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ON THE WEAKLY ALMOST PERIODIC SOLUTIONS OF CERTAIN ABSTRACT DIFFERENTIAL EQUATIONS

ARIBINDI SATYANARAYAN RAO, Montreal

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1. Suppose X is a Banach space and X^* is the dual space of X. Let J be the interval $-\infty < t < \infty$. A continuous function $f : J \to X$ is said to be (Bochner or strongly) almost periodic if, given $\varepsilon > 0$, there exists a positive real number $l = l(\varepsilon)$ such that any interval of the real line of length l contains at least one point τ for which

$$(1.1) \qquad \sup_{t \in J} \|f(t + \tau) - f(t)\| \leqq \varepsilon .$$

Maak's criterion for almost periodicity (MAAK [4], pp. 93–96 and 151–153) is as follows:

A continuous function $f : J \to X$ is almost periodic if and only if, given $\varepsilon > 0$, there is a partition $J = E_1 \cup \ldots \cup E_m$ such that

$$(1.2) \qquad \|f(\xi + t) - f(\eta + t)\| < \varepsilon \quad \text{for all} \quad t \in J\,, \quad \xi, \eta \in E_i\,, \quad i = 1, 2, \ldots, m\,.$$

We say that a function $f : J \to X$ is weakly almost periodic if the scalar-valued function $\langle x^*, f(t) \rangle = x^* f(t)$ is almost periodic for each $x^* \in X^*$.

For $1 \leqq p < \infty$, a function $f \in L^p_{\mathrm{loc}}(J; X)$ is said to be Stepanov-bounded or S^p-bounded if

$$(1.3) \qquad \|f\|_{S^p} = \sup_{t \in J} \left[\int_t^{t+1} \|f(s)\|^p \, \mathrm{d}s \right]^{1/p} < \infty\,.$$

For $1 \leqq p < \infty$, a function $f \in L^p_{\mathrm{loc}}(J; X)$ is said to be Stepanov almost periodic or S^p-almost periodic if, given $\varepsilon > 0$, there is a positive real number $l = l(\varepsilon)$ such that any interval of the real line of length l contains at least one point τ for which

$$(1.4) \qquad \sup_{t \in J} \left[\int_t^{t+1} \|f(s + \tau) - f(s)\|^p \, \mathrm{d}s \right]^{1/p} \leqq \varepsilon\,.$$

We denote by $\mathscr{L}(X, X)$ the set of all bounded linear operators of X into itself. An operator-valued function $G : J \to \mathscr{L}(X, X)$ is called a (strongly) continuous group if

$$(1.5) \qquad G(0) = I = \text{the identity operator of } X\,;$$

$$(1.6) \qquad G(t_1 + t_2) = G(t_1)\, G(t_2) \quad \text{for all} \quad t_1, t_2 \in J\,;$$

$$(1.7) \qquad \text{for each} \quad x \in X\,, \quad G(t)\, x\,, \quad t \in J \to X \quad \text{is continuous.}$$

The infinitesimal generator A of $G(t)$ is a closed linear operator, with domain $D(A)$ dense in X, defined by

$$(1.8) \qquad Ax = \lim_{t \to 0} \frac{G(t)\, x - x}{t} \quad \text{for all} \quad x \in D(A)$$

(see DUNFORD and SCHWARTZ [3]).

The function $G : J \to \mathscr{L}(X, X)$ is said to be weakly almost periodic if $G(t)\, x$, $t \in J \to X$ is weakly almost periodic for each $x \in X$.

Our main result is as follows (see Theorem 4, ZAIDMAN [6]).

Theorem 1. *Let A be the infinitesimal generator of a weakly almost periodic continuous group $G : J \to \mathscr{L}(X, X)$. Suppose $T \in \mathscr{L}(X, X)$ is a compact operator commuting with $G(t)$ for all $t \in J$, T^{-1} exists on a dense set in X, and the adjoint operator $(T^{-1})^*$ is defined on a dense set in X^*. Further, suppose that, for $1 \leqq p < \infty$, $f : J \to X$ is an S^p-almost periodic continuous function, and that $u : J \to D(A)$ is a (strong) solution of the differential equation*

$$(1.9) \qquad u'(t) = A\, u(t) + f(t) \quad on \quad J\,.$$

Then, if u is S^p-bounded on J, it is weakly almost periodic from J to the Banach space X.

