Kar-Ping Shum; Patrick N. Stewart
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COMPLETELY PRIME IDEALS AND IDEMPOTENTS IN MOBS

K. P. SHUM, Hong Kong and P. N. STEWART,[1]) Halifax

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A subset I of a semigroup S is an *ideal* if $SI \subseteq I$ and $IS \subseteq I$. An ideal of a semigroup S is (i) *compressed* if for all positive integers n, $x_1^2 \ldots x_n^2 \in I$ implies $x_1 \ldots x_n \in \in I$, (ii) *completely semiprime* if $x^2 \in I$ implies $x \in I$, and (iii) *completely prime* if $xy \in I$ implies $x \in I$ or $y \in I$. Let S be a topological semigroup, an ideal I of S is *topologically semiprime* if $x \notin I$ implies $\Gamma(x) \cap I = \emptyset$ where $\Gamma(x)$ = the closure of $\{x, x^2, x^3, \ldots\}$. ADAMS [1] and CORNISH [2] have shown that every completely semiprime ideal in a semigroup is the intersection of completely prime ideals. ISEKI [6] proved that every compressed ideal of a semigroup is the intersection of completely prime ideals, and SHUM [11] proved that an ideal in a semigroup is compressed if and only if it is completely semiprime. So another proof of the Adams-Cornish theorem can be obtained from the work of Iseki and Shum. Shum [11] has also proved that every open completely semiprime ideal in a *compact mob* (a compact Hausdorff topological semigroup) is the intersection of open completely prime ideals. In this paper we study ideals which are the intersection of open completely prime ideals and strengthen the above result of Shum.

Definition 1. Let Q be a subset of a semigroup S.

(i) E = the set of idempotents of S,

(ii) $E(Q) = E \backslash Q$,

(iii) an idempotent $e \in S$ is *Q-primitive* if $e \notin Q$ and e is the only idempotent in $eSe \backslash Q$,

(iv) $\tilde{E}(Q)$ = the set of Q-primitive idempotents,

(v) $J_0(Q)$ = the largest ideal contained in Q,

(vi) when Q is a completely semiprime ideal of S and $x \in S$, $\mathrm{tod}_Q\, x$ denotes the ideal $\{s \in S : sx \in Q\} = \{s \in S : xs \in Q\}$.

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Lemma 2. *Let S be a compact mob and Q an ideal of S.*

(i) *If Q is topologically semiprime and $aSb \nsubseteq Q$, then $aSb \cap E(Q) \neq \emptyset$.*
(ii) *If Q is open completely semiprime and $aSb \nsubseteq Q$, then $aSb \cap \tilde{E}(Q) \neq \emptyset$.*
(iii) *If Q is completely semiprime and $x \in S$, then $\operatorname{tod}_Q x$ is a completely semiprime ideal.*
(iv) *If Q is topologically semiprime and $e \in \tilde{E}(Q)$, then $\operatorname{tod}_Q e = J_0(S\backslash e)$ and this ideal is open completely prime.*

Proof. (i) For each $x \in aSb\backslash Q$, there is an idempotent $e \in \Gamma(x) \subseteq aSb\backslash Q$.

(ii) Since open completely semiprime ideals are topologically semiprime there is an $e \in aSb \cap E(Q)$. The set $eSe\backslash Q$ is a closed subset of $aSb\backslash Q$ and so, because S is compact, we may apply Zorn's Lemma to obtain e^*Se^* minimal in $\{fSf : f \in E(Q)$ and $fSf \subseteq eSe\}$. Suppose $g = g^2 \in e^*Se^*$. By the minimality of e^*Se^* either $gSg = e^*Se^*$ (and hence $g = e^*$) or $g \in Q$. Thus $e^* \in aSb \cap \tilde{E}(Q)$.

(iii) This is 2.1(i) in [11].

(iv) Clearly $\operatorname{tod}_Q e \subseteq J_0(S\backslash e)$. Suppose that $\operatorname{tod}_Q e \subsetneqq I$ where I is an ideal of S. Let $x \in I\backslash\operatorname{tod}_Q e$. Then $ex \notin \operatorname{tod}_Q e$. Since a topologically semiprime ideal is completely semiprime we may apply (iii) to see that $exex \notin \operatorname{tod}_Q e$. Thus $exexe \notin Q$. By (i) there is an $f \in exSxe \cap E(Q)$. Since $e \in \tilde{E}(Q)$, $f = e$ and so $e \in exSxe \subseteq I$. Thus $\operatorname{tod}_Q e = J_0(S\backslash e)$. We now verify that $\operatorname{tod}_Q e$ is completely prime. Suppose that $abe \in Q$ and $be \notin Q$. Because Q is completely semiprime $eSbe \nsubseteq Q$ and so by (i) there is an idempotent $f \in eSbe\backslash Q$. Because $e \in \tilde{E}(Q)$, $f = e$ and so $e = exbe$ for some $x \in S$. Since Q is completely semiprime it follows from $abe \in Q$ that $aexbe \in Q$. Thus $ae \in Q$ and so $\operatorname{tod}_Q e$ is completely prime. Finally, $J_0(S\backslash e)$ is open by Lemma 1 in [5].

