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ARCHIMEDEAN CLASSES IN AN ORDERED SEMIGROUP II

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The terminology and notation of our previous paper [7] are used throughout. In particular, we denote by S an ordered semigroup and by $\mathcal{C}$ the set of all archimedean classes of S.

The original purpose of this paper is to study the behavior of the set product $AB$ of two archimedean classes $A$ and $B$ such that $A \leq B$ and the $\delta$-class in $\mathcal{C}$ containing $A$ and $B$ is torsion-free. Thus in this note let $A, B \in \mathcal{C}$ satisfying these conditions. Moreover we assume $A < B$. Also let $T$ be the subset of $S$ consisting of all elements $x$ such that the archimedean class containing the element $x$ lies between $A$ and $B$. Then $T$ is a convex subsemigroup of $S$, which contains the subsemigroup of $S$ generated by $A$ and $B$.

In order to consider the behavior of $AB$, in this paper we shall construct an $o$-homomorphism of $T$ into the ordered additive group of real numbers such that its images are negative on $A$, are positive on $B$ and are zero on $T \setminus (A \cup B)$.

By [7] Theorem 3.5, we have the following

**Lemma 1.** $A$ is a negative torsion-free archimedean class and $B$ is a positive torsion-free archimedean class of $S$. Moreover the $\delta$-class $A \delta$ of $\mathcal{C}$ consists of just two elements $A$ and $B$.

Two elements $a$ and $b$ of $S$ are said to form an anomalous pair if either $a^n < b^{n+1}$ and $b^n < a^{n+1}$ or $a^n > b^{n+1}$ and $b^n > a^{n+1}$ for every natural number $n$.

**Lemma 2.** There exists an $o$-homomorphism $w_1$ of $A$ into the ordered additive semigroup of negative real numbers such that two elements of $S$ have the same image if and only if they form an anomalous pair. Also there exists an $o$-homomorphism $w_2$ of $B$ into the ordered additive semigroup of positive real numbers such that two elements of $S$ have the same image if and only if they form an anomalous pair.

**Proof.** The second assertion follows from [3] Theorem or [4] Theorem 1. Dually we have the first assertion.
Lemma 3. Let $a \in A$ and $b \in B$. Put

$\mathcal{L}(a,b) = \{ r \text{ positive real number; there exist natural numbers } p \text{ and } q \text{ such that } r \leq q/p \text{ and } a^p b^q \in A \}$;

$\mathcal{U}(a,b) = \{ r \text{ positive real number; there exist natural numbers } p \text{ and } q \text{ such that } q/p \leq r \text{ and } a^p b^q \in B \}$.

Then

1. $\mathcal{L}(a,b) \neq \emptyset$ and $\mathcal{U}(a,b) \neq \emptyset$;
2. if $r \in \mathcal{L}(a,b)$ and $p$ and $q$ are natural numbers such that $q/p \leq r$, then $a^p b^q \in A$;
3. if $r \in \mathcal{U}(a,b)$ and $p$ and $q$ are natural numbers such that $r \leq q/p$, then $a^p b^q \in B$;
4. if $r \in \mathcal{L}(a,b)$, then $r' \in \mathcal{L}(a,b)$ for every positive real number $r'$ such that $r' < r$;
5. if $r \in \mathcal{U}(a,b)$, then $r' \in \mathcal{U}(a,b)$ for every positive real number $r'$ such that $r < r'$;
6. $\mathcal{L}(a,b) \cap \mathcal{U}(a,b) = \emptyset$;
7. if $r \in \mathcal{L}(a,b)$ and $r' \in \mathcal{U}(a,b)$, then $r < r'$;
8. if $r$ is a positive real number such that $r \notin \mathcal{L}(a,b)$ and $p$ and $q$ are natural numbers such that $r < q/p$, then $a^p b^q \in B$;
9. if $r$ is a positive real number such that $r \notin \mathcal{U}(a,b)$ and $p$ and $q$ are natural numbers such that $q/p < r$, then $a^p b^q \in A$;
10. $\sup \mathcal{L}(a,b) = \inf \mathcal{U}(a,b)$. (This common positive real number is denoted by $r(a,b)$);
11. $\mathcal{L}(a,b)$ has no greatest element and $\mathcal{U}(a,b)$ has no least element;
12. for natural numbers $p$ and $q$, $a^p b^q \in A$ if and only if $q/p < r(a,b)$, and if and only if $q/p \in \mathcal{L}(a,b)$;
13. for natural numbers $p$ and $q$, $a^p b^q \in B$ if and only if $r(a,b) < q/p$, and if and only if $q/p \in \mathcal{U}(a,b)$;
14. for natural numbers $p$ and $q$, $a^p b^q \notin A \cup B$ if and only if $r(a,b)$ is a rational number and $r(a,b) = q/p$.

