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A SET FUNCTOR WHICH COMMUTES WITH ALL HOMFUNCTORS IS A HOMFUNCTOR

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0. INTRODUCTION

The aim of the present paper is to prove under the GCH (generalized continuum hypothesis): given a covariant set functor F such that for each covariant homfunctor Q, $F \circ Q$ and $Q \circ F$ are naturally equivalent, the functor F is itself equivalent to a homfunctor.

The first part contains preliminaries; in the second one we prove the theorem for functors from the category $\mathbf{S}_n$ of all sets of cardinality less than n, n being a cardinal inaccessible in the sense: if $a, b < n$ then $a^b < n$ (n is not assumed to be regular). In the last part, the theorem is proved for small functors – and, under the generalized continuum hypothesis for all functors – from the category of sets into itself.

1. CONVENTIONS, DEFINITIONS AND PRELIMINARY LEMMAS

Given sets A, B and a mapping $f : A \to B$, $|A|$ denotes the cardinality of A, $A \simeq B$ ($A < B$, $A \leqq B$) stands for $|A| = |B|$ ($|A| < |B|$, $|A| \leqq |B|$, respectively). The set $\{f(x); x \in X\}$ is denoted by $\operatorname{Im} f$. If $X \subset A$ then i_A^X denotes the inclusion mapping of X into A. Each cardinal m will be viewed as a set (with $m = |m|$).

Q_A denotes the covariant homfunctor from the category **Set** of sets into itself: $Q_A = \operatorname{Hom}(A, -)$. Clearly $Q_A \sim Q_{|A|}$ ($\sim$ denotes the natural equivalence of functors) so that we shall consider Q_m (m is a cardinal) only. If n is a cardinal then $\mathbf{S}_n$ is the category of sets of cardinality $<n$. The word functor as well as the letter F (or G, H etc.) will stand for a covariant functor from **Set** to **Set**, or from $\mathbf{S}_n$ to $\mathbf{S}_n$, respectively.

Let F be a functor. Let A, X be sets, $A \subset FX$. (A, X) is a *reaching couple* for F if for each set Y and each $y \in FY$ there are $a \in A$ and $f : X \to Y$ with $Ff(a) = y$. F is said to be *small* if it possesses a reaching couple; the minimal cardinality of A is denoted by δF. Clearly, δF is the smallest cardinal m such that there exists a system $\{\varepsilon_\alpha : Q_X \to F;\ \alpha \in m\}$ which is collectively epimorphic, i.e., if $\mu \circ \varepsilon_\alpha = \nu \circ \varepsilon_\alpha$ for some transformations $\mu, \nu : F \to G$ and for each α then $\mu = \nu$; equivalently: $\operatorname{Im}(\varepsilon_\alpha)^X$ cover FX for each X.

Cardinal n is called an *unattainable cardinal* of F provided that there is $x \in Fn$ such that $x \notin \operatorname{Im} Ff$ for any $f : X \to n$ with $X < n$; $\mathscr{A}_F$ denotes the class of all unattainable cardinals of F.

For every X and $x \in FX$, put $\mathscr{F}_F^X(x) = \{Y \subset X;\ x \in \operatorname{Im} Fi_X^Y\}$. $\mathscr{F}_F^X(x)$ is a filter on X [5] ($\exp X = \{Y;\ Y \subset X\}$ is also considered a filter). Denote $\varphi F = \sup \chi(\mathscr{F}_F^X(x))$ (if it exists) where $\chi\mathscr{F}$ ($\mathscr{F}$ being a filter) is the character of $\mathscr{F}$, i.e., the minimal cardinality of a base of $\mathscr{F}$.

A filter $\mathscr{F}$ is called *trivial* if $\chi\mathscr{F} = 1$, i.e., if $\bigcap\mathscr{F} \in \mathscr{F}$.

