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Archimedean classes in an ordered semigroup. II


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The terminology and notation of our previous paper [7] are used throughout. In particular, we denote by S an ordered semigroup and by \( ^{\delta} \) the set of all archimedean classes of S.

The original purpose of this paper is to study the behavior of the set product \( AB \) of two archimedean classes \( A \) and \( B \) such that \( A \delta B \) and the \( \delta \)-class in \( ^{\delta} \) containing \( A \) and \( B \) is torsion-free. Thus in this note let \( A, B \in ^{\delta} \) satisfying these conditions. Moreover we assume \( A < B \). Also let \( T \) be the subset of \( S \) consisting of all elements \( x \) such that the archimedean class containing the element \( x \) lies between \( A \) and \( B \). Then \( T \) is a convex subsemigroup of \( S \), which contains the subsemigroup of \( S \) generated by \( A \) and \( B \).

In order to consider the behavior of \( AB \), in this paper we shall construct an \( o \)-homomorphism of \( T \) into the ordered additive group of real numbers such that its images are negative on \( A \), are positive on \( B \) and are zero on \( T \setminus (A \cup B) \).

By [7] Theorem 3.5, we have the following

**Lemma 1.** A is a negative torsion-free archimedean class and \( B \) is a positive torsion-free archimedean class of \( S \). Moreover the \( \delta \)-class \( A\delta \) of \( ^{\delta} \) consists of just two elements \( A \) and \( B \).

Two elements \( a \) and \( b \) of \( S \) are said to form an anomalous pair if either \( a^n < b^{n+1} \) and \( b^n < a^{n+1} \) or \( a^n > b^{n-1} \) and \( b^n > a^{n+1} \) for every natural number \( n \).

**Lemma 2.** There exists an \( o \)-homomorphism \( w_1 \) of \( A \) into the ordered additive semigroup of negative real numbers such that two elements of \( S \) have the same image if and only if they form an anomalous pair. Also there exists an \( o \)-homomorphism \( w_2 \) of \( B \) into the ordered additive semigroup of positive real numbers such that two elements of \( S \) have the same image if and only if they form an anomalous pair.

**Proof.** The second assertion follows from [3] Theorem or [4] Theorem 1. Dually we have the first assertion.
Lemma 3. Let \( a \in A \) and \( b \in B \). Put

\[ L(a, b) = \{ r \text{ positive real number; there exist natural numbers } p \text{ and } q \text{ such that } r \leq q/p \text{ and } a^p b^q \in A \}; \]

\[ U(a, b) = \{ r \text{ positive real number; there exist natural numbers } p \text{ and } q \text{ such that } q/p \leq r \text{ and } a^p b^q \in B \}. \]

Then

1. \( L(a, b) \neq \emptyset \) and \( U(a, b) \neq \emptyset \);
2. if \( r \in L(a, b) \) and \( p \) and \( q \) are natural numbers such that \( q/p \leq r \), then \( a^p b^q \in A \);
3. if \( r \in U(a, b) \) and \( p \) and \( q \) are natural numbers such that \( r \leq q/p \), then \( a^p b^q \in B \);
4. if \( r \in L(a, b) \), then \( r' \in L(a, b) \) for every positive real number \( r' \) such that \( r' < r \);
5. if \( r \in U(a, b) \), then \( r' \in U(a, b) \) for every positive real number \( r' \) such that \( r < r' \);
6. \( L(a, b) \cap U(a, b) = \emptyset \);
7. if \( r \in L(a, b) \) and \( r' \in U(a, b) \), then \( r < r' \);
8. if \( r \) is a positive real number such that \( r \notin L(a, b) \) and \( p \) and \( q \) are natural numbers such that \( r < q/p \), then \( a^p b^q \in B \);
9. if \( r \) is a positive real number such that \( r \notin U(a, b) \) and \( p \) and \( q \) are natural numbers such that \( q/p < r \), then \( a^p b^q \in A \);
10. \( \sup L(a, b) = \inf U(a, b) \). (This common positive real number is denoted by \( r(a, b) \));
11. \( L(a, b) \) has no greatest element and \( U(a, b) \) has no least element;
12. for natural numbers \( p \) and \( q \), \( a^p b^q \in A \) if and only if \( q/p < r(a, b) \), and if and only if \( q/p \in L(a, b) \);
13. for natural numbers \( p \) and \( q \), \( a^p b^q \in B \) if and only if \( r(a, b) < q/p \), and if and only if \( q/p \in U(a, b) \);
14. for natural numbers \( p \) and \( q \), \( a^p b^q \notin A \cup B \) if and only if \( r(a, b) \) is a rational number and \( r(a, b) = q/p \).

