Kar-Ping Shum; Patrick N. Stewart
Completely prime ideals and idempotents in mobs


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A subset $I$ of a semigroup $S$ is an *ideal* if $SI \subseteq I$ and $IS \subseteq I$. An ideal of a semigroup $S$ is (i) *compressed* if for all positive integers $n$, $x_1^2 \ldots x_n^2 \in I$ implies $x_1 \ldots x_n \in I$, (ii) *completely semiprime* if $x^2 \in I$ implies $x \in I$, and (iii) *completely prime* if $xy \in I$ implies $x \in I$ or $y \in I$. Let $S$ be a topological semigroup, an ideal $I$ of $S$ is *topologically semiprime* if $x \notin I$ implies $\overline{\{x, x^2, x^3, \ldots\}} \cap I = \emptyset$ where $\overline{\{x\}}$ is the closure of $\{x, x^2, x^3, \ldots\}$. Adams [1] and Cornish [2] have shown that every completely semiprime ideal in a semigroup is the intersection of completely prime ideals. Iseki [6] proved that every compressed ideal of a semigroup is the intersection of completely prime ideals, and Shum [11] proved that an ideal in a semigroup is compressed if and only if it is completely semiprime. So another proof of the Adams-Cornish theorem can be obtained from the work of Iseki and Shum. Shum [11] has also proved that every open completely semiprime ideal in a *compact mob* (a compact Hausdorff topological semigroup) is the intersection of open completely prime ideals. In this paper we study ideals which are the intersection of open completely prime ideals and strengthen the above result of Shum.

**Definition 1.** Let $Q$ be a subset of a semigroup $S$.

(i) $E = \text{the set of idempotents of } S$,

(ii) $E(Q) = E \setminus Q$,

(iii) an idempotent $e \in S$ is $Q$-**primitive** if $e \notin Q$ and $e$ is the only idempotent in $eS \setminus Q$,

(iv) $E(Q) = \text{the set of } Q\text{-primitive idempotents}$,

(v) $J_0(Q) = \text{the largest ideal contained in } Q$,

(vi) when $Q$ is a completely semiprime ideal of $S$ and $x \in S$, $\text{tod}_Q x$ denotes the ideal $\{s \in S : sx \in Q\} = \{s \in S : xs \in Q\}$.

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Lemma 2. Let $S$ be a compact mob and $Q$ an ideal of $S$.

(i) If $Q$ is topologically semiprime and $aSb \not\subseteq Q$, then $aSb \cap E(Q) \neq \emptyset$.
(ii) If $Q$ is open completely semiprime and $aSb \not\subseteq Q$, then $aSb \cap \overline{E(Q)} \neq \emptyset$.
(iii) If $Q$ is completely semiprime and $x \in S$, then $\text{tod}_Q x$ is a completely semiprime ideal.
(iv) If $Q$ is topologically semiprime and $e \in \overline{E(Q)}$, then $\text{tod}_Q e = J_0(S\{e\})$ and this ideal is open completely prime.

Proof. (i) For each $x \in aSb \setminus Q$, there is an idempotent $e \in \Gamma(x) \subseteq aSb \setminus Q$.

(ii) Since open completely semiprime ideals are topologically semiprime there is an $e \in aSb \cap E(Q)$. The set $eS\setminus Q$ is a closed subset of $aSb \setminus Q$ and so, because $S$ is compact, we may apply Zorn’s Lemma to obtain $e^*S e^*$ minimal in $\{fSf : f \in E(Q)\}$ and $fSf \subseteq eSe$. Suppose $g = g^2 \in e^*S e^*$. By the minimality of $e^*S e^*$ either $gSg = e^*S e^*$ (and hence $g = e^*$) or $g \in Q$. Thus $e^* \in aSb \cap \overline{E(Q)}$.

(iii) This is 2.1(i) in [11].