Let $\mathscr{F}$ be a filter on a set A; let $\mathscr{F}_a$ ($a \in A$) be filters on a set X. Denote by

$$\bigcup_{\mathscr{F}} \mathscr{F}_a$$

the filter whose base consists of sets of the form

$$\bigcup_{a \in Z} Z_a$$

where $Z \in \mathscr{F}$ and $Z_a \in \mathscr{F}_a$ ($a \in A$).

Lemma 1.1. *Let $\mathscr{F}$ be a trivial filter. Let there exist $F_a \in \mathscr{F}_a$ such that $\{F_a;\ a \in A\}$ is a disjoint family. Then*

$$\chi \bigcup_{\mathscr{F}} \mathscr{F}_a = \prod_{a \in \cap\mathscr{F}} (\chi\mathscr{F}_a)\,.$$

In particular, if $\chi\mathscr{F}_a > 1$ for each a and $m \simeq \bigcap\mathscr{F}$ then

$$\chi \bigcup_{\mathscr{F}} \mathscr{F}_a \geqq 2^m\,.$$

Lemma 1.2. *Conversely, if all $\mathscr{F}_a$ are trivial then*

$$\chi \bigcup_{\mathscr{F}} \mathscr{F}_a \leqq \chi\mathscr{F}\,.$$

Lemma 1.3. *Let F, G be functors. Then*

$$\mathscr{F}_{F\circ G}^X(x) = \bigcup_{\mathscr{F}_F^{GX}(x)} \mathscr{F}_G^X(a) \quad (a \in GX)\,.$$

Lemma 1.4. *For each $f \in Q_m X$ (i.e. $f : m \to X$),*

$$\mathscr{F}_{Q_m}^X(f) = \{Z \subset X;\ Z \supset \operatorname{Im} f\}\,.$$

Thus all the filters $\mathscr{F}_{Q_m}^X(f)$ are trivial.

Proposition 1.5. *Let $\varepsilon : F \to G$ be an epitransformation, $x \in FX$, $\varepsilon^X(x) = y$. Then*

$$\mathscr{F}^X_G(y) - \{\emptyset\} = \bigcup_{\varepsilon^X(z)=y} \mathscr{F}^X_F(z) - \{\emptyset\} = \{Z;\ Z \in \mathscr{F}^X_F(z),\ \varepsilon^X(z) = y\} - \{\emptyset\}.$$

In particular, $\mathscr{F}^X_F(x) \subset \mathscr{F}^X_G(y)$; if moreover $\varepsilon^X(x) \neq \varepsilon^X(z)$ for every $z \neq x$, then $\mathscr{F}^X_F(x) = \mathscr{F}^X_G(y)$.

Proposition 1.6. *If $Fm \simeq m$ for an infinite m then $m \notin \mathscr{A}_F$.*

Lemma 1.7. *Let $\bar{F}, \bar{G}$ be domain-restrictions of $F, G : \mathbf{Set} \to \mathbf{Set}$, respectively, to $\mathbf{S}_n$, where $\sup \mathscr{A}_F$, $\sup \mathscr{A}_G < n$. If $\bar{F} \sim \bar{G}$ then $F \sim G$.*

Proofs of the above propositions, except 1,6, are straightforward computations. Concerning 1,6: It is proved in [2] that, for any infinite $m \in \mathscr{A}_F$, $Fm \geqq |\mathfrak{D}|$ where $\mathfrak{D}$ is an almost-disjoint system of subsets of m. It is well-known (e.g. [1]) that $\mathfrak{D}$ can be found such that $|\mathfrak{D}| > m$.

Lemma 1.8. [4]. *F preserves intersections iff each $\mathscr{F}^X_F(x)$ is trivial.*

Lemma 1.9 [2]. *If $Ff(x) = y$ for some $x \in FX$, $y \in FY$, $f : X \to Y$, then $Z \in \mathscr{F}^X_F(x) \Rightarrow f(Z) \in \mathscr{F}^Y_F(y)$.*
If, moreover, f is one-to-one on a set of $\mathscr{F}^X_F(x)$, the converse is also true.