Proof. (1) follows from [7] Theorem 2.4.

(2) Suppose \( r \in L(a, b) \) and \( q/p \leq r \). Then there exists a positive rational number \( v/u \) such that \( r \leq v/u \) and \( a^v b^u \in A \). By [7] Lemma 2.3, we have \( a^u b^v \in A \). Since \( q/p \leq r \leq v/u \), we have \( uq \leq vp \). Hence

\[ a^p b^q = a^{p-uq}(a^u b^v) \in A, \]
where, if \( vp - uq = 0 \), we assume that \( a^{v_p - u_q} \) is the empty symbol. Hence, again by [7] Lemma 2.3, we have \( a^p b^q \in A \).

(3) can be proved in a similar way.

(4) Suppose \( r \in L(a, b) \) and \( r' < r \). We take a positive rational number \( q/p \) such that \( r' < q/p < r \). Then, by (2), we have \( a^p b^q \in A \) and so \( r' \in L(a, b) \).

(5) can be proved in a similar way.

(6) By way of contradiction, we assume there exists \( r \in L(a, b) \cap U(a, b) \). Then there exist positive rational numbers \( q/p \) and \( v/u \) such that \( v/u \leq r \leq q/p \), \( a^p b^q \in B \) and \( a^p b^q \in A \). But, by definition, \( q/p \in L(a, b) \) and, by (2), we have \( a^p b^q \in A \). This contradicts the fact that \( A < B \).

(7) follows immediately from (4) and (6).

(8) Suppose \( r \notin L(a, b) \) and \( r < q/p \). We take a positive rational number \( v/u \) such that \( r < v/u < q/p \). Then, by (4), we have \( v/u \notin L(a, b) \) and so \( a^p b^q \notin A \). Hence, by [7] Lemma 2.3, \( a^{v/u} b^{q/p} \notin A \). First we suppose \( a^p b^q \in B \). Then

\[
(a^{v/u} b^{q/p}) b^{q/p - v/u} \in B.
\]

Hence, by [7] Lemma 2.3, we have \( a^p b^q \in B \). Next we suppose that \( a^p b^q \notin B \). Then, by [7] Lemma 2.3, \( a^{v/u} b^{q/p} \notin B \). Let \( C \) be the archimedean class containing the element \( a^{v/u} b^{q/p} \). Then, since

\[
a^{v/u + q/p} \leq a^p b^{q/p} \leq b^{v/u + q/p},
\]

and since \( a^{v/u} b^{q/p} \notin A \) and \( a^{v/u} b^{q/p} \notin B \), we have \( A < C < B \). By [7] Lemma 5.6, we have \( A \delta = B \delta = A \delta \wedge B \delta \leq C \delta \) and, by Lemma 1, \( A \notin B \delta \) and so \( B \notin A \). Hence, by [7] Theorem 6.1, we have \( C B \subseteq B \). Hence

\[
a^{v/u} b^{q/p} = (a^{v/u} b^{q/p}) b^{q/p - v/u} \in C B \subseteq B
\]

and so \( a^p b^q \in B \).

(9) can be proved in a similar way.

(10) By (1) and (7), we have \( \sup L(a, b) \leq \inf U(a, b) \). By way of contradiction, we assume \( \sup L(a, b) < \inf U(a, b) \). We take positive rational numbers \( q/p \) and \( v/u \) such that

\[
\sup L(a, b) < q/p < v/u < \inf U(a, b).
\]

Then \( q/p \notin L(a, b) \) and, by (8), \( a^p b^q \in B \). Hence \( v/u \in U(a, b) \), contradicting \( v/u < \inf U(a, b) \).