(iv) Clearly $\text{tod}_Q e \subseteq J_0(S\{e\})$. Suppose that $\text{tod}_Q e \subseteq I$ where $I$ is an ideal of $S$. Let $x \in I \setminus \text{tod}_Q e$. Then $ex \notin \text{tod}_Q e$. Since a topologically semiprime ideal is completely semiprime we may apply (iii) to see that $ex e \notin \text{tod}_Q e$. Thus $exexe \notin Q$. By (i) there is an $f \in exS e \cap E(Q)$. Since $e \in \overline{E(Q)}$, $f = e$ and so $e \in exS e \subseteq I$. Thus $\text{tod}_Q e = J_0(S\{e\})$. We now verify that $\text{tod}_Q e$ is completely prime. Suppose that $abe \in Q$ and $be \notin Q$. Because $Q$ is completely semiprime $eSeb \not\subseteq Q$ and so by (i) there is an idempotent $f \in eSeb \setminus Q$. Because $e \in \overline{E(Q)}$, $f = e$ and so $e = exbe$ for some $x \in S$. Since $Q$ is completely semiprime it follows from $abe \in Q$ that $axbe \in Q$. Thus $ae \in Q$ and so $\text{tod}_Q e$ is completely prime. Finally, $J_0(S\{e\})$ is open by Lemma 1 in [5].

We note that (iv) above generalizes Theorem 2.8 in [11].

Although every open completely semiprime ideal is clearly topologically semiprime the converse is not true. This, and that it is not sufficient to assume that $Q$ is topologically semiprime in (ii) above, is illustrated by the following example.

Example 3. Let $S$ be the minthread; that is, $S = [0, 1]$ with the usual topology and multiplication $\ast$ defined by $x \ast y = \min \{x, y\}$. Then $\{0\}$ is a topologically semiprime ideal which is not open and $\overline{E(\{0\})} = \emptyset$.

Theorem 4. Let $S$ be a compact mob and $Q$ a topologically semiprime ideal of $S$. For each $e \in E(Q)$, $J_0(S\{e\})$ is an open prime ideal which contains a completely prime ideal $P_e \subseteq Q$. Moreover,

$$Q = \bigcap\{P_e : e \in E(Q)\} = \bigcap\{\text{tod}_Q e : e \in E(Q)\} = \bigcap\{J_0(S\{e\}) : e \in E(Q)\} = J_0(S|E(Q)).$$

Proof. It is shown in [7] that $J_0(S\{e\})$ is an open prime ideal for each $e \in E(Q)$. If $Q$ is proper, then $E(Q) \neq \emptyset$ and $Q = \bigcap\{J_0(S\{e\}) : e \in E(Q)\}$. Clearly $\bigcap\{J_0(S\{e\}) : e \in E(Q)\} = J_0(S|E(Q))$. We note that (iv) above generalizes Theorem 2.8 in [11].

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Corollary 5. (Shum and Hoo [10]). If $S$ is a compact abelian mob with 0 and $N = \{x \in S : 0 \in \Gamma(x)\}$, then $N = J_0(S^\wedge E(\{0\})) = \bigcap \{J_0(S^\wedge e) \mid e \in E(\{0\})\}$.

Corollary 6. (Shum [12]). If $I$ is an open ideal in a compact abelian mob $S$, then $A(I) = \{x \in S : x^k \in I \text{ for some integer } k \geq 1\}$ is an open ideal of $S$.

Proof. Clearly $A(I)$ is a topologically semiprime ideal of $S$, so by the theorem $A(I) = J_0(S^\wedge E(A(I)))$. Also, $E(A(I)) = E(I) = E \cap (S^\wedge I)$ is closed and so $A(I)$ is open by [5].

We now consider how the conclusion of Theorem 4 can be strengthened when assume that the ideal $Q$ is not only topologically semiprime but also open.

Proposition 7. Let $S$ be a compact mob, $Q$ an open completely semiprime ideal of $S$ and $e \in E(Q)$.

(i) (Koch [4]). The following are equivalent:
   (a) $e \in E(Q)$,
   (b) $eS^\wedge Q$ is a group,
   (c) $S^\wedge e$ is a closed left ideal minimum with respect to not being contained in $Q$,
   (d) $S^\wedge eS$ is a closed (two-sided) ideal minimum with respect to not being con­
   tained in $Q$,
   (e) every idempotent in $S^\wedge eS^\wedge Q$ is in $E(Q)$.

(ii) The following are equivalent:
   (a) $P$ is minimal in the set of prime ideals containing $Q$ and $P$ is open,
   (b) $P$ is minimal in the set of open prime ideals containing $Q$,
   (c) $P = J_0(S^\wedge e)$ for some $e \in E(Q)$,
   (d) $P = \text{tod}_Q e$ for some $e \in E(Q)$.

Proof. (a) $\rightarrow$ (b) is clear.