Proposition 1.10 [2]. *A functor $F : \mathbf{Set} \to \mathbf{Set}$ (or $F : \mathbf{S}_n \to \mathbf{S}_n$) is small iff $\mathscr{A}_F$ is a set (or $\sup \mathscr{A}_F < n$, respectively).*

Proposition 1.11 [2]. *If $X > n = \sup \mathscr{A}_F$, then $FX \leqq \max\{Fn, X^n\}$.*

2. FUNCTORS FROM $\mathbf{S}_n$ TO $\mathbf{S}_n$

Convention. Throughout this part, F denotes a small functor of $\mathbf{S}_n$ into itself. We shall suppose

$$a, b < n \Rightarrow a^b < n.$$

Thus each covariant homfunctor Q_a $(a < n)$ maps the category $\mathbf{S}_n$ into itself; we may and we shall consider it as a functor from $\mathbf{S}_n$ to $\mathbf{S}_n$.

Lemma 2.1. *For each set $A < n$, $\delta(F \circ Q_A) \leqq \delta F$.*

Proof. If $\{\varepsilon_\alpha : Q_X \to F;\ \alpha \in I\}$ is a collectively epimorphic system, so is $\{\varepsilon_\alpha Q_A : Q_X \circ Q_A \to F \circ Q_A;\ \alpha \in I\}$. As $Q_X \circ Q_A \sim Q_{X \times A}$, our lemma follows.

Lemma 2.2. *If $\delta F > 1$ then there exists $m < n$ such that $\delta(Q_m \circ F) > \delta F$.*

Proof. Put $m = \delta F$. Let us suppose $\delta(Q_m \circ F) \leqq \delta F = m$. Then there exists a reaching couple (A_1, X_1) for $Q_m \circ F$, where $A_1 = \{a_\alpha; \alpha \in m\}$. As X_1 can be chosen arbitrarily large, we may assume that (FX_1, X_1) is a reaching couple for F. Write each a_α in the form $a_\alpha = \{a_\alpha^\beta; \beta \in m\}$, $\alpha \in m$, where $a_\alpha^\beta \in FX_1$ for $\alpha, \beta \in m$. As $\delta F > 1$ and (FX_1, X_1) is a reaching couple for F, for every $x \in FX_1$ there is $y \in FX_1$ such that $y = Ff(x)$ holds for no $f : X_1 \to X_1$. Hence for each $\alpha \in m$ we can choose y_α such that $y_\alpha \neq Ff(a_\alpha^\alpha)$ for any $f : X_1 \to X_1$. Thus, putting $y = \{y_\alpha; \alpha \in m\} \in Q_m \circ F(X_1)$ we have $y \neq Q_m \circ Ff(a_\alpha)$ for any $f : X_1 \to X_1$ and $\alpha \in m$ which is a contradiction because (A_1, X_1) is a reaching couple for $Q_m \circ F$.

Proposition 2.3. *Let $F \circ Q_m \sim Q_m \circ F$ for each $m \in \mathbf{S}_n$. Then F is a factorfunctor of some Q_a $(a \in \mathbf{S}_n)$.*

Proof follows from Lemmas 2,1 and 2,2.

Lemma 2.4. *For each $a \in \mathbf{S}_n$, $\varphi(F \circ Q_a) \leqq \varphi F$.*

Proof. See 1,2, 1,3 and 1,4.