2. We shall require the following lemmas.

Lemma 1. *Consider the differential equation*

$$(2.1) \qquad u'(t) = (A + B)\, u(t) + f(t) \quad on \quad J\,,$$

where B is a bounded linear operator of X into itself. Any solution of (2.1) admits the representation

$$(2.2) \qquad u(t) = G(t)\, u(0) + \int_0^t G(t - s)\,[B\, u(s) + f(s)]\, \mathrm{d}s \quad on \quad J\,.$$

Proof. By applying the operator $G(t - s)$ to (2.1) (with an arbitrary but fixed $t \in J$), we get

$$(2.3) \qquad G(t - s)\,[u'(s) - A\, u(s)] = G(t - s)\,[B\, u(s) + f(s)] \quad \text{for} \quad s \in J\,.$$

Also, we have

$$(2.4) \qquad \frac{\mathrm{d}}{\mathrm{d}s}[G(t - s)\, u(s)] = G(t - s)\,[u'(s) - A\, u(s)]\,.$$

So, integrating (2.3) from 0 to t, we obtain the representation (2.2).

Lemma 2. *If $g : J \to X$ is almost periodic, and if $G : J \to \mathscr{L}(X, X)$ is weakly almost periodic, then $G(t)\,g(t)$ is weakly almost periodic from J to X.*

Proof. For an arbitrary but fixed $x^* \in X^*$, $\{x^*\,G(t)\}_{t\in J}$ is a family of bounded linear functionals on X. Under the assumption made on G, for each $x \in X$, the scalar-valued function $x^*\,G(t)\,x$ is almost periodic, and so is bounded on J. Hence, by the uniform boundedness principle,

$$\sup_{t\in J} \|x^*\,G(t)\| = M < \infty\,. \tag{2.5}$$

Since g is almost periodic, its range $g(J)$ is relatively compact. Consequently, given $\varepsilon > 0$, there exist finitely many $y_1, y_2, \ldots, y_k \in g(J)$ which form an $(\varepsilon/4M)$-net for $g(J)$. Now, by Maak's criterion, we can find a partition $E_1, E_2, \ldots, E_m$ of J such that, for all $t \in J$ and $\xi, \eta \in E_i$, $i = 1, 2, \ldots, m$,

$$\|g(\xi + t) - g(\eta + t)\| < \varepsilon/4M\,, \quad |x^*G(\xi + t)\,y_j - x^*G(\eta + t)\,y_j| < \varepsilon/4\,, \quad j = 1, 2, \ldots, k\,. \tag{2.6}$$

For fixed $t \in J$ and for fixed $\xi, \eta \in E_i$ with fixed $i = 1, 2, \ldots, m$, there is y_ν in the $(\varepsilon/4M)$-net for $g(J)$ such that

$$\|g(\eta + t) - y_\nu\| < \varepsilon/4M\,. \tag{2.7}$$

Now, by $(2.5)-(2.7)$, we have

$$\begin{aligned} &|x^*G(\xi + t)\,g(\xi + t) - x^*G(\eta + t)\,g(\eta + t)| \leqq \\ &\leqq \|x^*G(\xi + t)\| \,.\, \|g(\xi + t) - g(\eta + t)\| + \|x^*G(\xi + t)\| \,.\, \|g(\eta + t) - y_\nu\| + \\ &+ |x^*G(\xi + t)\,y_\nu - x^*G(\eta + t)\,y_\nu| + \|x^*G(\eta + t)\| \,.\, \|y_\nu - g(\eta + t)\| < \\ &< M\,.\,(\varepsilon/4M) + M\,.\,(\varepsilon/4M) + \varepsilon/4 + M\,.\,(\varepsilon/4M) = \varepsilon\,. \end{aligned} \tag{2.8}$$

Similarly, we can demonstrate the continuity of $x^*\,G(t)\,g(t)$. Thus the desired conclusion follows.