(b) $\rightarrow$ (c). From [7] we obtain an idempotent $f \in E(Q)$ such that $P = J_0(S^\wedge f)$. By Lemma 2(ii) there is an $e \in fS^\wedge \cap E(Q)$. Thus $ef = e$ and so $J_0(S^\wedge e) \subseteq J_0(S^\wedge f)$. Also, from [7] again, $J_0(S^\wedge e)$ is an open prime ideal. Therefore, by the minimality of $P$, $P = J_0(S^\wedge e)$.

(c) $\rightarrow$ (d). Lemma 2 (iv).
(d) → (a). By Lemma 2(iv) $\text{tod}_Q e$ is an open prime ideal containing $Q$. Suppose $Q \subseteq P \subseteq \text{tod}_Q e$ where $P$ is a prime ideal of $S$. Then $eS(\text{tod}_Q e) \subseteq Q \subseteq P$ and so, since $e \notin P$, $\text{tod}_Q e \subseteq P$. This establishes (a).

**Definition 8.** Let $Q$ be a subset of a semigroup $S$.

(i) If $x$ and $y$ are two distinct elements of $S$, then $x$ is $Q$-orthogonal to $y$ if $xy \in Q$.
(ii) A subset $X \subseteq S$ is $Q$-orthogonal if $xy \in Q$ for all $x, y \in X$, $x \neq y$.
(iii) A subset $X \subseteq S$ is $Q$-complete if for every $s \in S \setminus Q$ there is an $x \in X$ such that $xs \notin Q$.

Let $S$ be a compact mob and $Q$ an open completely semiprime ideal of $S$. If there is a $Q$-orthogonal subset of $\mathcal{E}(Q)$ then by Zorn’s Lemma we may choose a maximal $Q$-orthogonal subset $E_0 \subseteq \mathcal{E}(Q)$. Lemma 2(ii) implies that $E_0$ must also be $Q$-complete. So given any open completely semiprime ideal of a compact mob $S$ there are $Q$-complete $Q$-orthogonal subsets of $\mathcal{E}(Q)$. Notice that if $S$ is quasi normal (idempotents commute), then $\mathcal{E}(Q)$ is $Q$-orthogonal and $\mathcal{E}(Q)$ is the only $Q$-complete $Q$-orthogonal set of $Q$-primitive idempotents.

**Theorem 9.** Let $S$ be a compact mob, $Q$ an open completely semiprime ideal of $S$ and $E_0$ a $Q$-complete $Q$-orthogonal subset of $\mathcal{E}(Q)$. Then

$$Q = \bigcap \{\text{tod}_Q e : e \in E_0\} = \bigcap \{J_0(S\{e\}) : e \in E_0\} = J_0(S|E_0)$$

and these intersections are irredundant.

**Proof.** Clearly $Q \subseteq \bigcap \{\text{tod}_Q e : e \in E_0\}$, and since $E_0$ is $Q$-complete, $Q = \bigcap \{\text{tod}_Q e : e \in E_0\}$. By Lemma 2(iv), $\text{tod}_Q e = J_0(S\{e\})$ for all $e \in E_0$. Finally, the intersections are irredundant because $E_0$ is $Q$-orthogonal.

**Corollary 10** (Shum [11]). If $Q$ is an open completely semiprime ideal in a compact mob $S$, then $Q$ is an intersection of open completely prime ideals.

**Theorem 11.** Let $S$ be a compact mob and $I$ an ideal of $S$ which is an intersection of open completely prime ideals. Then $I$ is topologically semiprime,

$$I = \bigcap \{\text{tod}_Q e : e \in E(I)\} = \bigcap \{J_0(S\{e\}) : e \in E(I)\} = J_0(S|E(I))$$

and $I$ is open if and only if $E(I)$ is closed.

**Proof.** Suppose that $I = \bigcap \{P_\lambda : \lambda \in \Lambda\}$ where $P_\lambda$ is an open completely prime ideal for all $\lambda \in \Lambda$. If $x \notin I$, then $x \notin P_\lambda$ for some $\lambda \in \Lambda$ and so $I(x) \subseteq S\setminus P_\lambda \subseteq S\setminus I$. Thus $I$ is topologically semiprime and so by Theorem 4 $I = \bigcap \{\text{tod}_Q e : e \in E(I)\} = \bigcap \{J_0(S\{e\}) : e \in E(I)\} = J_0(S|E(I))$. Since $I = J_0(S|E(I))$, it follows from [5] that $I$ is open if $E(I)$ is closed; and if $I$ is open, $E(I) = E \cap (S\setminus I)$ is closed.

