Czechoslovak Mathematical Journal

Jaroslav Milota

Interpolation in a Banach space

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 1, 84–92

Persistent URL: http://dml.cz/dmlcz/101375

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

INTERPOLATION IN A BANACH SPACE

JAROSLAV MILOTA, Praha

(Received March 25, 1974)

1. INTRODUCTION

Let E be a Banach space and let Φ be a linear subspace of its dual space E^*. A linear subspace $L \subset E$ is said to be *Φ-interpolative* if for every $x \in E$ there exists one and only one $y \in L$ such that $\langle x, \varphi \rangle = \langle y, \varphi \rangle$ for all $\varphi \in \Phi$. If it is this case we denote by J_L the operator $J_L : x \to y$. It is obvious that J_L is a projection onto L.

In Section 2 we shall prove some simple conditions on L in order to be Φ-interpolative and closed (in this case J_L is continuous). If Φ is a finite dimensional subspace of a reflexive space E we shall show that there exists a Φ-interpolative subspace with smallest possible norm of J_L and in such a way we shall generalize a result due to AUBIN [1]. We shall also give a dual interpretation of this fact.

To relate the notion of Φ-interpolative subspace with the notion of the n-width (see e.g. [5], [6], [9]) we define for $M \subset E$ and a Φ-interpolative L

$$\sigma_\Phi(M, L) = \sup_{x \in M} \|x - J_L x\| \tag{1}$$

and

$$\sigma_\Phi(M) = \inf_L \sigma_\Phi(M, L)\,, \tag{2}$$

where the greatest lower bound is taken over all Φ-interpolative L's. We shall say that $\sigma_\Phi(M)$ is the *Φ-interpolative width* of M. Using a similar method to that of GARKAVI [4], who has proved the existence of the best n-dimensional approximation for bounded M, we shall prove in Section 3 that this fact is valid in a reflexive space E also for the Φ-interpolative width if the dimension of Φ is finite.

2. Φ-INTERPOLATIVE SUBSPACES

Throughout the paper we shall use the following notation: If $L \subset E$ then $L^\perp = \{f \in E^*;\ \langle x, f \rangle = 0 \text{ for all } x \in L\}$, if $\Phi \subset E^*$ then $\Phi_\perp = \{x \in E;\ \langle x, \varphi \rangle = 0$ for all $\varphi \in \Phi\}$. It is easy to prove that $L^\perp$ is a w^*-closed subspace of E^* and $\Phi_\perp$ is a w-closed subspace of E.

Lemma 1. *Let L be a linear subspace of E. Then $(L^{\perp})_{\perp}$ is the closure of L.*

Proof. It was noted that $(L^{\perp})_{\perp}$ is w-closed and therefore closed. As $L \subset (L^{\perp})_{\perp}$ it is $\bar{L} \subset (L^{\perp})_{\perp}$. If there exists $x_0 \in (L^{\perp})_{\perp} \setminus \bar{L}$ then, by using the Hahn-Banach theorem, we can find $f \in E^*$ such that $\langle x_0, f \rangle \neq 0$ and $f(\bar{L}) = 0$, what contradicts $x_0 \in (L^{\perp})_{\perp}$.

Lemma 2. *Let Φ be a linear subspace of E^*. Then $(\Phi_{\perp})^{\perp}$ is the w^*-closure of Φ. If E is moreover reflexive then $(\Phi_{\perp})^{\perp}$ is the closure of Φ.*

Proof. It was noted that $(\Phi_{\perp})^{\perp}$ is w^*-closed. Let Ψ denote the w^*-closure of Φ. Then $\Psi \subset (\Phi_{\perp})^{\perp}$. If $f_0 \notin \Psi$ then, by virtue of one theorem of Banach (see e.g. [2], p. 122, or [8]), there exists $x_0 \in \Psi_{\perp}$ such that $\langle x_0, f_0 \rangle = 1$. The element x_0 belongs to $\Phi_{\perp}$ and therefore $f_0 \notin (\Phi_{\perp})^{\perp}$. Thus $\Psi = (\Phi_{\perp})^{\perp}$. The second statement follows now from the first one by using the Mazur theorem.

The following proposition yields a very simple condition for L in order to be Φ-interpolative.

