)$ has pushouts and coequalizers.*

Theorem 3.3*. *$A(G, F)$ has limits over a given scheme if and only if the scheme is connected. In this case the limits are preserved by the forgetful functor $\square$.*

Corollary 3,4*. *$A(G, F)$ has pullbacks and equalizers.*

As a consequence of the results of both parts of the paper we obtain:

Theorem 3,5. *Let H_1, H_2 be non-constant set functors with different variances. Whenever the category $A(H_1, H_2)$ has limits or colimits over a given diagram scheme then they are preserved by the forgetful functor $\square$.*

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