We note that (iv) above generalizes Theorem 2.8 in [11].

Although every open completely semiprime ideal is clearly topologically semiprime the converse is not true. This, and that it is not sufficient to assume that Q is topologically semiprime in (ii) above, is illustrated by the following example.

Example 3. Let S be the minthread; that is, $S = [0, 1]$ with the usual topology and multiplication $*$ defined by $x * y = \min\{x, y\}$. Then $\{0\}$ is a topologically semiprime ideal which is not open and $\tilde{E}(\{0\}) = \emptyset$.

Theorem 4. *Let S be a compact mob and Q a topologically semiprime ideal of S. For each $e \in E(Q)$, $J_0(S\backslash e)$ is an open prime ideal which contains a completely prime ideal $P_e \supseteq Q$. Moreover,*

$$\begin{aligned} Q &= \bigcap\{P_e : e \in E(Q)\} = \bigcap\{\operatorname{tod}_Q e : e \in E(Q)\} \\ &= \bigcap\{J_0(S\backslash e) : e \in E(Q)\} = J_0(S\backslash E(Q)) . \end{aligned}$$

Proof. It is shown in [7] that $J_0(S\backslash e)$ is an open prime ideal for each $e \in E(Q)$. If Q is proper, then $E(Q) \neq \emptyset$ and $Q = \bigcap\{J_0(S\backslash e) : e \in E(Q)\}$. Clearly $\bigcap\{J_0(S\backslash e) :$

$: e \in E(Q)\} = J_0(S \backslash E(Q))$. Since $Q \subseteq \mathrm{tod}_Q\, e \subseteq J_0(S \backslash e)$ for all $e \in E(Q)$, $Q = \bigcap\{\mathrm{tod}_Q\, e : e \in E(Q)\}$. The proof will be complete if we can show that for each $e \in E(Q)$, $J_0(S \backslash e)$ contains a completely prime ideal $P_e \supseteq Q$. Let $e \in E(Q)$. By Zorn's Lemma we may choose M maximal in $\{I : I$ is a completely semiprime ideal and $Q \subseteq I \subseteq J_0(S \backslash e)\}$. Since $e \notin \mathrm{tod}_M\, e$ it follows from Lemma 2(iii) that $M = \mathrm{tod}_M\, e$. Suppose $ab \in M$. If $\mathrm{tod}_M\, b \subseteq J_0(S \backslash e)$, then applying Lemma 2(iii) again we see that $M = \mathrm{tod}_M\, b$ and so $a \in M$. If $\mathrm{tod}_M\, b \nsubseteq J_0(S \backslash e)$, then $e \in \mathrm{tod}_M\, b$ and so $b \in \mathrm{tod}_M\, e = M$. Thus $a \in M$ or $b \in M$ and so M is completely prime.

Corollary 5. (Shum and Hoo [10]). *If S is a compact abelian mob with* 0 *and $N = \{x \in S : 0 \in \Gamma(x)\}$, then $N = J_0(S \backslash E(\{0\})) = \bigcap\{J_0(S \backslash e) \mid e \in E(\{0\})\}$.*

Corollary 6. (Shum [12]). *If I is an open ideal in a compact abelian mob S, then $\mathscr{R}(I) = \{x \in S : x^k \in I$ for some integer $k \geqq 1\}$ is an open ideal of S.*

Proof. Clearly $\mathscr{R}(I)$ is a topologically semiprime ideal of S, so by the theorem $\mathscr{R}(I) = J_0(S \backslash E(\mathscr{R}(I)))$. Also, $E(\mathscr{R}(I)) = E(I) = E \cap (S \backslash I)$ is closed and so $\mathscr{R}(I)$ is open by [5].

We now consider how the conclusion of Theorem 4 can be strengthened when assume that the ideal Q is not only topologically semiprime but also open.