Lemma 3. *If $h : J \to X$ is a bounded function such that $x^*\,h(t)$ is almost periodic for a dense set of elements x^* in the dual space X^*, then $h(t)$ is weakly almost periodic from J to X.*

This result is a consequence of the fact that a uniformly convergent sequence of almost periodic functions has an almost periodic limit.

Lemma 4. *Suppose that, for $1 \leqq p < \infty$, a continuous function Φ is S^p-almost periodic from J to a reflexive space Y. Let*

$$\Phi(t) = \int_0^t \Phi(s)\,\mathrm{d}s \quad \text{on} \quad J\,. \tag{2.9}$$

Then, if Φ is S^p-bounded, it is almost periodic from J to Y.

Proof. See Note (ii), Rao [5].

3. Proof of Theorem 1. From (2.2) with $B = 0$, we obtain

$$u(t) = G(t)\,u(0) + G(t)\int_0^t G(-s)\,f(s)\,ds \quad \text{on} \quad J\,. \tag{3.1}$$

Consider the functions

$$f_\delta(t) = \frac{1}{\delta}\int_0^\delta f(t+s)\,ds \quad \text{for} \quad \delta > 0\,. \tag{3.2}$$

Since f is S^p-almost periodic, and hence is S^1-almost periodic, it follows easily that f_δ is almost periodic for each fixed $\delta > 0$. As shown for scalar-valued functions in Besicovitch [2], pp. 80–81, we can prove that $f_\delta \to f$ as $\delta \to 0$ in the S^1 sense, that is,

$$\sup_{t\in J}\int_t^{t+1} \|f(s) - f_\delta(s)\|\,ds \to 0 \quad \text{as} \quad \delta \to 0\,.$$

Obviously, $G(-s), s \in J \to \mathscr{L}(X, X)$ is weakly almost periodic. Now, for an arbitrary but fixed $x^* \in X^*$, we have

$$x^*G(-s)\,f(s) = x^*G(-s)\,[f(s) - f_\delta(s)] + x^*G(-s)\,f_\delta(s)\,, \tag{3.3}$$

and, by (2.5),

$$\sup_{t\in J}\int_t^{t+1} |x^*G(-s)\,[f(s) - f_\delta(s)]|\,ds \leqq \tag{3.4}$$

$$\leqq M \sup_{t\in J}\int_t^{t+1} \|f(s) - f_\delta(s)\|\,ds \to 0 \quad \text{as} \quad \delta \to 0\,.$$

By Lemma 2, the functions $x^*G(-s)\,f_\delta(s)$ are almost periodic from J to the scalars. So it follows from (3.3)–(3.4) that $x^*G(-s)\,f(s)$ is S^1-almost periodic from J to the scalars.

By (3.1), we have

$$x^*G(-t)\,u(t) = x^*\,u(0) + \int_0^t x^*G(-s)\,f(s)\,ds \quad \text{on} \quad J\,. \tag{3.5}$$

By our assumption, u is S^p-bounded, and hence is S^1-bounded. Consequently, by (2.5), $x^*G(-t)\,u(t)$ is S^1-bounded. Thus, by Lemma 4, $x^*G(-t)\,u(t)$ is almost periodic from J to the scalars. Hence it follows that $G(-t)\,u(t)$ is weakly almost periodic from J to X.

From (2.5), again by the uniform boundedness principle, it follows that

$$\sup_{t \in J} \|G(t)\| = K < \infty . \tag{3.6}$$

Consequently, $u(t) = G(t)[G(-t)u(t)]$ is bounded on J.

Since T is a bounded linear operator of X into itself, $TG(-t)u(t)$ is also weakly almost periodic from J to X. T being a compact operator, the range of $TG(-t)u(t)$ is relatively compact. Therefore, by Theorem 10, p. 45, AMERIO and PROUSE [1], $TG(-t)u(t)$ is almost periodic from J to X. Thus, again by Lemma 2, $G(t)TG(-t) . u(t) = Tu(t)$ is weakly almost periodic from J to X.