(11) Suppose \( r \in L(a, b) \). Then there exists a positive rational number \( q/p \) such that \( r \leq q/p \) and \( a^p b^q \in A \). Since \( a^2 \in A \) and \( A \) is negative torsion-free, there exists a natural number \( n > 1 \) such that \( (a^p b^q)^n < a^2 \). First suppose that \( ab \leq ba \). Then
\[ a^{np} b^n \leq (a^p b^q)^n < a^2 \] and so
\[ a^{np-1+q} \leq a^{np-1} b^n < a. \]
Hence \( a^{np-1} b^n \in A \). Next suppose that \( ba \leq ab \). Then \( b^q a^{np} \leq (a^p b^q)^n < a^2 \) and so
\[ a^{np-1+q} \leq b^n a^{np-1} < a. \]
Hence \( b^n a^{np-1} \in A \) and so, by [7] Lemma 2.3, we obtain the same result \( a^{np-1} b^n \in A \). Therefore always we have \( nq/(np - 1) \in L(a, b) \) with \( \epsilon \leq q/p < nq/(np - 1) \). This proves the first assertion. The second assertion can be proved in a similar way.

(12) First suppose \( a^p b^q \in A \). Then \( q/p \in L(a, b) \), by definition. Next suppose \( q/p \in L(a, b) \). Then, by (11), there exists \( r \in L(a, b) \) such that \( q/p < r \). Hence
\[ q/p < r \leq \sup L(a, b) = r(a, b) . \]
Finally suppose \( q/p < r(a, b) \). Then there exists \( r \in L(a, b) \) such that \( q/p < r \). Hence, by (2), \( a^p b^q \in A \).

(13) can be proved in a similar way.

(14) follows from (12) and (13).

**Lemma 4.** (1) Let \( a \in A \). Then the positive real number \( r(a, b) w_2(b) \) is determined uniquely irrespective of the choice of \( b \in B \).

(2) Let \( b \in B \). Then the negative real number \( r(a, b) w_1(a) \) is determined uniquely irrespective of the choice of \( a \in A \).

**Proof.** (1) Let \( a \in A \) and \( b, b' \in B \). Let \( r \) be an arbitrary positive real number such that \( r < (r(a, b) w_2(b))/w_2(b') \). Then there exist natural numbers \( p, q, u \) and \( v \) such that \( r < qv/pu, q/p < r(a, b) \) and \( u/v < w_2(b)/w_2(b') \). Hence
\[ w_2(b'^v) = v w_2(b') < u w_2(b) = w_2(b^u) \]
and so \( b'^u < b^v \). By Lemma 3 (12), we have \( a^p b^q \in A \) and, by [7] Lemma 2.3, \( a^{pu} b^{qv} \in A \). Hence
\[ a^{pu+v} \leq a^{pu} b^{qv} \leq a^{pu} b^{qu} \in A \]
and so \( a^{pu} b^{qv} \in A \). Hence \( qv/pu \in L(a, b') \) and \( r \in L(a, b') \). Hence \( r \leq \sup L(a, b') = r(a, b') \). Therefore
\[ (r(a, b) w_2(b))/w_2(b') \leq r(a, b') \]
and so \( r(a, b) w_2(b) \leq r(a, b') w_2(b') \). The converse inequality can be proved in a similar way. Thus we have the assertion (1).

(2) can be proved in a similar way.
Lemma 5. (1) For \(a, a' \in A\) and \(b \in B\), \(r(a, b) + r(a', b) = r(aa', b)\).
(2) For \(a \in A\) and \(b, b' \in B\), \((1/r(a, b)) + (1/r(a, b')) = 1/r(a, bb')\).

Proof. (1) By Lemma 4 (2), there exists a negative real number \(k\) such that \(r(a, b) = k w_1(a)\), \(r(a', b) = k w_1(a')\) and \(r(aa', b) = k w_1(aa')\). Hence
\[
r(aa', b) = k w_1(aa') = k(w_1(a) + w_1(a')) = k w_1(a) + k w_1(a') = r(a, b) + r(a', b).
\]
(2) can be proved in a similar way.

Lemma 6. For \(a \in A\) and \(b \in B\) such that \(ab \in A\), \(1 + r(ab, b) = r(a, b)\).