Proposition 7. *Let S be a compact mob, Q an open completely semiprime ideal of S and $e \in E(Q)$.*

(i) (Koch [4]). *The following are equivalent:*
  (a) $e \in \tilde{E}(Q)$,
  (b) *$eSe \backslash Q$ is a group,*
  (c) *Se is a closed left ideal minimum with respect to not being contained in Q,*
  (d) *SeS is a closed (two-sided) ideal minimum with respect to not being contained in Q,*
  (e) *every idempotent in $SeS \backslash Q$ is in $\tilde{E}(Q)$.*

(ii) *The following are equivalent:*
  (a) *P is minimal in the set of prime ideals containing Q and P is open,*
  (b) *P is minimal in the set of open prime ideals containing Q,*
  (c) *$P = J_0(S \backslash e)$ for some $e \in \tilde{E}(Q)$,*
  (d) *$P = \mathrm{tod}_Q\, e$ for some $e \in \tilde{E}(Q)$.*

Proof. (a) $\to$ (b) is clear.

(b) $\to$ (*c*). From [7] we obtain an idempotent $f \in E(Q)$ such that $P = J_0(S \backslash f)$. By Lemma 2(ii) there is an $e \in fSf \cap \tilde{E}(Q)$. Thus $ef = e$ and so $J_0(S \backslash e) \subseteq J_0(S \backslash f)$. Also, from [7] again, $J_0(S \backslash e)$ is an open prime ideal. Therefore, by the minimality of P, $P = J_0(S \backslash e)$.

(c) $\to$ (*d*). Lemma 2 (iv).

(d) $\to$ (a). By Lemma 2(iv) $\mathrm{tod}_Q\, e$ is an open prime ideal containing Q. Suppose $Q \subseteq P \subseteq \mathrm{tod}_Q\, e$ where P is a prime ideal of S. Then $eS(\mathrm{tod}_Q\, e) \subseteq Q \subseteq P$ and so, since $e \notin P$, $\mathrm{tod}_Q\, e \subseteq P$. This establishes (a).

Definition 8. Let Q be a subset of a semigroup S.

(i) If x and y are two distinct elements of S, then x is *Q-orthogonal* to y if $xy \in Q$.
(ii) A subset $X \subseteq S$ is *Q-orthogonal* if $xy \in Q$ for all $x, y \in X$, $x \neq y$.
(iii) A subset $X \subseteq S$ is *Q-complete* if for every $s \in S \backslash Q$ there is an $x \in X$ such that $xs \notin Q$.

Let S be a compact mob and Q an open completely semiprime ideal of S. If there is a Q-orthogonal subset of $\tilde{E}(Q)$ then by Zorn's Lemma we may choose a maximal Q-orthogonal subset $E_0 \subseteq \tilde{E}(Q)$. Lemma 2(ii) implies that E_0 must also be Q-complete. So given any open completely semiprime ideal of a compact mob S there are Q-complete Q-orthogonal subsets of $\tilde{E}(Q)$. Notice that if S is *quasi normal* (idempotents commute), then $\tilde{E}(Q)$ is Q-orthogonal and $\tilde{E}(Q)$ is the only Q-complete Q-orthogonal set of Q-primitive idempotents.

Theorem 9. *Let S be a compact mob, Q an open completely semiprime ideal of S and E_0 a Q-complete Q-orthogonal subset of $\tilde{E}(Q)$. Then*

$$Q = \bigcap\{\mathrm{tod}_Q\, e : e \in E_0\} = \bigcap\{J_0(S\backslash e) : e \in E_0\} = J_0(S\backslash E_0)$$

and these intersections are irredundant.

Proof. Clearly $Q \subseteq \bigcap\{\mathrm{tod}_Q\, e : e \in E_0\}$, and since E_0 is Q-complete, $Q = \bigcap\{\mathrm{tod}_Q\, e : e \in E_0\}$. By Lemma 2(iv), $\mathrm{tod}_Q\, e = J_0(S\backslash e)$ for all $e \in E_0$. Finally, the intersections are irredundant because E_0 is Q-orthogonal.

Corollary 10 (Shum [11]). *If Q is an open completely semiprime ideal in a compact mob S, then Q is an intersection of open completely prime ideals.*

Theorem 11. *Let S be a compact mob and I an ideal of S which is an intersection of open completely prime ideals. Then I is topologically semiprime,*

$$I = \bigcap\{\mathrm{tod}_Q\, e : e \in E(I)\} = \bigcap\{J_0(S\backslash e) : e \in E(I)\} = J_0(S\backslash E(I))$$

and I is open if and only if $E(I)$ is closed.