We have shown in Theorem 4 that every topologically semiprime ideal in a compact mob is the intersection of open prime ideals (but not conversely, see the example on
From the above theorem we see that an intersection of open completely prime ideals is topologically semiprime. This leads to the following.

Problem. Is every topologically semiprime ideal in a compact mob the intersection of open completely prime ideals?

Notice that the answer to this problem is "yes" if the prime ideals $J_0(S \backslash e)$ of our Theorem 4 are completely prime; for instance, if $S$ is normal ($xS = Sx$ for all $x \in S$).

The Frattini ideal of a semigroup $S$ is $\Phi(S) = \bigcap\{M : M$ is a maximal ideal of $S\}$. When $S$ has no maximal ideals $\Phi(S) = S$. If a maximal ideal $M$ in a compact mob is completely semiprime, then $M$ is completely prime (we can apply Corollary 10 because $M$ is open [5]) and $M = J_0(S \backslash e)$ for any $e \in \mathcal{E}(M)$ (this follows from Theorem 9).

Corollary 12. Let $S$ be a compact mob in which each maximal ideal is completely semiprime. Then $\Phi(S)$ is topologically semiprime,

$$\Phi(S) = \bigcap\{\text{tod}_{\Phi(S)} e : e \in E(\Phi(S))\} = \bigcap\{J_0(S \backslash e) : e \in E(\Phi(S))\} = J_0(S \backslash E(\Phi(S)))$$

and $\Phi(S)$ is open if and only if $E(\Phi(S))$ is closed.

A maximal ideal $M$ in a compact mob $S$ is completely semiprime if and only if $S \backslash M$ is a disjoint union of groups [3]. The case when $S \backslash M$ is actually a group is also considered in [3], where it is shown that in this case there is a unique idempotent $e \notin M$ and then, of course, $M = J_0(S \backslash e)$. We shall call an idempotent $e$ $g$-maximal if $S \backslash J_0(S \backslash e)$ is a group. Notice that if $e$ is a $g$-maximal idempotent, then $J_0(S \backslash e)$ must be a maximal ideal which is open and completely prime. Let $\Phi_g(S) = \bigcap\{J_0(S \backslash e) : e \text{ is a $g$-maximal idempotent}\}$. When $S$ has no $g$-maximal idempotents $\Phi_g(S) = S$.

By Theorem 11 $\Phi_g(S)$ is topologically semiprime.

Lemma 13. Let $S$ be a semigroup.

(i) Every idempotent in $E(\Phi_g(S))$ is $g$-maximal.

(ii) If $e$ is $g$-maximal and $e \neq f \in E(\Phi(S))$, then $ef \in \Phi(S)$.

(iii) $E(\Phi_g(S)) = E(\Phi_g(S))$.

Proof. (i) Let $f \in E(\Phi_g(S))$. Then there is a $g$-maximal idempotent $e$ such that $f \notin J_0(S \backslash e)$. Hence $f = e$ is $g$-maximal.

(ii) Let $e$ be a $g$-maximal idempotent and $e \neq f \in E(\Phi(S))$. Then there is a maximal ideal $M$ such that $f \in S \backslash M$ and $M \neq J_0(S \backslash e)$ because $f \neq e$. Thus $ef \in [S \backslash J_0(S \backslash e)] [S \backslash M]$ which is contained in $\Phi(S)$ by Theorem 2(e) in [9].
(iii) It is immediate from (ii) that every idempotent in \(E(\Phi(S))\) is \(\Phi(S)\)-primitive.

**Proposition 14.** Let \(S\) be a compact mob and \(Q\) a topologically semiprime ideal of \(S\) containing \(\Phi(S)\). Then \(E(Q) = \bar{E}(Q)\) is a \(Q\)-complete \(Q\)-orthogonal set, each idempotent in \(E(Q)\) is \(g\)-maximal, \(\text{tod}_Q e = J_0(S|e)\) is an open completely prime ideal for all \(e \in E(Q)\) and

\[
Q = \bigcap \{\text{tod}_Q e : e \in E(Q)\}
\]

is an irredundant intersection. Moreover, \(Q\) is open if and only if \(E(Q)\) is closed.