Proof. Let \(r \in L(ab, b)\). Then there exists a positive rational number \(q/p\) such that \(r \leq q/p\) and \((ab)^p b^q \in A\). If \(ab \leq ba\), then \(a^2 p + q \leq a^p b^q \leq (ab)^p b^q\) with \(a^{2p+q}\), \((ab)^p b^q \in A\) and so \(a^p b^q \in A\). Also, if \(ba \leq ab\), then \(a^{2p+q} \leq b^p a^q \leq (ab)^p b^q\) and so \(b^p a^q \in A\), whence, by [7] Lemma 2.3, we obtain again \(a^p b^q \in A\). Therefore
\[
1 + r \leq 1 + (q/p) = (p + q)/p \leq \sup L(a, b) = r(a, b)
\]
and so \(1 + r(ab, b) = \sup (1 + L(ab, b)) \leq r(a, b)\). By taking an arbitrary element in \(U(ab, b)\) instead of an element in \(L(ab, b)\), we obtain in a similar way that \(1 + r(ab, b) \geq r(a, b)\). Hence we have the assertion.

In a similar way, we can prove

Lemma 7. For \(a \in A\) and \(b \in B\) such that \(ab \in B\), \(1 + (1/r(a, ab)) = 1/r(a, b)\).

Lemma 8. (1) Let \(x \in T\setminus (A \cup B)\) and \(y \in B\). Then \(xy, yx \in B\) and the pairs \(\{xy, y\}\) and \(\{yx, y\}\) form anomalous pairs.
(2) Let \(x \in T\setminus (A \cup B)\) and \(y \in A\). Then \(xy, yx \in A\) and the pairs \(\{xy, y\}\) and \(\{yx, y\}\) form anomalous pairs.

Proof. (1) Let \(X\) be the archimedean class containing the element \(x\). Then, since \(x \in T\setminus (A \cup B)\), we have \(A \leq X < B\). By assumption \(A \subseteq B\) and so, by [7] Lemma 4.3, we have \(B \subseteq X\). Also, by Lemma 1, we have \(B\) non-\(\delta\) \(X\). Hence, by [7] Theorem 6.1, \(xy \in XB \subseteq B\) and \(yx \in BX \subseteq B\). Let \(n\) be an arbitrary natural number. Since \(x^{2n} \in A\) and \(y \in B\), we have \(x^{2n} < y\). First suppose \(xy \leq y\). Then
\[
(yx)^{2n} = y^{2n} x^{2n} \leq y^{2n+1} < y^{2n+2}
\]
and so \((xy)^{2n} \leq (yx)^{n+1} < y^{n+1}\). By way of contradiction, we suppose \((xy)^{n+1} \leq y^n\). Then
\[
(x^{n+1} y)^{n+1} = x^{n+1} y^{n+1} \leq (xy)^{n+1} \leq y^n
\]
with \(x^{n+1} y \in XB \subseteq B\) and \(y^n \in B\). This contradicts [5] Theorem 6. Hence \(y^n <
Hence \((xy)^{n+1} \leq (yx)^{n+1}\). Hence \(\{xy, y\}\) and \(\{yx, y\}\) form anomalous pairs. In the case when \(yx \leq xy\), we obtain the same conclusion in a similar way.

(2) can be proved in a similar way.

**Theorem 9.** There exists an \(o\)-homomorphism \(v\) of \(T\) into the additive ordered group of real numbers such that

- if \(x \in A\), then \(v(x) < 0\);
- if \(x \in T \setminus (A \cup B)\), then \(v(x) = 0\);
- if \(x \in B\), then \(v(x) > 0\),

and, for \(x, y \in T\), \(v(x) = v(y)\) if and only if either \(x\) and \(y\) form an anomalous pair or \(x, y \in T \setminus (A \cup B)\).

**Proof.** We define the mapping \(v\) of \(T\) into the set of real numbers by:

- if \(x \in A\), then \(v(x) = -r(x, b)w_2(b)\) where \(b \in B\);
- if \(x \in T \setminus (A \cup B)\), then \(v(x) = 0\);
- if \(x \in B\), then \(v(x) = w_2(x)\).