Proposition 1. *Let L be a linear subspace of E and let Φ be a linear subspace of E^*. Then L is Φ-interpolative if and only if $E = L \oplus \Phi_{\perp}$ (algebraic direct sum).*

Proof. Let L be Φ-interpolative. From the definition of J_L it is obvious that $x - J_L x \in \Phi_{\perp}$ for all $x \in E$. If $x_0 \in L \cap \Phi_{\perp}$ then $\langle x, \varphi \rangle = \langle J_L x + x_0, \varphi \rangle$ for all $x \in E$, $\varphi \in \Phi$. From the requirement of the uniqueness of $J_L x$ it follows that $x_0 = 0$. Hence $E = L \oplus \Phi_{\perp}$. The sufficient part of the proposition is obvious.

Corollary. *Let Φ be a finite dimensional subspace of E^* with a base $\varphi_1, \ldots, \varphi_n$. Then the following conditions are equivalent:*

(i) *L is a Φ-interpolative subspace of E.*

(ii) *There exists a base $x_1, \ldots, x_n$ of L such that*

$$\langle x_i, \varphi_j \rangle = \delta_{ij}, \quad i, j = 1, \ldots, n. \tag{3}$$

(iii) *$E = L + \Phi_{\perp}$ and $\dim L = n$.*

Proof. (i) $\Rightarrow$ (ii). It is a well known fact that there exists a biorthogonal sequence $y_1, \ldots, y_n$ to $\varphi_1, \ldots, \varphi_n$. Put $x_i = J_L y_i$, $i = 1, \ldots, n$. These elements belong to L, satisfy the condition (3) and therefore they are linearly independent. Now

$$\langle J_L x - \sum_{i=1}^{n} \langle x, \varphi_i \rangle x_i, \varphi_j \rangle = 0$$

for $j = 1, \ldots, n$ and all $x \in E$. Hence

$$J_L x = \sum_{i=1}^{n} \langle x, \varphi_i \rangle x_i \tag{4}$$

and this proves that $x_1, \ldots, x_n$ form a base of L.

(ii) $\Rightarrow$ (iii). We have only to prove the first condition. But it is obvious from (3), (4) that L is Φ-interpolative. It remains to use Proposition 1.

(iii) $\Rightarrow$ (i). By the assumption on the dimension of Φ, it follows that $\Phi = (\Phi_\perp)^\perp$ (Lemma 2). Being $(E/\Phi_\perp)^*$ isomorphic to $(\Phi_\perp)^\perp = \Phi$, $E/\Phi_\perp$ is a space of the dimension n. Therefore L cannot contain a proper subspace that is a direct complement of $\Phi_\perp$. This proves that $E = L \oplus \Phi_\perp$ and, by Proposition 1, L is a Φ-interpolative subspace.

Proposition 2. *Let Φ be a subspace of E^*. Then a linear subspace of E is Φ-interpolative if and only if it is $(\Phi_\perp)^\perp$-interpolative.*

Proof. By virtue of Lemma 2, it is $\Phi_\perp = [(\Phi_\perp)^\perp]_\perp$ and therefore the statement follows immediately from Proposition 1.

We remark that J_L is a bounded linear operator if L is a closed Φ-interpolative subspace (evidently, the finite dimension of Φ is sufficient for this). This fact is a simple consequence of the Banach closed graph theorem. Further, it is known (see [7]) that there exists a Banach space E (e.g. ℓ_p, $p \neq 2$) with a closed linear subspace X having no closed complement. Setting $X^\perp = \Phi$ we obtain an example of a w^*-closed subspace of E^* having no closed Φ-interpolative subspaces, what follows directly from Proposition 1 and Lemma 1.

Proposition 3. *Let Φ be a subspace of E^*. Then a Φ-interpolative subspace L is closed if and only if $E^* = (\Phi_\perp)^\perp \oplus L^\perp$.*

Proof. Suppose first L is a closed Φ-interpolative subspace and let $f \in E^*$. As J_L is a continuous linear map the functional $g = f \circ J_L$ is an element of E^* and moreover $g \in (\Phi_\perp)^\perp$. Further, for $x \in L$ we have $\langle x, g\rangle = \langle J_L x, f\rangle = \langle x, f\rangle$ and therefore $f - g \in L^\perp$. Now, if $f \in (\Phi_\perp)^\perp \cap L^\perp$ then $\langle x, f\rangle = \langle x - J_L x, f\rangle + \langle J_L x, f\rangle = 0$ for all $x \in E$. Thus $f = 0$ what finishes the proof of the necessity part.