Lemma 2.5. *If $n > \varphi F > 1$ then there exists $m \in \mathbf{S}_n$ such that $\varphi(Q_m \circ F) > \varphi F$.*

Proof. As $n > \varphi F > 1$, there is Y and $y \in FY$ with $\chi \mathscr{F}_F^Y(y) > 1$. Put $m = \varphi F$. Further, φF is infinite so that we can choose monomorphisms ψ_ι $(\iota \in m)$ from Y to Y such that $\iota \neq \iota' \Rightarrow \operatorname{Im} \psi_\iota \cap \operatorname{Im} \psi_{\iota'} = \emptyset$. Put $x_\iota = F\psi_\iota(y)$. By 1,9, $\chi \mathscr{F}_F^Y(x_\iota) > 1$. Thus, putting $x = \{x_\iota; \iota \in m\} \in Q_m \circ FY$, we get

$$\varphi Q_m \circ F \geqq \chi \mathscr{F}^Y_{Q_m \circ F}(x) \geqq 2^m > m = \varphi F$$

(see 1,1 and 1,3).

Proposition 2.6. Let $F \circ Q_m \sim Q_m \circ F$ *for each $m < n$. Then $\varphi F = 1$, i.e. each filter $\mathscr{F}_F^X(x)$ is trivial, equivalently: F preserves intersections.*

Proof. See 1,8, 2,4 and 2,5.

Lemma 2.7. *Let F be a factorfunctor of some Q_a $(a \in \mathbf{S}_n)$ such that F preserves intersections. Then there exists $V \in \mathbf{S}_n$ and an epitransformation $\varepsilon : Q_V \to F$ such that $\mathscr{F}_F^V(\varepsilon^V(1_V)) = \{V\}$.*

Proof. Let $\nu : Q_a \to F$ be an epitransformation. As F preserves intersections, each filter $\mathscr{F}_F^X(x)$ is trivial. In particular, there is $V \subset a$ such that $\mathscr{F}_F^a(\nu^a(1_a)) = \{Y \subset a;\ Y \supset V\}$. Define $\varepsilon : Q_V \to F$ by $\varepsilon^V(1_V) = u$, where u is the (only) element of FV satisfying $Fi_a^V(u) = \nu^a(1_a)$. Then ε is evidently an epitransformation and $\mathscr{F}_F^V(u) = \{V\}$ (see 1,9).

Definition. For F satisfying the assumptions of 2,7 the epitransformation ε from 2,7 will be called the *minimal factorization.*

Lemma 2.8. *Let $\varepsilon : Q_V \to F$ be the minimal factorization. Let $f : V \to X$ be a monomorphism, $g : V \to X$ an arbitrary mapping $(X < n)$. If $\varepsilon^X(f) = \varepsilon^X(g)$ then* $\operatorname{Im} g \supset \operatorname{Im} f$.

Proof. Let $x \in (\operatorname{Im} f - \operatorname{Im} g)$. Choose $h : X \to V$ such that $h \circ f = 1_V$ and $|h^{-1}(h(x))| = 1$. As

$$(\operatorname{Im} h \circ g) \in \mathscr{F}^V_{Q_V}(h \circ g),$$

we get by 1,5 that

$$(\operatorname{Im} h \circ g) \in \mathscr{F}^V_F(\varepsilon^V(h \circ g)) = \mathscr{F}^V_F(\varepsilon^V(1_V)) = \{V\}$$

(it is easily seen, that $\varepsilon^V(h \circ g) = \varepsilon^V(h \circ f) = \varepsilon^V(1_V)$). Thus $\operatorname{Im} h \circ g = V$ which is a contradiction because $h(x) \notin \operatorname{Im} h \circ g$.

Lemma 2.9. *Let $\varepsilon : Q_V \to F$ be the minimal factorization. Let $\varepsilon^V(1_V) \neq \varepsilon^V(f)$ for every $f : V \to V$, $f \neq 1_V$. Let $u \in FX$ such that $(\{u\}, X)$ is a reaching couple for F. Then $|(\varepsilon^X)^{-1}(u)| = 1$ and $((\varepsilon^X)^{-1}(u), X)$ is a reaching couple for Q_V.*