Proof. Suppose that $I = \bigcap\{P_\lambda : \lambda \in \Lambda\}$ where P_λ is an open completely prime ideal for all $\lambda \in \Lambda$. If $x \notin I$, then $x \notin P_\lambda$ for some $\lambda \in \Lambda$ and so $\Gamma(x) \subseteq S\backslash P_\lambda \subseteq S\backslash I$. Thus I is topologically semiprime and so by Theorem 4 $I = \bigcap\{\mathrm{tod}_Q\, e : e \in E(I)\} = \bigcap\{J_0(S\backslash e) : e \in E(I)\} = J_0(S\backslash E(I))$. Since $I = J_0(S\backslash E(I))$, it follows from [5] that I is open if $E(I)$ is closed; and if I is open, $E(I) = E \cap (S\backslash I)$ is closed.

We have shown in Theorem 4 that every topologically semiprime ideal in a compact mob is the intersection of open prime ideals (but not conversely, see the example on page 51 in [8]). From the above theorem we see that an intersection of open *completely* prime ideals is topologically semiprime. This leads to the following.

Problem. *Is every topologically semiprime ideal in a compact mob the intersection of open completely prime ideals?*

Notice that the answer to this problem is "yes" if the prime ideals $J_0(S\backslash e)$ of our Theorem 4 are completely prime; for instance, if S is *normal* ($xS = Sx$ for all $x \in S$).

The *Frattini ideal* of a semigroup S is $\Phi(S) = \bigcap\{M : M$ is a maximal ideal of $S\}$. When S has no maximal ideals $\Phi(S) = S$. If a maximal ideal M in a compact mob is completely semiprime, then M is completely prime (we can apply Corollary 10 because M is open [5]) and $M = J_0(S\backslash e)$ for any $e \in \tilde{E}(M)$ (this follows from Theorem 9).

Corollary 12. *Let S be a compact mob in which each maximal ideal is completely semiprime. Then $\Phi(S)$ is topologically semiprime,*

$$\begin{aligned}\Phi(S) &= \bigcap\{\operatorname{tod}_{\Phi(S)} e : e \in E(\Phi(S))\}\\ &= \bigcap\{J_0(S\backslash e) : e \in E(\Phi(S))\} = J_0(S\backslash E(\Phi(S)))\end{aligned}$$

and $\Phi(S)$ is open if and only if $E(\Phi(S))$ is closed.

A maximal ideal M in a compact mob S is completely semiprime if and only if $S\backslash M$ is a disjoint union of groups [3]. The case when $S\backslash M$ is actually a group is also considered in [3], where it is shown that in this case there is a unique idempotent $e \notin M$ and then, of course, $M = J_0(S\backslash e)$. We shall call an idempotent e *g-maximal* if $S\backslash J_0(S\backslash e)$ is a group. Notice that if e is a g-maximal idempotent, then $J_0(S\backslash e)$ must be a maximal ideal which is open and completely prime. Let $\Phi_g(S) = \bigcap\{J_0(S\backslash e) : e$ is a g-maximal idempotent$\}$. When S has no g-maximal idempotents $\Phi_g(S) = S$. By Theorem 11 $\Phi_g(S)$ is topologically semiprime.

Lemma 13. *Let S be a semigroup.*

(i) *Every idempotent in $E(\Phi_g(S))$ is g-maximal.*
(ii) *If e is g-maximal and $e \neq f \in E(\Phi(S))$, then $ef \in \Phi(S)$.*
(iii) $E(\Phi_g(S)) = \tilde{E}(\Phi_g(S))$.

Proof. (i) Let $f \in E(\Phi_g(S))$. Then there is a g-maximal idempotent e such that $f \notin J_0(S\backslash e)$. Hence $f = e$ is g-maximal.

(ii) Let e be a g-maximal idempotent and $e \neq f \in E(\Phi(S))$. Then there is a maximal ideal M such that $f \in S\backslash M$ and $M \neq J_0(S\backslash e)$ because $f \neq e$. Thus

$$ef \in [S\backslash J_0(S\backslash e)]\,[S\backslash M]$$

which is contained in $\Phi(S)$ by Theorem 2(e) in [9].

(iii) It is immediate from (ii) that every idempotent in $E(\Phi_g(S))$ is $\Phi_g(S)$-primitive.

Proposition 14. *Let S be a compact mob and Q a topologically semiprime ideal of S containing $\Phi_g(S)$. Then $E(Q) = \tilde{E}(Q)$ is a Q-complete Q-orthogonal set, each idempotent in $E(Q)$ is g-maximal,* $\operatorname{tod}_Q e = J_0(S\backslash e)$ *is an open completely prime ideal for all $e \in E(Q)$ and*

$$Q = \bigcap\{\operatorname{tod}_Q e : e \in E(Q)\}$$

is an irredundant intersection. Moreover, Q is open if and only if $E(Q)$ is closed.