Now, for each $x^* \in D((T^{-1})^*)$, we have

$$x^* u(t) = x^* T^{-1} T u(t) = (x^* T^{-1})(T u(t)) = [(T^{-1})^* x^*](T u(t)), \tag{3.7}$$

with $[(T^{-1})^* x^*](Tu(t))$ being almost periodic from J to the scalars. So, by Lemma 3, u is weakly almost periodic from J to X, completing the proof of the theorem.

4. Here we prove the following result.

Theorem 2. *Suppose that G, T and f are defined as in Theorem* 1. *Let $u : J \to D(A)$ be a solution of the differential equation*

$$u'(t) = (A + B)u(t) + f(t) \quad on \quad J, \tag{4.1}$$

where B is a bounded linear operator of X into itself. Then, if u is S^p-almost periodic from J to X, it is also weakly almost periodic (X a Banach space).

Proof. From (2.2), we obtain

$$u(t) = G(t)u(0) + G(t)\int_0^t G(-s)[Bu(s) + f(s)] \, ds \quad \text{on} \quad J. \tag{4.2}$$

So, for an arbitrary but fixed $x^* \in X^*$, we have

$$x^* G(-t)u(t) = x^* u(0) + \int_0^t x^* G(-s)[Bu(s) + f(s)] \, ds \quad \text{on} \quad J. \tag{4.3}$$

Obviously, $Bu(t) + f(t)$, $t \in J \to X$ is S^p-almost periodic. As shown in the proof of Theorem 1, we can prove that $x^* G(-t)u(t)$ and $x^* G(-t)[Bu(t) + f(t)]$ are S^1-almost periodic from J to the scalars. By Theorem 8, p. 79, Amerio and Prouse [1], $x^* G(-t)u(t)$ is uniformly continuous on J. Consequently, by Theorem 7, p. 78, Amerio and Prouse [1], $x^* G(-t)u(t)$ is almost periodic from J to the scalars. So it follows that $G(-t)u(t)$ is weakly almost periodic from J to X. Now the remaining part of the proof is analogous to that of Theorem 1.

Remark 1. We note that, if, for some complex number λ, $(\lambda I - A)^{-1}$ is a compact linear operator of X, and if the adjoint operator A^* is densely defined in X^*, then we may take $(\lambda I - A)^{-1}$ for T in Theorems 1 and 2, since

$$(\lambda I - A)^{-1} G(t) = G(t) (\lambda I - A)^{-1} \quad \text{for all} \quad t \in J .$$

Remark 2. Theorems 1 and 2 remain valid if f is weakly almost periodic instead of S^p-almost periodic, with u being bounded on J.

Proof. (a) By (3.1), we have

$$TG(-t)\, u(t) = Tu(0) + \int_0^t G(-s)\, (Tf)\, (s)\, ds \quad \text{on} \quad J . \tag{4.4}$$

Since $(Tf)\,(t)$ is almost periodic, $G(-t)\,(Tf)\,(t)$, $t \in J \to X$ is weakly almost periodic (by Lemma 2).

By our assumption, $u(t)$ is bounded on J, and hence $G(-t)\, u(t)$ and $TG(-t)\, u(t)$ are bounded on J (by (3.6)).

So, by Bohl-Bohr's theorem, $TG(-t)\, u(t)$ is weakly almost periodic, and hence is almost periodic. Now the remainder of the proof parallels that of Theorem 1.

(b) By (4.2), we have

$$TG(-t)\, u(t) = Tu(0) + \int_0^t G(-s)\, [TB\, u(s) + Tf(s)]\, ds \quad \text{on} \quad J . \tag{4.5}$$

Hence $Tf(t)$ is almost periodic and $TB\, u(t)$ is S^p-almost periodic. Hence it follows that $TG(-t)\, u(t)$ is weakly almost periodic. So the remaining part of the proof is again similar to that of Theorem 1.

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Author's address: Department of Mathematics, University of Montreal, Montreal, Quebec, Canada.