Proof. As $(\{u\}, X)$ is a reaching couple for F, there is $h : X \to V$ with $Fh(u) = \varepsilon^X(1_V)$. Let $f, g \in (\varepsilon^X)^{-1}(u)$. Then

$$\varepsilon^V(Q_V h(f)) = F h(u) = \varepsilon^V(1_V).$$

Thus $Q_V h(f) = 1_V$, i.e. $h \circ f = 1_V$; analogously $h \circ g = 1_V$. Further, $\operatorname{Im} f = \operatorname{Im} g$ by 2,8. Clearly, if two one-to-one mappings have common retraction and the same image, then they must be equal; thus $f = g$. As $Q_V h(f) = 1_V$, $(\{f\}, X)$ is a reaching couple for Q_V.

Lemma 2.10. *Let $\varepsilon : Q_V \to F$ be the minimal factorization and let $|(\varepsilon^V)^{-1}(\varepsilon^V(1_V))| = 1$. Given X, $m \in \mathbf{S}_n$ and a reaching couple $(\{u\}, X)$ for $F \circ Q_m$. Then $(\{u\}, Q_m X)$ is a reaching couple for F and there are monomorphisms $g_i : m \to X$ such that* $\operatorname{Im} g_i \cap \operatorname{Im} g_j = \emptyset$ *for $i \neq j$ and such that*

$$\varepsilon^{Q_m X}(\{g_i;\ i \in V\}) = u .$$

Proof. To prove that $(\{u\}, Q_m X)$ is a reaching couple for F, consider any Y and $y \in FY$. Take an epimorphism $k : Q_m Y \to Y$ and choose $z \in F \circ Q_m Y$ with $F k(z) = y$. Then $z = F \circ Q_m h(u)$ for some $h : X \to Y$ so that $y = F(k \circ Q_m h)(u)$. By 2,9 there exists exactly one element $g = \{g_i;\ i \in V\}$ such that

$$\varepsilon^{Q_m X}(g) = u$$

where $g_i \in Q_m X$, i.e. $g_i : m \to X$ for $i \in V$. Let $h_i : m \to Z$ $(i \in V)$ be arbitrary monomorphisms such that $i \neq j \Rightarrow \operatorname{Im} h_i \cap \operatorname{Im} h_j = \emptyset$. Then $h = \{h_i;\ i \in V\} \in Q_V(Q_m Z)$ so that there is $p : X \to Z$ with $Q_V \circ Q_m\, p(g) = h$ (by 2,9, $(\{g\}, Q_m X)$ is a reaching couple for Q_V). Thus $p \circ g_i = h_i$, $i \in V$. As h_i are monomorphisms, so are g_i, as h_i have disjoint images, so have g_i.

Given $f : V \to V$ and m ($V, m \in \mathbf{S}_n$), denote by $\tilde{f}_i$ ($i \in m$) mappings from $V \times m$ defined as follows:

$$\tilde{f}_i(x, j) = (x, j) \quad \text{for} \quad j \neq i\,, \quad \tilde{f}_i(x, i) = (f(x), i)\,.$$

Lemma 2.11. *Let $\varepsilon : Q_V \to F$ be a transformation. Let $f, g : V \to V$ with $\varepsilon^V(f) = \varepsilon^V(g)$. Then*

$$Q_m \circ F(\tilde{f}_i)\left((Q_m\varepsilon)^{m \times V}(1_{m \times V})\right) = Q_m \circ F(\tilde{g}_i)\left((Q_m\varepsilon)^{m \times V}(1_{m \times V})\right).$$

Proof. Straightforward computation.