We remark that it follows from Lemma 4 that, for \(x \in A\), \(v(x)\) is determined uniquely irrespective of the choice of \(b \in B\). Now we show that, for \(x, y \in T\), \(v(xy) = v(x) + v(y)\) by dividing into the following cases.

(a) The case when \(x, y \in A\):

In this case \(xy \in A\). We take \(b \in B\) arbitrarily. Then, by Lemma 5 (1),

\[
v(xy) = -r(xy, b)w_2(b) = -(r(x, b) + r(y, b))w_2(b) = -r(x, b)w_2(b) - r(y, b)w_2(b) = v(x) + v(y).
\]

(b) The case when \(x, y \in B\):

In this case \(xy \in B\) and, by Lemma 2,

\[
v(xy) = w_2(xy) = w_2(x) + w_2(y) = v(x) + v(y).
\]

(c) The case when \(x \in A\), \(y \in B\) and \(xy \in A\):

By Lemma 6, we have

\[
v(xy) = -r(xy, y)w_2(y) = -r(x, y)w_2(y) + w_2(y) = v(x) + v(y).
\]

(d) The case when \(x \in B\), \(y \in A\) and \(xy \in A\):

By [7] Lemma 2.3, we have \(yx \in A\) and, by (c), \(v(yx) = v(y) + v(x)\). Also, by (a),

\[
v(y) + v(xy) = v(y) = v(yx) = v(yx) + v(y).
\]

Hence \(v(xy) = v(yx) = v(x) + v(y)\).
(e) The case when \( x \in A, \ y \in B \) and \( xy \in B \):

By Lemmas 4 (1) and 7, we have

\[
v(xy) = w_2(xy) = \left(\frac{r(x, y)}{r(x, xy)}\right) w_2(y) = (1 - r(x, y)) w_2(y) = -r(x, y) w_2(y) + w_2(y) = v(x) + v(y).
\]

(f) The case when \( x \in B, \ y \in A \) and \( xy \in B \):

We have \( yx \in B \) and, by (e), \( v(yx) = v(y) + v(x) \). Also, by (b), \( v(xy) + v(x) = v(xy) = v(x) + v(y) \). Hence \( v(xy) = v(yx) = v(x) + v(y) \).

(g) The case when \( x \in A, \ y \in B \) and \( xy \in T \setminus (A \cup B) \):

By way of contradiction, we assume \( r(x, y) > 1 \). We take a real number \( r \) such that \( 1 < r < r(x, y) \). Then \( r \in L(x, y) \) and so there exists a rational number \( \frac{q}{p} \) such that \( r < \frac{q}{p} \) and \( \frac{x^q y^p}{y} \in A \). Since \( 1 < r < \frac{q}{p} \), we have \( p < q \) and so

\[
x^q y^p = x^{q-r} (x^p y^q) \in A.
\]

Hence, by [7] Lemma 2.3, we have \( xy \in A \), which is a contradiction. Similarly we can prove that \( r(x, y) < 1 \) implies a contradiction. Hence \( r(x, y) = 1 \) and so

\[
v(x) + v(y) = v(y) + v(x) = v(xy) = 0 = v(xy).
\]

(h) The case when \( x \in B, \ y \in A \) and \( xy \in T \setminus (A \cup B) \):

By [7] Lemma 2.3, \( yx \in T \setminus (A \cup B) \) and, by (g),

\[
v(x) + v(y) = v(y) + v(x) = v(xy) = 0 = v(xy).
\]

(i) The case when either \( x \in T \setminus (A \cup B) \) and \( y \in B \) or \( x \in B \) and \( y \in T \setminus (A \cup B) \):

It follows from Lemmas 2 and 8 (1) that, if \( x \in T \setminus (A \cup B) \) and \( y \in B \), then

\[
v(xy) = w_2(xy) = w_2(y) = 0 + w_2(y) = v(x) + v(y).
\]

and, if \( x \in B \) and \( y \in T \setminus (A \cup B) \), then

\[
v(xy) = w_2(xy) = w_2(x) = 0 + w_2(x) = v(x) + v(y).
\]

(j) The case when either \( x \in A \) and \( y \in T \setminus (A \cup B) \) or \( x \in T \setminus (A \cup B) \) and \( y \in A \):

Suppose \( x \in A \) and \( y \in T \setminus (A \cup B) \). Then, by Lemmas 2 and 8 (2), we have \( w_1(x) = w_1(xy) \). Let \( b \in B \). By Lemma 4 (2), we have \( r(x, b)/w_1(x) = r(xy, b)/w_1(xy) \). Hence \( r(x, b) = r(xy, b) \) and so

\[
v(xy) = -r(xy, b) w_2(b) = -r(x, b) w_2(b) + 0 = v(x) + v(y).
\]

The case when \( x \in T \setminus (A \cup B) \) and \( y \in A \) can be treated in a similar way.
(k) The case when \( x, y \in T \setminus (A \cup B) \):

Let \( X, Y \) and \( Z \) be the archimedean classes containing \( x, y \) and \( xy' \), respectively. Then \( A \prec X \prec B \) and \( A \prec Y \prec B \). Since \( xy \) lies between \( x^2 \) and \( y^2 \), \( Z \) lies between \( X \) and \( Y \). Hence \( A \prec Z \prec B \) and so \( xy' \in T \setminus (A \cup B) \). Therefore

\[
v(xy) = 0 = 0 + 0 = v(x) + v(y).
\]

This proves that \( v \) is a homomorphism of \( T \) into the additive ordered group of real numbers. By the definition of \( v \), \( v(x) < 0 \) if \( x \in A \), \( v(x) = 0 \) if \( x \in T \setminus (A \cup B) \) and \( v(x) > 0 \) if \( x \in B \). Also it follows from Lemma 2 that \( v \) is order-preserving and \( v(x) = v(y) \) if and only if either \( x \) and \( y \) form an anomalous pair or \( x, y \in T \setminus (A \cup B) \).

**Corollary 10.** The set \( T \setminus (A \cup B) \) is a convex subsemigroup of \( S \), if it is nonvoid.

**Corollary 11.** The following conditions are equivalent:

1. \( AB \subseteq A \cup B \);
2. \( BA \subseteq A \cup B \);
3. \( r(a, b) \neq 1 \) for every \( a \in A \) and \( b \in B \);
4. \( r(a, b) \) is irrational for every \( a \in A \) and \( b \in B \).

**Proof.** (1) \( \iff \) (2) follows from [7] Lemma 2.3. (1) \( \Rightarrow \) (4). By way of contradiction, we assume \( r(a, b) \) is equal to a rational number \( n/m \). Then, by Lemma 5, we obtain

\[
v(a^m b^n) = v(a^m) + r(b^n) = -r(a^m, b^n) w_2(b^n) + w_2(b^n) = 0.
\]

Hence \( a^m b^n \in T \setminus (A \cup B) \), contradicting Condition (1). (4) \( \Rightarrow \) (3) is clear. (3) \( \Rightarrow \) (1). Let \( a \in A \) and \( b \in B \). Then, by Condition (3), \( r(a, b) > 1 \) or \( r(a, b) < 1 \). If \( r(a, b) > 1 \), then

\[
v(ab) = v(a) + v(b) = -r(a, b) w_2(b) + w_2(b) < 0
\]

and so \( ab \in A \). If \( r(a, b) < 1 \), then

\[
v(ab) = -r(a, b) w_2(b) + w_2(b) > 0
\]

and so \( ab \in B \).

Finally we give an example which shows that there is no restriction for the structure of the ordered semigroup \( T \setminus (A \cup B) \).

**Example.** Let \( U \) be an arbitrary ordered semigroup and let \( S = R \times U \) be the lexicographic product of the ordered additive group \( R \) of real numbers and \( U \). Then, since \( R \) is cancellative, it follows from [6] Corollary 8 that \( S \) is an ordered
semigroup. Put

\[ A = \{(r, u) \in R \times U; \ r < 0\}, \quad B = \{(r, u) \in R \times U; \ r > 0\}. \]

Then \( A \) is the least and \( B \) is the greatest archimedean class on \( S \). It can be easily checked that both \( A \) and \( B \) are torsion-free, \( A \triangleleft B \) in \( \mathcal{C} \), and the ordered semigroup \( S \setminus (A \cup B) \) is \( o \)-isomorphic to \( U \).

\[ \begin{align*}
References \\
\end{align*} \]

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