Proof. $E(Q) \subseteq E(\Phi_g(S))$ and so it follows from the lemma 13(iii) that $E(Q) = \tilde{E}(Q)$ is a Q-orthogonal set of g-maximal idempotents. By Lemma 2(i), $E(Q)$ is Q-complete. By Lemma 2(iv), $\operatorname{tod}_Q e = J_0(S\backslash e)$ is an open completely prime ideal for all $e \in \tilde{E}(Q)$. By Theorem 4, $Q = \bigcap\{\operatorname{tod}_Q e : e \in E(Q)\}$ and the intersection is irredundant because $E(Q)$ is Q-orthogonal. The last statement of the proposition follows from Theorem 11.

We note that the ideal Q in the above proposition may not be open. In fact, the following example shows that $\Phi_g(S)$ may not be open.

Example 15. Let $S = \{0. e_1, e_2, \ldots\}$. Multiplication is commutative and satisfies $0 \,.\, e_i = 0$ and

$$e_i \,.\, e_j = \begin{cases} e_i & \text{if} \quad j = i \\ 0 & \text{if} \quad j \neq i \end{cases}$$

for all positive integers i, j. The topology on S is defined by specifying that every subset of $\{e_1, e_2, \ldots\}$ is open and also all subsets of the form $\{0\} \cup C$ where $\{e_1, e_2, \ldots\}\backslash C$ is finite. Then each e_i is g-maximal and $\Phi_g(S) = \{0\}$.

From Lemma 13(ii) we see that every g-maximal idempotent is $\Phi(S)$-orthogonal to every other idempotent in S. When S is a compact mob in which all maximal ideals that are prime are also completely prime, the converse is true as well (this is the case when conditions (R) and (L) of [7] are satisfied).

Proposition 16. *Let S be a compact mob in which all maximal ideals that are prime are also completely prime. Then an idempotent e is g-maximal if and only if $e \notin \Phi(S)$ and e is $\Phi(S)$-orthogonal to every idempotent in S.*

Proof. Suppose that $e = e^2 \notin \Phi(S)$ and e is $\Phi(S)$-orthogonal to every other idempotent in S. Then $J_0(S\backslash e)$ is a prime ideal [7], and so $J_0(S\backslash e)$ is a maximal ideal [9]. Now by assumption $J_0(S\backslash e)$ is completely prime. Since e is $\Phi(S)$-orthogonal to every other idempotent in S it follows that e is the only idempotent in $S\backslash J_0(S\backslash e)$. Thus e is g-maximal [3]. The converse follows from Lemma 13(ii).

In our final theorem we consider a case when the intersections in Theorem 9 are finite.

Theorem 17. *Let S be a compact mob and Q an open completely semiprime ideal.*

(i) *If Q is closed, then every Q-complete Q-orthogonal subset of $\tilde{E}(Q)$ is finite.*

(ii) *The ideal Q is closed if and only if there is a finite collection $\{P_i : i = 1, \ldots, n\}$ of open and closed completely prime ideals such that $Q = \bigcap_{i=1}^{n} P_i$.*

Proof. (i) Let E_0 be a Q-complete Q-orthogonal subset of $\tilde{E}(Q)$. By Theorem 9 $Q = \bigcap\{\text{tod}_Q\, e : e \in E_0\}$. Arguing as in Proposition 1.8 of [10] we see that because Q is closed, $\text{tod}_Q\, e$ is closed for all $e \in E_0$. Now since S is compact and Q is open there is a finite subset $E' \subseteq E_0$ such that $Q = \bigcap\{\text{tod}_Q\, e : e \in E'\}$. Because E_0 is Q-orthogonal we must have $E_0 = E'$ and so E_0 is finite.

(ii) Assume that Q is closed. By the remarks preceding Theorem 9 at least one Q-complete Q-orthogonal subset E_0 of $\tilde{E}(Q)$ exists and by (i) E_0 is finite. Thus $\{\text{tod}_Q\, e : e \in E_0\}$ is a finite collection of closed and open completely prime ideals and $Q = \bigcap\{\text{tod}_Q\, e : e \in E_0\}$.

The converse is clear.

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Author's addresses: K. P. Shum, Department of Mathematics, Science Centre, The Chinese University of Hong Kong, Hong Kong; P. N. Stewart, Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada.