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PARTITION OF NONDENumerable CLOSED SETS OF REALS

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In what follows every set mentioned is a subset of the set of all real numbers. Moreover, every item pertaining to measure is in the sense of Lebesgue. As usual, $m(S)$, $m^*(S)$ and $m_*(S)$ denote respectively the *measure*, the *outer* and the *inner measures* of the set S .

Lemma 1. *Let C be a closed set. Let B be a subset of C such that B has a nonempty intersection with every closed subset of C of positive measure. Then*

$$(1) \quad m^*(B) = m(C).$$

Proof. Assume on the contrary that $m^*(B) < m(C)$. But then there exists a covering H of B by pairwise disjoint open intervals such that $C - \bigcup H$ is a closed subset of C of positive measure. Clearly, B has no point in common with $C - \bigcup H$ which is a contradiction. Thus, (1) is established.

As usual, we identify every cardinal k with the set of all ordinals preceding k . Thus, k is a well ordered set and k is greater than the cardinality of every initial segment of k . Moreover, if $\bar{S} = k$ then S is well ordered by virtue of the equipollence between S and k . Based on this, we prove:

Lemma 2. *Let n be a cardinal and c be an infinite cardinal such that*

$$(2) \quad n \leq c.$$

Let $(A_i)_{i < n}$ be a (not necessarily disjoint) family of sets A_i such that

$$(3) \quad \bar{A}_i = c \quad \text{for every } i < n.$$

Then there exists a family $(a_i)_{i < n}$ of pairwise distinct real numbers a_i such that

$$(4) \quad a_i \in A_i \quad \text{for every } i < n.$$

Proof. Clearly, every A_i is well ordered by virtue of (3). We assert the existence of the family $(a_i)_{i < n}$ based on transfinite induction given by:

$$(5) \quad a_i = \text{the first element of } A_i - \bigcup_{j < i} \{a_j\} \text{ for every } i < n.$$

The above definition is justified since by (2), we see that $i < n$ implies $i < c$ and therefore $c - i = c$, which by (3), implies that $A_i - \bigcup_{j < i} \{a_j\}$, in (5), is nonempty. But then clearly (5) implies (4), as desired.

Remark. In what follows we let c denote the cardinality of the continuum (i.e., the set of all real numbers). We recall that every closed set P of positive measure (or for that matter every nondenumerable closed set) is of cardinality c . Moreover, the family of all the closed subsets of P of positive measure is also of cardinality c . Based on this, we prove:

Theorem 1. *Let P be a closed set of positive measure. Let c be the cardinal of the continuum and let k be any positive cardinal such that $k \leq c$. Then P is a disjoint union of k -many subsets B_j of P such that*

$$(6) \quad m^*(B_j) = m(P) \text{ for every } j < k.^1$$

Proof. Since c is infinite and $k \leq c$, we see that

$$(7) \quad kc = c.$$

In view of the Remark, we let $(P_i)_{i < c}$ denote the family of all the closed subsets P_i of P of positive measure. Again, in view of the Remark, we have $\bar{P}_i = c$ for every $i < c$ which, by (7) implies that every P_i is a disjoint union of k -many subsets A_{ij} such that

$$(8) \quad A_{ij} \subseteq P_i \text{ and } \bar{A}_{ij} = c \text{ for every } i < c \text{ and } j < k.$$

Let us consider the family A given by

$$(9) \quad A = \{A_{ij} \mid i < c \text{ and } j < k\}.$$

From (7) it follows that $kc \leq c$ and therefore, from (9) and (8), by Lemma 2 we see that there exists a family $(a_{ij})_{i < c}$ with $j < k$ of pairwise distinct real numbers a_{ij}

¹ The results presented strengthens some former results of Professor W. SIERPINSKI (*L equivalence par decomposition finite et la mesure extérieure des ensembles*, Fund. Math. XXXVII (1950), 209–212). In this paper Sierpinski proved for example the following assertion: *If $\aleph_1 = 2^{\aleph_0}$ and $E \subset R_m$ has positive measure, n is positive integer, then $E = \bigcup_{j=1}^n E_j$ (disjoint union) and the outer measure of each of the sets E_j is equal to the measure of the set E .* (The reviewer's remark.)

such that

$$(10) \quad a_{ij} \in A_{ij} \text{ for every } i < c \text{ and } j < k.$$

Let

$$(11) \quad B_0 = \{a_{i0} \mid i < c\} \cup (P - \{a_{ij} \mid i < c \text{ and } j < k\})$$

and

$$(12) \quad B_j = \{a_{ij} \mid i < c\} \text{ with } 0 < j < k.$$

From (11) and (12) we see that $(B_j)_{j < k}$ is a family of pairwise disjoint subsets B_j of P such that

$$(13) \quad P = \bigcup_{j < k} B_j.$$

Moreover, from (10) and (8), it follows that for every $j < k$ it is the case that B_j has a nonempty intersection with every closed subset P_i of P of positive measure. Hence, from Lemma 1 it follows that

$$(14) \quad m^*(B_j) = m(P) \text{ for every } j < k.$$

Thus, from (13) and (14) it follows that P is a disjoint union of k -many subsets B_j of P satisfying (6). Hence the Theorem is proved.

Corollary. *Let P be a closed set of positive measure. Let c be the cardinal of the continuum and k any cardinal such that $2 \leq k \leq c$. Then P is a disjoint union of k -many nonmeasurable subsets B_j of P such that*

$$(15) \quad m^*(B_j) = m(P) \text{ and } m_*(B_j) = 0 \text{ for every } j < k.$$

Proof. In view of the hypothesis of the Corollary, from Theorem 1 it follows that P is a disjoint union of k -many subsets B_j of P satisfying (6). On the other hand, since $k \geq 2$ we see that for every $j < k$ there exists $i < k$ such that $j \neq i$ and $B_i \subseteq (P - B_j)$ with

$$m^*(B_j) = m^*(P - B_j) = m(P)$$

which implies (15) and the nonmeasurability of B_j for every $j < k$.

Thus the Corollary is proved.

We observe that if C is a closed set of positive measure $m(C)$ then for every nonnegative extended real number r (i.e., $0 \leq r \leq +\infty$) such that $r \leq m(C)$ there exists a closed subset P of C such that $m(P) = r$.

Based on the above observation we prove:

Theorem 2. *Let C be a nondenumerable closed set. Let r be a nonnegative extended real number such that $r \leq m(C)$. Then C is a disjointed union of continuumly many subsets C_j of C such that $m^*(C_j) = r$.*

Proof. If $r = 0$ then the conclusion of the Theorem follows immediately since C is a disjoint union of (see the Remark) continuumly many of its singletons. Next, let $0 < r \leq m(C)$. Thus, C is a closed set of positive measure and we let (in view of the above observation) P be a closed subset of C such that $m(P) = r$. Let c be the cardinal of the continuum then since $c \leq c$, from Theorem 1 it follows that P is the union of a family $(B_j)_{j < c}$ of continuumly many pairwise disjoint subsets B_j of P such that $m^*(B_j) = m(P) = r$. Clearly, $\overline{C - P} = e \leq c$ and therefore $C - P$ is equal to the family $(b_j)_{j < e}$ of pairwise distinct real numbers b_j . But then letting

$$C_j = B_j \cup \{b_j\} \text{ if } j < e \text{ and } C_j = B_j \text{ if } e \leq j < c$$

we see that the above C_j 's satisfy the conclusion of the Theorem. Thus, the Theorem is proved.

For related results see the reference below.

Reference

Oxtoby, J. C.: Measure and Category, Springer-Verlag (1970), p. 79.

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