Lemma 2.12. *Let $\varepsilon : Q_V \to F$ be the minimal factorization such that $\varepsilon^V(f) = \varepsilon^V(1_V)$ for some $f : V \to V$. If $F \circ Q_m \sim Q_m \circ F$ for every $m < n$, then $f = 1_V$.*

Proof. Put $Y = \{t \in V;\ f(t) \neq t\}$. Choose $m > V^3$. Let $\mu : Q_m \circ F \to F \circ Q_m$ be a natural equivalence. Put $v = \mu^{m \times V}(u)$, where

$$u = (Q_m\varepsilon)^{m \times V}(1_{m \times V})\,.$$

By 2,11, $Q_m \circ F(\tilde{f}_i)(u) = u$ so that $F(Q_m\tilde{f}_i)(v) = v$ for each $i \in m$. As follows easily by 1,9, for any

$$g \in W = \bigcap \mathscr{F}_F^{Q_m(m \times V)}(v)$$

and for any $i \in m$ there is

$$k_i \in \bigcap \mathscr{F}_F^{Q_m(m \times V)}(v)$$

with $g = Q_m\tilde{f}_i(k_i) = \tilde{f}_i \circ k_i$ $(i \in m)$. Let $i \in m$. Evidently, if $k_i = k_j$ for some $j \neq i$ then $\tilde{f}_i \circ g = g$, because $\tilde{f}_i \circ g = \tilde{f}_i \circ (\tilde{f}_i \circ k_i) = \tilde{f}_i \circ k_i = g$. Thus $\{i;\ \tilde{f}_i \circ g \neq g\} \subset \{i;\ k_i \neq k_j \text{ for every } j \neq i\} \subset \{k_i;\ i \in m\} \subset W$. Further, $W \leqq V$ as for any $x \in FX$, $\bigcap \mathscr{F}_F^X(x) \leqq V$ (see 1,5 and 1,6). We get $\{i;\ \tilde{f}_i \circ g \neq g\} \leqq V$ so that

$$\{i;\ \bigcup_{g \in W} \operatorname{Im} g \cap (\{i\} \times Y) \neq \emptyset\} \leqq V^2$$

and finally

$$\bigcup_{g \in W} \operatorname{Im} g \cap (m \times V) \leqq Y \times V^2 \leqq V^3\,.$$

On the other hand, using 1,3 and 1,4 we get

$$\bigcup_{g \in W} \operatorname{Im} g = \bigcap \mathscr{F}_{F \circ Q_m}^{m \times V}(v) = \bigcap \mathscr{F}_{Q_m \circ F}^{m \times V}(u)\,.$$

Since for any minimal factorization ε, $Q_m\varepsilon$ is a minimal factorization, too, so that the last intersection is $m \times V$. Hence

$$\bigcup_{g \in W} \operatorname{Im} g \cap (m \times Y) = m \times Y .$$

As shown above, the former set has cardinality $\leqq V^3 < m$; we get $Y = \emptyset$, i.e. $f = 1_V$.

Theorem 2.13. *Let F be a small functor of $\mathbf{S}_n$ into $\mathbf{S}_n$ such that for every $m < n$ $F \circ Q_m \sim Q_m \circ F$. Then $F \sim Q_r$ for some r.*

Proof. Let $\varepsilon : Q_V \to F$ be a minimal factorization. By 2,12, $|(\varepsilon^V)^{-1}(\varepsilon^V(1_V))| = 1$. It suffices to prove the following: if $\varepsilon^X(f) = \varepsilon^X(g)$ for some $f, g : V \to X$, then $f = g$. Choose m, $n > m > V^2$, and put $u = (Q_m\varepsilon)^{m \times V}(1_{m \times V})$. Let $\mu : Q_m \circ F \to F \circ Q_m$ be a natural equivalence. As $(\{u\}, m \times V)$ is a reaching couple for $Q_m \circ F$, so $(\{\mu(u)\}, m \times V)$ is a reaching couple for $F \circ Q_m$. By 2,10 $(\{\mu(u)\}, Q_m(m \times V))$ is a reaching couple for F, and there are monomorphisms $h_i : m \to m \times V$ with disjoint images such that $h = \{h_i;\ i \in V\}$ is the only element of $Q_V \circ Q_m(m \times V)$ with