**Proof.** \(E(Q) \subseteq E(\Phi(S))\) and so it follows from the lemma 13(iii) that \(E(Q) = \bar{E}(Q)\) is a \(Q\)-orthogonal set of \(g\)-maximal idempotents. By Lemma 2(i), \(E(Q)\) is \(Q\)-complete. By Lemma 2(iv), \(\text{tod}_Q e = J_0(S|e)\) is an open completely prime ideal for all \(e \in E(Q)\). By Theorem 4, \(Q = \bigcap \{\text{tod}_Q e : e \in E(Q)\}\) and the intersection is irredundant because \(E(Q)\) is \(Q\)-orthogonal. The last statement of the proposition follows from Theorem 11.

We note that the ideal \(Q\) in the above proposition may not be open. In fact, the following example shows that \(\Phi(S)\) may not be open.

**Example 15.** Let \(S = \{0, e_1, e_2, \ldots\}\). Multiplication is commutative and satisfies \(0 \cdot e_i = 0\) and

\[
e_i \cdot e_j = \begin{cases} e_i & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}
\]

for all positive integers \(i, j\). The topology on \(S\) is defined by specifying that every subset of \(\{e_1, e_2, \ldots\}\) is open and also all subsets of the form \(\{0\} \cup C\) where \(\{e_1, e_2, \ldots\}\) \(\setminus\) \(C\) is finite. Then each \(e_i\) is \(g\)-maximal and \(\Phi(S) = \{0\}\).

From Lemma 13(ii) we see that every \(g\)-maximal idempotent is \(\Phi(S)\)-orthogonal to every other idempotent in \(S\). When \(S\) is a compact mob in which all maximal ideals that are prime are also completely prime, the converse is true as well (this is the case when conditions (R) and (L) of [7] are satisfied).

**Proposition 16.** Let \(S\) be a compact mob in which all maximal ideals that are prime are also completely prime. Then an idempotent \(e\) is \(g\)-maximal if and only if \(e \notin \Phi(S)\) and \(e\) is \(\Phi(S)\)-orthogonal to every idempotent in \(S\).

**Proof.** Suppose that \(e = e^2 \notin \Phi(S)\) and \(e\) is \(\Phi(S)\)-orthogonal to every other idempotent in \(S\). Then \(J_0(S|e)\) is a prime ideal [7], and so \(J_0(S|e)\) is a maximal ideal [9]. Now by assumption \(J_0(S|e)\) is completely prime. Since \(e\) is \(\Phi(S)\)-orthogonal to every other idempotent in \(S\) it follows that \(e\) is the only idempotent in \(S \setminus J_0(S|e)\). Thus \(e\) is \(g\)-maximal [3]. The converse follows from Lemma 13(ii).

In our final theorem we consider a case when the intersections in Theorem 9 are finite.
Theorem 17. Let $S$ be a compact mob and $Q$ an open completely semiprime ideal.

(i) If $Q$ is closed, then every $Q$-complete $Q$-orthogonal subset of $E(Q)$ is finite.

(ii) The ideal $Q$ is closed if and only if there is a finite collection $\{P_i : i = 1, \ldots, n\}$ of open and closed completely prime ideals such that $Q = \bigcap_{i=1}^n P_i$.

Proof. (i) Let $E_0$ be a $Q$-complete $Q$-orthogonal subset of $E(Q)$. By Theorem 9 $Q = \bigcap \{\text{tod}_Q e : e \in E_0\}$. Arguing as in Proposition 1.8 of [10] we see that because $Q$ is closed, $\text{tod}_Q e$ is closed for all $e \in E_0$. Now since $S$ is compact and $Q$ is open, there is a finite subset $E' \subseteq E_0$ such that $Q = \bigcap \{\text{tod}_Q e : e \in E'\}$. Because $E_0$ is $Q$-orthogonal we must have $E_0 = E'$ and so $E_0$ is finite.

(ii) Assume that $Q$ is closed. By the remarks preceding Theorem 9 at least one $Q$-complete $Q$-orthogonal subset $E_0$ of $E(Q)$ exists and by (i) $E_0$ is finite. Thus $\{\text{tod}_Q e : e \in E_0\}$ is a finite collection of closed and open completely prime ideals and $Q = \bigcap \{\text{tod}_Q e : e \in E_0\}$.

The converse is clear.

References


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