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A NECESSARY AND SUFFICIENT CONDITION FOR CONTINUITY OF ADDITIVE FUNCTIONS

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(Received January 10, 1973)

In the sequel, a real-valued function \( f \) defined on the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) is called to be additive if it satisfies the Cauchy functional equation

\[
f(x + y) = f(x) + f(y)
\]

for all \( x, y \in \mathbb{R}^n \).

R. GER and M. KUCZMA [2] introduced the following set classes:

A set \( T \subset \mathbb{R}^n \) belongs to the class \( \mathcal{B} \) if and only if each additive function upper-bounded on \( T \) is continuous.

A set \( T \subset \mathbb{R}^n \) belongs to the class \( \mathcal{C} \) if and only if each additive function bounded (bilaterally) on \( T \) is continuous.

It is known that \( \mathcal{B} \subset \mathcal{C} \) but \( \mathcal{B} \neq \mathcal{C} \), see e.g. [2]. M. Kuczma [4] posed the problem to find some characterizations of the classes \( \mathcal{B} \) and \( \mathcal{C} \). The class \( \mathcal{C} \) has been characterized in [5]. The main aim of the present note is to give a characterization of \( \mathcal{B} \); this result is complemented by an example of a strange set belonging to \( \mathcal{B} \).

Throughout the paper, the set of rational numbers will be denoted by \( \mathbb{Q} \). The symbols \( +, - \) denote always the algebraic operations.

A set \( A \subset \mathbb{R}^n \) is called \( \mathbb{Q} \)-radial at a point \( x_0 \) if for each \( x \in \mathbb{R}^n \) there is a real \( c_x > 0 \) such that \( x_0 + \alpha x \in A \) whenever \( \alpha \in \mathbb{Q}, 0 \leq \alpha < c_x \).

A set \( A \subset \mathbb{R}^n \) is called \( \mathbb{Q} \)-convex if for each \( x, y \in A \), and each \( \alpha \in \mathbb{Q}, 0 \leq \alpha \leq 1, \alpha x + (1 - \alpha) y \in A \). The \( \mathbb{Q} \)-convex hull of a set \( B \subset \mathbb{R}^n \) (i.e. the minimal \( \mathbb{Q} \)-convex set containing \( B \)) will be denoted by \( \mathbb{Q}(B) \).

Now we are able to prove the main result.

**Theorem.** Let \( T \) be a subset of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Then each additive function \( f : \mathbb{R}^n \to \mathbb{R} \) upper-bounded on \( T \) is continuous if and only if for each subset \( A \) of \( \mathbb{R}^n \), \( \mathbb{Q} \)-radial at a certain point, the set \( \mathbb{Q}(T - A) \) contains a sphere.

In other words, \( T \in \mathcal{B} \) if and only if for each subset \( A \) of \( \mathbb{R}^n \), \( \mathbb{Q} \)-radial at a point, the \( \mathbb{Q} \)-convex hull of \( T - A \) contains a ball.
Proof of the theorem is based on the following result of M. E. Kuczma [3]:

Let $C$ be a $Q$-convex subset of $\mathbb{R}^n$, $Q$-radial at a point; then either $C$ contains a ball or there exists a discontinuous additive function upper-bounded on $C$.

Let $T \subset \mathbb{R}^n$ and let $A$ be a subset of $\mathbb{R}^n$, $Q$-radial at $x_0$ such that $C = Q(T - A)$ contains no ball. We may without loss of generality assume $T \neq \emptyset$. Since $C$ is $Q$-convex and $Q$-radial (at each point of the set $T - x_0$) the above quoted result of M. E. Kuczma implies the existence of a discontinuous additive function $f : \mathbb{R}^n \to \mathbb{R}$ upper-bounded on $Q(T - A)$. Let $a$ be a fixed point from $A$. Since $T - a \subset C = Q(T - A)$, we conclude that $f$ is upper-bounded on $T - a$, and consequently, by the additivity of $f$, $f$ is upper-bounded on $T$. Thus $T \notin \mathcal{B}$.