Let now the condition of Proposition be satisfied. By Proposition 2, we can suppose that Φ is w^*-closed and therefore $E^* = \Phi \oplus L^\perp$. For $f \in E^*$ we have $f = g + h$, where $g \in \Phi$ and $h \in L^\perp$. If x is an element of the closure of L it follows from the assumptions and Lemma 1 that $\langle x, f\rangle = \langle x, g\rangle = \langle J_L x, g\rangle = \langle J_L x, f\rangle$. Hence $x = J_L x$ and $x \in L$.

Corollary. *Let L be a closed subspace of E and Φ be a subspace of E^*. Then the decomposition $E^* = (\Phi_\perp)^\perp \oplus L^\perp$ is a necessary condition for L to be Φ-interpolative. If E is moreover a reflexive space then this condition is also sufficient.*

Proof. We have to prove only the second statement. By the decomposition of E^*, $L^\perp$ is a closed $\Phi_\perp$-interpolative subspace of E^* ($\Phi_\perp$ is considered as a subset of E^{**}). Proposition 3 and reflexivity of E yield the decomposition of E in the form $E = [(\Phi_\perp)^\perp]_\perp \oplus (L^\perp)_\perp$. Using now Lemma 1 and Proposition 1 we finish the proof.

If Φ is a finite dimensional subspace of E we need not to assume reflexivity of E for the validity of the last corollary because of the following proposition.

Proposition 4. *Let Φ be a finite dimensional subspace of E^*. Then a subspace L of E is Φ-interpolative if and only if $E^* = \Phi \oplus L^{\perp}$.*

Proof. By the assumption on the dimension of Φ and Lemma 2, it follows that $\Phi = (\Phi_{\perp})^{\perp}$. Suppose L is Φ-interpolative. According to Corollary of Proposition 1 the dimension of L is finite, i.e. L is a closed subspace of E. Hence the decomposition $E^* = \Phi \oplus L^{\perp}$ follows from Proposition 3.

Suppose now $E^* = \Phi \oplus L^{\perp}$. Being L^* isomorfic to $E^*/L^{\perp}$, the dimension of L is finite. For the sake of simplicity we denote $E^* = X$, $L^{\perp} = A$, i.e. we have $X = \Phi \oplus A$. As A is closed $A = (A^{\perp})_{\perp}$ according to Lemma 1. By using Proposition 1, the decomposition of X means that Φ is $A^{\perp}$-interpolative and therefore, by Proposition 3, we obtain that $X^* = \Phi^{\perp} \oplus A^{\perp}$. Let Q denote the canonical imbedding of E into E^{**}. By virtue of Lemma 1 in [3], § I,5, we have $Q(L) = A^{\perp}$ (L is a finite dimensional subspace) and the above decomposition of X^* can be rewritten in the form

$$E^{**} = \Phi^{\perp} \oplus Q(L)\,. \tag{5}$$

Let x be an element of E. Then there exist $\xi \in \Phi^{\perp}$ and $z \in L$ such that $Qx = \xi + Qz$. It means that $x - z \in \Phi_{\perp}$ and hence $E = L + \Phi_{\perp}$. By (5), it is obvious that $L \cap \Phi_{\perp} = \{0\}$. Using Proposition 1 it finishes the proof.

Lemma 3. *Let Φ be a subspace of E^* and L be a closed Φ-interpolative subspace of E. Then J_L^* (the adjoint operator to J_L) is the projection onto $(\Phi_{\perp})^{\perp}$ which is parallel to $L^{\perp}$.*

Proof. J_L is a bounded linear operator and hence J_L^* exists and it is bounded. By the definition,

$$\langle J_L x, f\rangle = \langle x, J_L^* f\rangle \quad \text{for all} \quad x \in E\,, \quad f \in E^*\,. \tag{6}$$

Putting x to be an element of $\Phi_{\perp}$ we find $\langle x, J_L^* f\rangle = 0$ for all $f \in E^*$ and therefore $J_L^*(E^*) \subset (\Phi_{\perp})^{\perp}$. Now, let g be an element of $(\Phi_{\perp})^{\perp}$. Then $\langle x, g\rangle = \langle J_L x, g\rangle$ for all $x \in E$ (Proposition 2), what proves that $g = J_L^* g$. Thus J_L^* is a projection onto $(\Phi_{\perp})^{\perp}$. Setting f to be an element of $L^{\perp}$ in (6) we find $\langle x, J_L^* f\rangle = 0$ for every $x \in E$. It proves the rest of the statement.

Definition. Let Φ be a subspace of E^*. If there exists a closed Φ-interpolative subspace $\tilde{L}$ of E such that

$$\|J_{\tilde{L}}\| = \inf_L \|J_L\|\,,$$

where the greatest lower bound is taken over all Φ-interpolative subspaces L, then $\tilde{L}$ is called the *best Φ-interpolative subspace.*

The following theorems yield the existence and the characterization of the best Φ-interpolative subspace and they can be considered as a generalization of analogous results due to Aubin [1] for Hilbert spaces.

Theorem 1. *Let E be a reflexive Banach space and let Φ be a finite dimensional subspace of E^*. Then there exists the best Φ-interpolative subspace.*

Proof. Denote $\sigma = \inf_L \|J_L\|$, where the greatest lower bound is taken over all Φ-interpolative subspaces. As σ is finite there exists a sequence $(L^{(n)})$ of Φ-interpolative subspaces such that

$$\sigma \leqq \|J_{L^{(n)}}\| < \sigma + \frac{1}{n}\,. \tag{7}$$

Let $\varphi_1, \ldots, \varphi_m$ be a base of Φ. According to Corollary of Proposition 1 let $x_1^{(n)}, \ldots, x_m^{(n)}$ be the base of $L^{(n)}$ with the property (3). Then $x_i^{(n)} = J_{L^{(n)}} x_i^{(1)}$, $i = 1, \ldots, m$, and therefore

$$\|x_i^{(n)}\| \leqq \|J_{L^{(n)}}\| \,.\, \|x_i^{(1)}\| \leqq (\sigma + 1)\,\|x_i^{(1)}\|\,, \quad i = 1, \ldots, m\,.$$

By virtue of the Eberlein-Smulyan theorem (see e.g. [3]), the sequences $(x_i^{(n)})_n$, $i = 1, \ldots, m$, are w-sequentially compact and, by it, there exist subsequences $(x_i^{(n_j)})_j$, $i = 1, \ldots, m$, such that

$$w\text{-}\lim_j x_i^{(n_j)} = \tilde{x}_i\,, \quad i = 1, \ldots, m\,. \tag{8}$$

In particular, $\tilde{x}_1, \ldots, \tilde{x}_m$ is biorthogonal to $\varphi_1, \ldots, \varphi_m$. By Corollary of Proposition 1, $\tilde{x}_1, \ldots, \tilde{x}_m$ generate a Φ-interpolative subspace which we denote by $\tilde{L}$. By (4), (8) we further have

$$w\text{-}\lim_j J_{L^{(n_j)}} x = w\text{-}\lim_j \sum_{i=1}^{m} \langle x, \varphi_i \rangle\, x_i^{(n_j)} = \sum_{i=1}^{m} \langle x, \varphi_i \rangle\, \tilde{x}_i$$

for all $x \in E$. Therefore

$$\|J_{\tilde{L}} x\| \leqq \liminf_j \|J_{L^{(n_j)}} x\| \leqq \lim_j \left(\sigma + \frac{1}{n_j}\right) \|x\|\,.$$

Thus the estimate $\|J_{\tilde{L}}\| \leqq \sigma$ is valid. This inequality completes the proof.

Theorem 2. *Let E be a reflexive Banach space and let Φ be such a subspace of E^* that $(\Phi_\perp)^\perp$ admits a bounded projection onto itself. Then $\tilde{L}$ is the best Φ-interpolative subspace if and only if $J_{\tilde{L}}^*$ is a projection onto $(\Phi_\perp)^\perp$ with the smallest possible norm, i.e. $\|J_{\tilde{L}}^*\| = \inf_P \|P\|$, where the greatest lower bound is taken over all bounded projections P of E onto $(\Phi_\perp)^\perp$.*

Proof. First, by the assumptions on Φ, E and Corollary of Proposition 3, there exists at least one closed Φ-interpolative subspace. For, if P is a bounded projection onto $(\Phi_\perp)^\perp$ and $N = P_{-1}(0)$ then $N = (N_\perp)^\perp$ (Lemma 2). Using Corollary of Proposition 3 we obtain that $L = N_\perp$ is a closed Φ-interpolative subspace. Let now $\tilde{L}$ be the best Φ-interpolative subspace. By virtue of Lemma 3, $J_{\tilde{L}}^*$ is a bounded projection onto $(\Phi_\perp)^\perp$. Suppose that there exists a projection P onto $(\Phi_\perp)^\perp$ such that $\|P\| < \|J_{\tilde{L}}^*\|$. We put L as above. L is a Φ-interpolative subspace and, by Lemma 3, J_L^* is the projection onto $(\Phi_\perp)^\perp$ which is parallel to N and therefore $J_L^* = P$. It means that $\|J_L\| = \|P\| < \|J_{\tilde{L}}^*\| = \|J_{\tilde{L}}\|$, a contradiction. To prove the sufficient part suppose $\tilde{P}$ is a projection onto $(\Phi_\perp)^\perp$ with the least possible norm. As above, we obtain $\tilde{L} = [\tilde{P}_{-1}(0)]$ which is a closed Φ-interpolative subspace. If here exists a closed Φ-interpolative subspace L such that $\|J_L\| < \|J_{\tilde{L}}\|$ we get, by using Lemma 3, a projection J_L^* onto $(\Phi_\perp)^\perp$ which norm is less than the norm of $\tilde{P}$. This contradiction finishes the proof.

3. Φ-INTERPOLATIVE WIDTH

The definition of the Φ-interpolative width was given by (1) and (2). Throughout this section we shall suppose that Φ is of the dimension n and we shall choose some base of Φ which will be denoted by $\varphi_1, \ldots, \varphi_n$. For a subset M of E we use the following notation:

(a) $K(M)$ is the absolute convex hull of M, i.e.

$$K(M) = \{\sum_{i=1}^m a_i x_i;\ x_1, \ldots, x_m \in M,\ \sum_{i=1}^m |a_i| \leqq 1,\ m \text{ is any positive integer}\}\ .$$

(b) If L is a subspace of E then we put

$$d(M, L) = \sup_{x \in M} \inf_{y \in L} \|x - y\|\ .$$

(c) $d_n(M)$ denotes the n-width of M (see e.g. [5], [6], [9]), i.e. $d_n(M) = \inf_L d(M, L)$, where the greatest lower bound is taken over all subspaces L of E such that $\dim L = n$.

The following proposition yields very simple properties of the Φ-interpolative width.

Proposition 5. *Let Φ be a finite dimensional subspace of E and let M, N be subsets of E. Then:*

(i) *If $M \subset N$ then $\sigma_\Phi(M) \leqq \sigma_\Phi(N)$.*
(ii) *If N is the closure of M then $\sigma_\Phi(M) = \sigma_\Phi(N)$.*
(iii) *If M is bounded set then $\sigma_\Phi(M)$ is finite.*
(iv) $\sigma_\Phi(M) = \sigma_\Phi(K(M))$.

(v) *If L is a closed Φ-interpolative subspace of E then*

$$d(M, L) \leqq \sigma_{\Phi}(M, L) \leqq (1 + \|J_L\|)\, d(M, L)\,.$$

(vi) *If* $\dim \Phi = n$ *then* $d_n(M) \leqq \sigma_{\Phi}(M)$.

Proof. (i) It is clear.

(ii), (iii) It is also obvious from the continuity of J_L for any Φ-interpolative subspace L.

(iv) Let L be Φ-interpolative and $x \in K(M)$, i.e. $x = \sum_{i=1}^{m} a_i x_i$, where $x_1, \ldots, x_m \in M$ and $\sum_{i=1}^{m} |a_i| \leqq 1$. Then

$$\|x - J_L x\| = \|\sum_{i=1}^{m} a_i(x_i - J_L x_i)\| \leqq \sum_{i=1}^{m} |a_i|\, \sigma_{\Phi}(M, L) \leqq \sigma_{\Phi}(M, L)\,.$$

By (i), we have $\sigma_{\Phi}(K(M), L) = \sigma_{\Phi}(M, L)$ and taking the greatest lower bound we obtain the result.