Proof. (1) follows from [7] Theorem 2.4.

(2) Suppose $r \in \mathcal{L}(a,b)$ and $q/p \leq r$. Then there exists a positive rational number $v/u$ such that $r \leq v/u$ and $a^v b^u \in A$. By [7] Lemma 2.3, we have $a^{uv} b^{vu} \in A$. Since $q/p \leq r \leq v/u$, we have $uq \leq vp$. Hence

$$a^{vp} b^{uq} = a^{vp - uq} (a^{uq} b^{vu}) \in A,$$

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where, if \( \nu p - uq = 0 \), we assume that \( a^{\nu p - uq} \) is the empty symbol. Hence, again by [7] Lemma 2.3, we have \( a^{p}b^{q} \in A \).

(3) can be proved in a similar way.

(4) Suppose \( r \in L(a, b) \) and \( r' < r \). We take a positive rational number \( q/p \) such that \( r' < q/p < r \). Then, by (2), we have \( a^{p}b^{q} \in A \) and so \( r' \in L(a, b) \).

(5) can be proved in a similar way.

(6) By way of contradiction, we assume there exists \( r \in L(a, b) \cap U(a, b) \). Then there exist positive rational numbers \( q/p \) and \( v/u \) such that \( v/u \leq r \leq q/p \), \( a^{p}b^{q} \in B \) and \( a^{p}b^{q} \in A \). But, by definition, \( q/p \in L(a, b) \) and, by (2), we have \( a^{p}b^{q} \in A \). This contradicts the fact that \( A < B \).

(7) follows immediately from (4) and (6).

(8) Suppose \( r \notin L(a, b) \) and \( r < q/p \). We take a positive rational number \( v/u \) such that \( r < v/u < q/p \). Then, by (4), we have \( v/u \notin L(a, b) \) and so \( a^{p}b^{q} \notin A \). Hence, by [7] Lemma 2.3, \( a^{p}b^{v} \notin A \). First we suppose \( a^{p}b^{v} \in B \). Then

\[
\quad \quad a^{p}b^{u} = (a^{p}b^{v})^{b^{u} - v} \in B .
\]

Hence, by [7] Lemma 2.3, we have \( a^{p}b^{q} \in B \). Next we suppose that \( a^{p}b^{q} \notin B \). Then, by [7] Lemma 2.3, \( a^{p}b^{v} \notin B \). Let \( C \) be the archimedean class containing the element \( a^{p}b^{v} \). Then, since

\[
\quad \quad a^{p}b^{v} \leq a^{p}b^{v} \leq b^{p} + v ,
\]

and since \( a^{p}b^{v} \notin A \) and \( a^{p}b^{v} \notin B \), we have \( A \neq C \neq B \). By [7] Lemma 5.6, we have \( A \delta = B \delta = A \delta \wedge B \delta \leq C \delta \), and, by Lemma 1, \( C \notin B \delta \) and so \( B \) non \( \delta \) \( C \). Hence, by [7] Theorem 6.1, we have \( CB \leq B \). Hence

\[
\quad \quad a^{p}b^{u} = (a^{p}b^{v})^{b^{u} - v} \in CB \leq B
\]

and so \( a^{p}b^{q} \in B \).

(9) can be proved in a similar way.

(10) By (1) and (7), we have \( \sup L(a, b) \leq \inf U(a, b) \). By way of contradiction, we assume \( \sup L(a, b) < \inf U(a, b) \). We take positive rational numbers \( q/p \) and \( v/u \) such that

\[
\quad \quad \sup L(a, b) < q/p < v/u < \inf U(a, b) .
\]

Then \( q/p \notin L(a, b) \) and, by (8), \( a^{p}b^{q} \in B \). Hence \( v/u \in U(a, b) \), contradicting \( v/u < \inf U(a, b) \).

(11) Suppose \( r \in L(a, b) \). Then there exists a positive rational number \( q/p \) such that \( r \leq q/p \) and \( a^{p}b^{q} \in A \). Since \( a^{2} \in A \) and \( A \) is negative torsion-free, there exists a natural number \( n > 1 \) such that \( (a^{p}b^{q})^{n} < a^{2} \). First suppose that \( ab \leq ba \). Then
\[ a^{np}b^n \leq (a^p b^q)^n < a^2 \] and so
\[ a^{np-1+n^q} \leq a^{np-1}b^n < a \, . \]

Hence \[ a^{np-1}b^n \in A \]. Next suppose that \( ba \leq ab \). Then \( b^\sigma a^{np} \leq (a^p b^q)^n < a^2 \) and so
\[ a^{np-1+n^q} \leq b^\sigma a^{np-1} < a \, . \]

Hence \( b^\sigma a^{np-1} \in A \) and so, by [7] Lemma 2.3, we obtain the same result \( a^{np-1}b^n \in A \). Therefore always we have \( nq/(np - 1) \in L(a, b) \) with \( r \leq q/p < nq/(np - 1) \). This proves the first assertion. The second assertion can be proved in a similar way.

(12) First suppose \( a^p b^q \in A \). Then \( q/p \in L(a, b) \), by definition. Next suppose \( q/p \in L(a, b) \). Then, by (11), there exists \( r \in L(a, b) \) such that \( q/p < r \). Hence
\[ q/p < r \leq \sup L(a, b) = r(a, b) \, . \]

Finally suppose \( q/p < r(a, b) \). Then there exists \( r \in L(a, b) \) such that \( q/p < r \). Hence, by (2), \( a^p b^q \in A \).

(13) can be proved in a similar way.

(14) follows from (12) and (13).

**Lemma 4.** (1) Let \( a \in A \). Then the positive real number \( r(a, b) w_2(b) \) is determined uniquely irrespective of the choice of \( b \in B \).

(2) Let \( b \in B \). Then the negative real number \( r(a, b)/w_1(a) \) is determined uniquely irrespective of the choice of \( a \in A \).

**Proof.** (1) Let \( a \in A \) and \( b, b' \in B \). Let \( r \) be an arbitrary positive real number such that \( r < (r(a, b) w_2(b))/w_2(b') \). Then there exist natural numbers \( p, q, u \) and \( v \) such that \( r < qv/pu, q/p < r(a, b) \) and \( v/u < w_2(b)/w_2(b') \). Hence
\[ w_2(b'^v) = v w_2(b') < u w_2(b) = w_2(b^p) \]
and so \( b'^v < b^u \). By Lemma 3 (12), we have \( a^p b^q \in A \) and, by [7] Lemma 2.3, \( a^{pu} b^{qu} \in A \). Hence
\[ a^{pu+qv} \leq a^{pu} b^{qu} \leq a^{pu} b^{qu} \in A \]
and so \( a^{pu} b^{qu} \in A \). Hence \( qv/pu \in L(a, b') \) and \( r \in L(a, b') \). Hence \( r \leq \sup L(a, b') = r(a, b') \). Therefore
\[ (r(a, b) w_2(b))/w_2(b') \leq r(a, b') \]
and so \( r(a, b) w_2(b) \leq r(a, b') w_2(b') \). The converse inequality can be proved in a similar way. Thus we have the assertion (1).

(2) can be proved in a similar way.
Lemma 5. (1) For \( a, a' \in A \) and \( b \in B \), \( r(a, b) + r(a', b) = r(aa', b) \).

(2) For \( a \in A \) and \( b, b' \in B \), \( (1/r(a, b)) + (1/r(a, b')) = 1/r(a, bb') \).

Proof. (1) By Lemma 4 (2), there exists a negative real number \( k \) such that \( r(a, b) = kw_1(a) \), \( r(a', b) = kw_1(a') \) and \( r(aa', b) = kw_1(aa') \). Hence

\[
\begin{align*}
r(aa', b) &= kw_1(aa') = k(w_1(a) + w_1(a')) = kw_1(a) + kw_1(a') = \\
&= r(a, b) + r(a', b).
\end{align*}
\]

(2) can be proved in a similar way.

Lemma 6. For \( a \in A \) and \( b \in B \) such that \( ab \in A \), \( 1 + r(ab, b) = r(a, b) \).

Proof. Let \( r \in L(ab, b) \). Then there exists a positive rational number \( q/p \) such that \( r \leq q/p \) and \( (ab)^p b^q \in A \). If \( ab \leq ba \), then \( a^{2p+q} \leq ab^{p+q} \leq (ab)^p b^q \) with \( a^{2p+q} \), \( (ab)^p b^q \in A \) and so \( a^{2p+q} \in A \). Also, if \( ba \leq ab \), then \( a^{2p+q} \leq b^{p+q} a^q \leq (ab)^p b^q \) and so \( b^{p+q} a^q \in A \), whence, by [7] Lemma 2.3, we obtain again \( a^{2p+q} \in A \). Therefore

\[
1 + r \leq 1 + (q/p) = (p + q)/p \leq \sup L(a, b) = r(a, b)
\]

and so \( 1 + r(ab, b) = \sup (1 + L(ab, b)) \leq r(a, b) \). By taking an arbitrary element in \( U(ab, b) \) instead of an element in \( L(ab, b) \), we obtain in a similar way that \( 1 + r(ab, b) \geq r(a, b) \). Hence we have the assertion.

In a similar way, we can prove

Lemma 7. For \( a \in A \) and \( b \in B \) such that \( ab \in B \), \( 1 + (1/r(a, ab)) = 1/r(a, b) \).

Lemma 8. (1) Let \( x \in T \setminus (A \cup B) \) and \( y \in B \). Then \( xy, yx \in B \) and the pairs \( \{xy, y\} \) and \( \{yx, y\} \) form anomalous pairs.

(2) Let \( x \in T \setminus (A \cup B) \) and \( y \in A \). Then \( xy, yx \in B \) and the pairs \( \{xy, y\} \) and \( \{yx, y\} \) form anomalous pairs.

Proof. (1) Let \( X \) be the archimedean class containing the element \( x \). Then, since \( x \in T \setminus (A \cup B) \), we have \( A < X < B \). By assumption \( A \delta B \) and so, by [7] Lemma 4.3, we have \( B \gamma X \). Also, by Lemma 1, we have \( B \) non \( \delta X \). Hence, by [7] Theorem 6.1, \( xy \in XB \leq B \) and \( yx \in BX \leq B \). Let \( n \) be an arbitrary natural number. Since \( x^{2n} \in A \) and \( y \in B \), we have \( x^{2n} < y \). First suppose \( xy \leq yx \). Then

\[
(yx)^{2n} \leq y^{2n} x^{2n} \leq y^{2n+1} < y^{2n+2}
\]

and so \( (xy)^n \leq y^{n+1} \). By way of contradiction, we suppose \( (xy)^{n+1} \leq y^n \). Then

\[
(xy^{n+1} y^n) y^n = x^{n+1} y^{n+1} \leq (xy)^{n+1} \leq y^n
\]

with \( x^{n+1} y \in XB \leq B \) and \( y^n \in B \). This contradicts [5] Theorem 6. Hence \( y^n < \)
Hence \( (xy)^{n+1} \leq (yx)^{n+1} \). Hence \( \{xy, y\} \) and \( \{yx, y\} \) form anomalous pairs. In the case when \( yx \leq xy \), we obtain the same conclusion in a similar way.

(2) can be proved in a similar way.

**Theorem 9.** There exists an \( o \)-homomorphism \( v \) of \( T \) into the additive ordered group of real numbers such that

- if \( x \in A \), then \( v(x) < 0 \);
- if \( x \in T \setminus (A \cup B) \), then \( v(x) = 0 \);
- if \( x \in B \), then \( v(x) > 0 \),

and, for \( x, y \in T \), \( v(x) = v(y) \) if and only if either \( x \) and \( y \) form an anomalous pair or \( x, y \in T \setminus (A \cup B) \).

**Proof.** We define the mapping \( v \) of \( T \) into the set of real numbers by:

- if \( x \in A \), then \( v(x) = -r(x, b)w_2(b) \) where \( b \in B \);
- if \( x \in T \setminus (A \cup B) \), then \( v(x) = 0 \);
- if \( x \in B \), then \( v(x) = w_2(x) \).

We remark that it follows from Lemma 4 that, for \( x \in A \), \( v(x) \) is determined uniquely irrespective of the choice of \( b \in B \). Now we show that, for \( x, y \in T \), \( v(xy) = v(x) + v(y) \) by dividing into the following cases.

(a) The case when \( x, y \in A \):

In this case \( xy \in A \). We take \( b \in B \) arbitrarily. Then, by Lemma 5 (1),

\[
v(xy) = -r(xy, b)w_2(b) = -(r(x, b) + r(y, b))w_2(b) = -r(x, b)w_2(b) - r(y, b)w_2(b) = v(x) + v(y).
\]

(b) The case when \( x, y \in B \):

In this case \( xy \in B \) and, by Lemma 2,

\[
v(xy) = w_2(xy) = w_2(x) + w_2(y) = v(x) + v(y).
\]

(c) The case when \( x \in A \), \( y \in B \) and \( xy \in A \):

By Lemma 6, we have

\[
v(xy) = -r(xy, y)w_2(y) = -r(x, y)w_2(y) + w_2(y) = v(x) + v(y).
\]

(d) The case when \( x \in B \), \( y \in A \) and \( xy \in A \):

By [7] Lemma 2.3, we have \( xy \in A \) and, by (c), \( v(xy) = v(y) + v(x) \). Also, by (a),

\[
v(y) + v(xy) = v(yxy) = v(yx) + v(y).
\]

Hence \( v(xy) = v(yx) = v(x) + v(y) \).
(e) The case when $x \in A$, $y \in B$ and $xy \in B$:

By Lemmas 4 (1) and 7, we have

\[
v(xy) = w_2(xy) = (r(x, y)/r(x, xy)) w_2(y) = (1 - r(x, y)) w_2(y) = -r(x, y) w_2(y) + w_2(y) = v(x) + v(y).
\]

(f) The case when $x \in B$, $y \in A$ and $xy \in B$:

We have $yx \in B$ and, by (e), $v(yx) = v(y) + v(x)$. Also, by (b), $v(xy) + v(x) = v(xyx) = v(x) + v(yx)$. Hence $v(xy) = v(yx) = v(x) + v(y)$.

(g) The case when $x \in A$, $y \in B$ and $xy \in T \setminus (A \cup B)$:

By way of contradiction, we assume $r(x, y) > 1$. We take a real number $r$ such that $1 < r < r(x, y)$. Then $r \in L(x, y)$ and so there exists a rational number $q/p$ such that $r \leq q/p$ and $x^q y^p \in A$. Since $1 < r \leq q/p$, we have $p < q$ and so

\[
x^q y^p = x^q r(x, y)^{q/p} \in A.
\]

Hence, by [7] Lemma 2.3, we have $xy \in A$, which is a contradiction. Similarly we can prove that $r(x, y) < 1$ implies a contradiction. Hence $r(x, y) = 1$ and so

\[
v(x) + v(y) = -r(x, y) w_2(y) + w_2(y) = -w_2(y) + w_2(y) = 0 = v(xy).
\]

(h) The case when $x \in B$, $y \in A$ and $xy \in T \setminus (A \cup B)$:

By [7] Lemma 2.3, $yx \in T \setminus (A \cup B)$ and, by (g),

\[
v(x) + v(y) = v(y) + v(x) = v(yx) = 0 = v(xy).
\]

(i) The case when either $x \in T \setminus (A \cup B)$ and $y \in B$ or $x \in B$ and $y \in T \setminus (A \cup B)$:

It follows from Lemmas 2 and 8 (1) that, if $x \in T \setminus (A \cup B)$ and $y \in B$, then

\[
v(xy) = w_2(xy) = w_2(y) = 0 + w_2(y) = v(x) + v(y).
\]

and, if $x \in B$ and $y \in T \setminus (A \cup B)$, then

\[
v(xy) = w_2(xy) = w_2(x) = 0 + v(x) + v(y).
\]

(j) The case when either $x \in A$ and $y \in T \setminus (A \cup B)$ or $x \in T \setminus (A \cup B)$ and $y \in A$:

Suppose $x \in A$ and $y \in T \setminus (A \cup B)$. Then, by Lemmas 2 and 8 (2), we have $w_1(x) = w_1(xy)$. Let $b \in B$. By Lemma 4 (2), we have $r(x, b)/w_1(x) = r(xy, b)/w_1(xy)$. Hence $r(x, b) = r(xy, b)$ and so

\[
v(xy) = -r(xy, b) w_2(b) = -r(x, b) w_2(b) + 0 = v(x) + v(y).
\]

The case when $x \in T \setminus (A \cup B)$ and $y \in A$ can be treated in a similar way.
Let $X, Y$ and $Z$ be the archimedean classes containing $x, y$ and $x'y$, respectively. Then $A < X < B$ and $A < Y < B$. Since $x'y$ lies between $x^2$ and $y^2$, $Z$ lies between $X$ and $Y$. Hence $A < Z < B$ and so $x'y \in T \setminus (A \cup B)$. Therefore

$$v(xy) = 0 = 0 + 0 = v(x) + v(y).$$

This proves that $v$ is a homomorphism of $T$ into the additive ordered group of real numbers. By the definition of $v$, $v(x) < 0$ if $x \in A$, $v(x) = 0$ if $x \in T \setminus (A \cup B)$ and $v(x) > 0$ if $x \in B$. Also it follows from Lemma 2 that $v$ is order-preserving and $v(x) = v(y)$ if and only if either $x$ and $y$ form an anomalous pair or $x, y \in T \setminus (A \cup B)$.

**Corollary 10.** The set $T \setminus (A \cup B)$ is a convex subsemigroup of $S$, if it is nonvoid.

**Corollary 11.** The following conditions are equivalent:

1. $AB \subseteq A \cup B$;
2. $BA \subseteq A \cup B$;
3. $v(a, b) \neq 1$ for every $a \in A$ and $b \in B$;
4. $v(a, b)$ is irrational for every $a \in A$ and $b \in B$.

**Proof.** $(1) \Leftrightarrow (2)$ follows from [7] Lemma 2.3. $(1) \Rightarrow (4)$. By way of contradiction, we assume $r(a, b)$ is equal to a rational number $n/m$. Then, by Lemma 5, we obtain

$$v(a^m b^n) = v(a^m) + v(b^n) = -r(a^m, b^n) w_2(b^n) + w_2(b^n) = 0.$$

Hence $a^m b^n \in T \setminus (A \cup B)$, contradicting Condition (1). $(4) \Rightarrow (3)$ is clear. $(3) \Rightarrow (1)$. Let $a \in A$ and $b \in B$. Then, by Condition (3), $r(a, b) > 1$ or $r(a, b) < 1$. If $r(a, b) > 1$, then

$$v(ab) = v(a) + v(b) = -r(a, b) w_2(b) + w_2(b) < 0$$

and so $ab \in A$. If $r(a, b) < 1$, then

$$v(ab) = -r(a, b) w_2(b) + w_2(b) > 0$$

and so $ab \in B$.

Finally we give an example which shows that there is no restriction for the structure of the ordered semigroup $T \setminus (A \cup B)$.

**Example.** Let $U$ be an arbitrary ordered semigroup and let $S = R \times U$ be the lexicographic product of the ordered additive group $R$ of real numbers and $U$. Then, since $R$ is cancellative, it follows from [6] Corollary 8 that $S$ is an ordered
semigroup. Put

\[ A = \{(r, u) \in R \times U; \ r < 0\} , \ B = \{(r, u) \in R \times U; \ r > 0\} . \]

Then \( A \) is the least and \( B \) is the greatest archimedean class on \( S \). It can be easily checked that both \( A \) and \( B \) are torsion-free, \( A \leq B \) in \( \preceq \), and the ordered semigroup \( S \setminus (A \cup B) \) is \( o \)-isomorphic to \( U \).

References


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