Now assume that $Q(T - A)$ contains a ball for each subset $A$ of $\mathbb{R}^n$, $Q$-radial at a certain point. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an additive function such that $f(x) < M$ for each $x \in T$. For each $y \in \mathbb{R}^n$ let $A_y = \{x \in Q, f(xy) > -1\}$, and put

$$A = \bigcup_{y \in \mathbb{R}^n} A_y.$$  

Clearly, $A$ is $Q$-radial at $0$. For each $u \in T, v \in A$, $f(u - v) = f(u) - f(v) < M + 1$, thus $f$ is upper-bounded on $T - A$ and consequently, $f$ is bounded on $Q(T - A)$ (see e.g. [1]). Thus $f$ is upper-bounded on a set with positive Lebesgue measure and so $f$ is continuous (see e.g. [2]), q.e.d.

Remark. It is easy to verify that in Theorem, the set $Q(T - A)$ can be replaced by $Q(T) - Q(A)$.

A set $A \subset \mathbb{R}^n$ is called midpoint convex if $\frac{1}{2}(A + A) = A$. R. Ger and M. Kuczma [2] have proved the following result: Let $T \subset \mathbb{R}^n$. If the set $J(T) - J(T)$ has a positive inner Lebesgue measure then $T \in \mathcal{C}$ (here $J(A)$ denotes the midpoint convex hull of $A$). The authors conjectured that this condition is not necessary for $T \in \mathcal{C}$. In [5] it is stated without proof that this conjecture is true. In the present note we give a somewhat stronger result, namely that this condition is not necessary for $T \in \mathcal{B}$.

Example. Let $H$ be a Hamel basis of the reals and let $T$ be the set of all numbers of the form $\sum x_i h_i$ (finite sum) where $h_i \in H$, and $x_i$ are dyadic rational numbers (i.e. $x_i = m_i \cdot 2^{n_i}$, where $m_i, n_i$ are integers).

It is easy to verify that $T \in \mathcal{B}$. Clearly $T$ is midpoint convex and so $J(T) - J(T) = T - T = T$. Now we show that the inner Lebesgue measure of $T$ is 0.

Since $H$ is a Hamel basis, $1$ can be written uniquely (up to the order of summands) as

$$1 = x_1 h_1 + x_2 h_2 + \ldots + x_n h_n,$$

where $h_i \in H, x_i \in Q, i = 1, 2, \ldots, n$. Assume that $x_i = \frac{u}{v}$, where $u, v$ are relatively prime integers. For each prime integer $q, q > u$, let $A_q$ be the set $T + q^{-1}$. We show that the sets $A_q$ are pairwise disjoint. Assume, on the contrary, that there are two
prime integers $p > q$ greater than $u$ such that $A_p \cap A_q$ is non-empty. Then $p^{-1} - q^{-1} = (p - q)/pq \in T$. On the other hand, from (1) we have

$$\frac{p - q}{pq} = \frac{p - q}{pq} \cdot \frac{u}{v} \cdot h_1 + \frac{p - q}{pq} \cdot (z_2 h_2 + \ldots + z_n h_n).$$

This representation of $(p - q)/pq$ is unique so $((p - q)/pq)(u/v)$ must be a dyadic rational number. But this is impossible since $(p - q)/u$ is not divisible by $p$. Thus the sets $A_q$ are pairwise disjoint. Now if the inner Lebesgue measure $m_1(T)$ of $T$ is positive then there is a finite interval $I \subset R$ and $\varepsilon > 0$ such that for each sufficiently large prime $q$, $m_1(I \cap A_q) > \varepsilon$. But in this case $m_1(I) = +\infty$ — a contradiction. Hence $m_1(T) = 0, \text{q.e.d.}$

References


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