$$\varepsilon^{Q_m(m \times V)}(h) = \mu^{Q_m(m \times V)}(u) .$$

By 1,4,

$$\mathscr{F}^{m \times V}_{Q_V \circ Q_m}(h)$$

contains

$$\bigcup_i \operatorname{Im} h_i$$

and so does

$$\mathscr{F}^{m \times V}_{F \circ Q_m}(\mu^{m \times V}(u))$$

(see 1,5). By 1,4, the last filter is equal to

$$\mathscr{F}^{m \times V}_{Q_V \circ Q_m}(1_{m \times V}) = \{m \times V\}$$

so that

$$\bigcup_i \operatorname{Im} h_i = m \times V .$$

Further, $\tilde{f}_i \circ h_j \neq \tilde{f}_i \circ h_k$ for every i, j, k, $j \neq k$ (indeed, $\tilde{f}_i(x) = \tilde{f}_i(y)$ for at most V^2 couples x, y with $x \neq y$; the equality $\tilde{f}_i \circ h_j = \tilde{f}_i \circ h_k$ would require m such couples, namely the couples $h_j(t)$, $h_k(t)$ for $t \in m$). Analogously $\tilde{g}_i \circ h_j \neq \tilde{g}_i \circ h_k$, $\tilde{f}_i \circ h_j \neq \tilde{g}_i \circ h_k$ for i, j, k as above. Let $p_1, p_2 \in Q_V(Q_m(m \times V))$, $p_1(j) = \tilde{f}_i \circ h_j$ for $j \in V$, $p_2(j) = \tilde{g}_i \circ h_j$ for $j \in V$, where $i \in m$ is arbitrary but fixed. As noted above, $j \neq k \Rightarrow p_1(j) \neq p_1(k)$ and analogously for p_2. Thus p_1, p_2 are monomorphisms. By 2,11, $Q_m \circ F\tilde{f}_i(u) = Q_m \circ F\tilde{g}_i(u)$ so that $F \circ Q_m\tilde{f}_i(\mu^{m \times V}(u)) = F \circ Q_m\tilde{g}_i(\mu^{m \times V}(u))$. Further $Q_V \circ Q_m(\tilde{f}_i)(h_j) = p_1$, $Q_V \circ Q_m(\tilde{g}_i)(h_j) = p_2$; hence

$$\varepsilon^{Q_m(m \times V)}(p_1) = F \circ Q_m\tilde{f}_i(\mu^{m \times V}(u)) = F \circ Q_m\tilde{g}_i(\mu^{m \times V}(u)) = \varepsilon^{Q_m(m \times V)}(p_2) .$$

By 2,8, $\operatorname{Im} p_1 = \operatorname{Im} p_2$. In other words, the set of all $\tilde{g}_i \circ h_j$ $(j \in V)$ is equal to the set of all $\tilde{f}_i \circ h_j$ $(j \in V)$. In particular, each $\tilde{f}_i \circ h_j$ is equal to some $\tilde{g}_i \circ h_k$; then necessarily $j = k$ (see above), i.e. $\tilde{g}_i \circ h_j = \tilde{f}_i \circ h_j$ for each j. As

$$\bigcup_j \operatorname{Im} h_j = m \times V,$$

we get $\tilde{g}_i = \tilde{f}_i$; thus $f = g$ which completes the proof.

3. FUNCTORS FROM **Set** TO **Set**

Let us define a transfinite sequence $\{\alpha_i\}$ of cardinals by the transfinite induction:

$$\alpha_0 = \aleph_0, \quad \alpha_{i+1} = 2^{\alpha_i}, \quad \alpha_i = \sup_{j<i} \alpha_j \text{ provided i is limit}.$$

Lemma 3.1. *Let i be an ordinal such that either i is limit or $i = 0$. Then $a^b < \alpha_i$ provided $a, b < \alpha_i$.*

Proof. The case $i = 0$ is easy. Let i be limit, $a, b < \alpha_i$. Choose j with $a, b < < \alpha_j < \alpha_i$. We have

$$a^b < \alpha_j^{\alpha_j} = 2^{\alpha_j} = \alpha_{j+1} < \alpha_i.$$

Lemma 3.2. *Let $F : \mathbf{Set} \to \mathbf{Set}$ be a small functor. Then there is a cardinal n such that $n > \sup \mathscr{A}_F$ and*

a) *F maps $\mathbf{S}_n$ into $\mathbf{S}_n$;*

b) *the restriction of F to S_n is a small functor;*

c) *for any two cardinals a, b, $a^b < n$ provided $a, b < n$.*

Proof. Let (A, X) be a reaching couple for F. Choose i such that $\alpha_i > FX$ and either $i = 0$ or i is a limit ordinal. Put $n = \alpha_i$. Now, c) and a) follow by 3,1 and 1,11; b) is obvious.

Lemma 3.3. *Assume the GCH. Let*

$$n = \aleph_{\alpha+\omega_0}.$$

Let $F : \mathbf{S}_n \to \mathbf{S}_n$ be a functor such that $F \circ Q_m \sim Q_m \circ F$ for each $m < n$. Then F is small.

Proof. For any natural k such that $F2 \leqq \aleph_{\alpha+k}$ we have

$$F(\aleph_{\alpha+k+1}) \simeq F(2^{\aleph_{\alpha+k}}) \simeq (F2)^{\aleph_{\alpha+k}} \simeq 2^{\aleph_{\alpha+k}} \simeq \aleph_{\alpha+k+1}$$

and so $\aleph_{\alpha+k+1} \notin \mathscr{A}_F$ by 1,6. Hence $\sup \mathscr{A}_F < n$ and F is small by 1,10.

Lemma 3.4. *Assume the GCH. Let F : **Set** $\to$ **Set** be a functor such that $F \circ Q_m \sim \sim Q_m \circ F$ for any m. Then* a), b), c) *of* 3,2 *take place for every $n = \aleph_{\alpha+\omega_0}$, where $\aleph_\alpha \geqq F\,2$.*

Proof. a) follows by 1,10, b) by a) and 3,3, c) by the GCH.

Theorem 3.5. *Let F : **Set** $\to$ **Set** be a small functor such that $F \circ Q_m \sim Q_m \circ F$ for every m. Then $F \sim Q_n$ for some n.*

Proof. See 1,7, 2,13 and 3,2.

Proposition 3.6. *Assume the GCH. Let*

$$n = \aleph_{\alpha+\omega_0},$$

arbitrary. Let $F : \mathbf{S}_n \to \mathbf{S}_n$ be a functor such that $F \circ Q_m \sim Q_m \circ F$ for every $m < n$. Then $F \sim Q_r$ for some r.

Proof. See 2,13 and 3,3.

Theorem 3.7. *Assume the GCH. Let F : **Set** $\to$ **Set** be a functor such that $F \circ Q_m \sim \sim Q_m \circ F$ for every m. Then $F \sim Q_r$ for some r.*

Proof. According to 3,4 and 3,6, for every α with $\aleph_\alpha \geqq F\,2$ the restriction $\bar{F}$ of F to $\mathbf{S}_n$, where

$$n = \aleph_{\alpha+\omega_0},$$

is naturally equivalent to some Q_r restricted to $\mathbf{S}_n$. The cardinal r does not depend on α, since it is uniquely determined by

$$2^r \simeq Q_r\,2 \simeq F\,2\,.$$

Thus, $r = \sup \mathscr{A}_F$ and our theorem follows by 1,7.

Remark. The above theorem can be proved under a little weaker set-theoretical assumptions than the GCH, viz: There is a proper class of cardinals α such that $\alpha^+ = 2^\alpha$ and $\alpha^{++} = 2^{\alpha^+}$ (+ denotes the follower).

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