(v) The left-hand side inequality is obvious from the definition of $d(M, L)$. Let $x \in M$ and $y_m \in L$ such that

$$\|x - y_m\| \leqq \inf_{y \in L} \|x - y\| + \frac{1}{m}\,.$$

Then $J_L y_m = y_m$ and we have

$$\|x - J_L x\| \leqq \|x - y_m\| + \|J_L(x - y_m)\| = (1 + \|J_L\|)\, \|x - y_m\|\,.$$

Therefore $\|x - J_L x\| \leqq (1 + \|J_L\|) \inf_{y \in L} \|x - y\|$. From this inequality the result follows immediately.

(vi) The inequality follows directly from the left-hand side inequality in (v).

Remark. The preceding proofs show that (i), (iv), (v) hold without the assumption upon the dimension of Φ.

Definition. Let Φ be a finite dimensional subspace of E^* and let M be a bounded set of E. If there exists a Φ-interpolative subspace $\underset{\sim}{L}$ such that $\sigma_{\Phi}(M, \underset{\sim}{L}) = \sigma_{\Phi}(M)$ then $\underset{\sim}{L}$ is called the *best Φ-interpolation* for M.

Our next aim is to prove the existence of a best Φ-interpolation. We fix some Φ-interpolative subspace for which we shall keep the notation N. Let $x_1, \ldots, x_n$ be a base of N with the properties (3), (4). A subset M of E is said to have the Φ-interpolative range m if $\dim \operatorname{Lin} J_N(M) = m$ (Lin denotes the linear hull). We remark that the Φ-interpolative range does not depend on the choice of N. For, let $y_1, \ldots, y_m$ be such elements of M that $J_N y_1, \ldots, J_N y_m$ form a base of $\operatorname{Lin} J_N(M)$.

This means that for each $x \in M$ there exist scalars $\xi_1, \ldots, \xi_m$ such that

$$J_N x = \sum_{i=1}^{m} \xi_i J_N y_i \,, \tag{9}$$

i.e. $x - \sum_{i=1}^{m} \xi_i y_i \in \Phi_\perp$. It follows that $J_L x = \sum_{i=1}^{m} \xi_i J_L y_i$ for a Φ-interpolative subspace L and therefore $\dim \operatorname{Lin} J_L(M) \leqq \dim \operatorname{Lin} J_N(M)$. Substituting N for L, we obtain the converse inequality.

We shall need the following lemma.

Lemma 4. *Let M be a subset of E with the Φ-interpolative range m. Then there exists a base $z_1, \ldots, z_n$ of N such that for each Φ-interpolative subspace L there exists a Φ-interpolative subspace L' having the following properties:*

(i) *L' has a base $c_1, \ldots, c_n$ with the decomposition $c_i = z_i + d_i$, $i = 1, \ldots, n$, where $d_1, \ldots, d_m$ are elements of Φ and $d_{m+1} = \ldots = d_n = 0$.*

(ii) *For all $x \in M$ there exist scalars $\xi_1, \ldots, \xi_m$ which do not depend on L such that*

$$J_{L'} x = \sum_{j=1}^{m} \xi_j c_j = J_L x \,. \tag{10}$$

Proof. Let $\{y_1, \ldots, y_m\}$ be the minimal set of M such that (9) is valid. We set $z_j = J_N y_j$, $j = 1, \ldots, m$. As these elements are linearly independent we can choose such elements $z_{m+1}, \ldots, z_n$ that $z_1, \ldots, z_n$ form a base of N. Let now L be a Φ-interpolative subspace. Then $J_L y_j = z_j + d_j$, $j = 1, \ldots, m$, where $d_1, \ldots, d_m$ belong to $\Phi_\perp$. We put $c_j = J_L y_j$, $j = 1, \ldots, m$ and $c_j = z_j$, $j = m + 1, \ldots, n$. By using Proposition 1, it can be easily proved that the subspace L' generated by $c_1, \ldots, c_n$ is Φ-interpolative. Further, $J_{L'} y_j = J_L y_j$, $j = 1, \ldots, m$ what follows that $J_{L'} x = J_L x$ for all $x \in M$. We have (10) with the same $\xi_1, \ldots, \xi_m$ as in (9).

For further purposes we denote by $\mathscr{L}_\Phi(K)$ the set of all Φ-interpolative subspaces L such that $\sigma_\Phi(M, L) \leqq K$.

Lemma 5. *Let M be a bounded subset of E with the Φ-interpolative range m. Let K be such that $K > \sigma_\Phi(M)$. Then there exists such a positive number A that for all $L \in \mathscr{L}_\Phi(K)$ the base $c_1, \ldots, c_n$ of L' from Lemma 4 has the property*

$$\|d_i\| \leqq A \,, \quad i = 1, \ldots, n \,.$$

Proof. By the proof of Lemma 4, we have $d_j = (J_L - J_N)\, y_j = (y_j - J_N y_j) - (y_j - J_L y_j)$ and thus

$$\|d_j\| \leqq \sigma_\Phi(M, N) + \sigma_\Phi(M, L) \leqq \sigma_\Phi(M, N) + K$$

for $j = 1, \ldots, m$.

Theorem 3. *Let M be a bounded set of a reflexive Banach space E and let Φ be a finite dimensional subspace of E. Then there exists a best Φ-interpolation for M.*

Proof. Let $(L^{(k)})$ be such a sequence of Φ-interpolative subspaces of E that

$$\sigma_\Phi(M) \leqq \sigma_\Phi(M, L^{(k)}) < \sigma_\Phi(M) + \frac{1}{k}.$$

Let M have the Φ-interpolative range m and let $(L^{(k)'})$ be the sequence of Φ-interpolative subspaces from Lemma 4. We denote the base of $L^{(k)'}$ with the properties of Lemma 4 by $c_1^{(k)}, \ldots, c_n^{(k)}$. Putting $K = \sigma_\Phi(M) + 1$ in Lemma 5 we find that $\|d_i^{(k)}\| \leqq A$ for $i = 1, \ldots, n$, $k = 1, \ldots$ By virtue of the w-sequential compactness of the unit ball of E, there exist subsequences $(d_i^{(k_j)})_j$ $i = 1, \ldots, n$, such that

$$w\text{-}\lim_j c_i^{(k_j)} = z_i + w\text{-}\lim_j d_i^{(k_j)} = z_i + \underset{\sim}{d}_i = \underset{\sim}{c}_i\,, \quad i = 1, \ldots, n\,.$$

As $d_i^{(k_j)} \in \Phi_\perp$ the elements $\underset{\sim}{d}_1, \ldots, \underset{\sim}{d}_n$ lie also in $\Phi_\perp$ and therefore $\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_n$ generate the Φ-interpolative subspace $\underset{\sim}{L}$. By virtue of the property (ii) of Lemma 4, we have $w\text{-}\lim_j J_{L^{(k_j)}}x = J_{\underset{\sim}{L}}x$ and hence

$$\|x - J_{\underset{\sim}{L}}x\| \leqq \liminf_j \|x - J_{L^{(k_j)}}x\| = \lim_j \sigma_\Phi(M, L^{(k_j)}) = \sigma_\Phi(M)$$

for all $x \in M$. Taking the least upper bound over $x \in M$ we obtain the required result.

References

[1] *Aubin J. P.:* Interpolation et approximation optimales et "spline functions", J. Math. Anal. Appl. *24* (1968), 1—24.

[2] *Banach S.:* Théorie des opérations linéaires, Warszawa 1932.

[3] *Day M. M.:* Normed linear spaces, Springer 1958.

[4] *Garkavi A. L.:* On the best net and the best section of a set in a normed linear space (in Russian), Izv. Akad. Nauk SSSR, ser. mat. *26* (1962), 87—106.

[5] *Kolmogorov A. N.:* Über die beste Annaherung von Funktionen einer gegeben Funktionenklasse, Ann. of Math. *37* (1936), 107—110.

[6] *Lorentz G. G.:* Approximations of functions, Holt, Rinehart and Winston 1966.

[7] *Murray F. J.:* On complementary manifolds and projections in spaces $\mathscr{L}_p$ and ℓ_p, Trans. Amer. Math. Soc. *41* (1937), 138—152.

[8] *Singer I.:* Quelques applications d'un dual du Théorème de Hahn-Banach, C.R. Acad. Sci. (Paris) *247* (1958), 846—849.

[9] *Tihomirov V. M.:* Widths of sets in functional spaces and the theory of best approximations (in Russian), Uspehi mat. nauk *15* (1960), No 3, 81—120.

Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).