Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

ARCHIMEDEAN CLASSES IN AN ORDERED SEMIGROUP II

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(Received October 7, 1974)

The terminology and notation of our previous paper [7] are used throughout. In particular, we denote by S an ordered semigroup and by $\mathscr{C}$ the set of all archimedean classes of S.

The original purpose of this paper is to study the behavior of the set product AB of two archimedean classes A and B such that $A\ \delta\ B$ and the δ-class in $\mathscr{C}$ containing A and B is torsion-free. Thus in this note let $A, B \in \mathscr{C}$ satisfying these conditions. Moreover we assume $A < B$. Also let T be the subset of S consisting of all elements x such that the archimedean class containing the element x lies between A and B. Then T is a convex subsemigroup of S, which contains the subsemigroup of S generated by A and B.

In order to consider the behavior of AB, in this paper we shall construct an o-homomorphism of T into the ordered additive group of real numbers such that its images are negative on A, are positive on B and are zero on $T \setminus (A \cup B)$.

By [7] Theorem 3.5, we have the following

Lemma 1. *A is a negative torsion-free archimedean class and B is a positive torsion-free archimedean class of S. Moreover the δ-class $A\delta$ of $\mathscr{C}$ consists of just two elements A and B.*

Two elements a and b of S are said to form an *anomalous pair* if either $a^n < b^{n+1}$ and $b^n < a^{n+1}$ or $a^n > b^{n+1}$ and $b^n > a^{n+1}$ for every natural number n.

Lemma 2. *There exists an o-homomorphism w_1 of A into the ordered additive semigroup of negative real numbers such that two elements of S have the same image if and only if they form an anomalous pair. Also there exists an o-homomorphism w_2 of B into the ordered additive semigroup of positive real numbers such that two elements of S have the same image if and only if they form an anomalous pair.*

Proof. The second assertion follows from [3] Theorem or [4] Theorem 1. Dually we have the first assertion.

Lemma 3. *Let $a \in A$ and $b \in B$. Put*

$L(a, b) = \{r$ *positive real number; there exist natural numbers* p *and* q *such that* $r \leqq q/p$ *and* $a^p b^q \in A\}$;

$U(a, b) = \{r$ *positive real number; there exist natural numbers* p *and* q *such that* $q/p \leqq r$ *and* $a^p b^q \in B\}$.

Then

(1) $L(a, b) \neq \square$ *and* $U(a, b) \neq \square$;

(2) *if* $r \in L(a, b)$ *and* p *and* q *are natural numbers such that* $q/p \leqq r$, *then* $a^p b^q \in A$;

(3) *if* $r \in U(a, b)$ *and* p *and* q *are natural numbers such that* $r \leqq q/p$, *then* $a^p b^q \in B$;

(4) *if* $r \in L(a, b)$, *then* $r' \in L(a, b)$ *for every positive real number* r' *such that* $r' < r$;

(5) *if* $r \in U(a, b)$, *then* $r' \in U(a, b)$ *for every positive real number* r' *such that* $r < r'$;

(6) $L(a, b) \cap U(a, b) = \square$;

(7) *if* $r \in L(a, b)$ *and* $r' \in U(a, b)$, *then* $r < r'$;

(8) *if* r *is a positive real number such that* $r \notin L(a, b)$ *and* p *and* q *are natural numbers such that* $r < q/p$, *then* $a^p b^q \in B$;

(9) *if* r *is a positive real number such that* $r \notin U(a, b)$ *and* p *and* q *are natural numbers such that* $q/p < r$, *then* $a^p b^q \in A$;

(10) $\sup L(a, b) = \inf U(a, b)$. (*This common positive real number is denoted by* $r(a, b)$);

(11) $L(a, b)$ *has no greatest element and* $U(a, b)$ *has no least element*;

(12) *for natural numbers* p *and* q, $a^p b^q \in A$ *if and only if* $q/p < r(a, b)$, *and if and only if* $q/p \in L(a, b)$;

(13) *for natural numbers* p *and* q, $a^p b^q \in B$ *if and only if* $r(a, b) < q/p$, *and if and only if* $q/p \in U(a, b)$;

(14) *for natural numbers* p *and* q, $a^p b^q \notin A \cup B$ *if and only if* $r(a, b)$ *is a rational number and* $r(a, b) = q/p$.

Proof. (1) follows from [7] Theorem 2.4.

(2) Suppose $r \in L(a, b)$ and $q/p \leqq r$. Then there exists a positive rational number v/u such that $r \leqq v/u$ and $a^u b^v \in A$. By [7] Lemma 2.3, we have $a^{uq} b^{vq} \in A$. Since $q/p \leqq r \leqq v/u$, we have $uq \leqq vp$. Hence

$$a^{vp} b^{vq} = a^{vp-uq}(a^{uq} b^{vq}) \in A ,$$

where, if $vp - uq = 0$, we assume that a^{vp-uq} is the empty symbol. Hence, again by [7] Lemma 2.3, we have $a^p b^q \in A$.

(3) can be proved in a similar way.

(4) Suppose $r \in L(a, b)$ and $r' < r$. We take a positive rational number q/p such that $r' < q/p < r$. Then, by (2), we have $a^p b^q \in A$ and so $r' \in L(a, b)$.

(5) can be proved in a similar way.

(6) By way of contradiction, we assume there exists $r \in L(a, b) \cap U(a, b)$. Then there exist positive rational numbers q/p and v/u such that $v/u \leqq r \leqq q/p$, $a^u b^v \in B$ and $a^p b^q \in A$. But, by definition, $q/p \in L(a, b)$ and, by (2), we have $a^u b^v \in A$. This contradicts the fact that $A < B$.

(7) follows immediately from (4) and (6).

(8) Suppose $r \notin L(a, b)$ and $r < q/p$. We take a positive rational number v/u such that $r < v/u < q/p$. Then, by (4), we have $v/u \notin L(a, b)$ and so $a^u b^v \notin A$. Hence, by [7] Lemma 2.3, $a^{up} b^{vp} \notin A$. First we suppose $a^u b^v \in B$. Then

$$a^{up} b^{uq} = (a^{up} b^{vp})\, b^{uq-vp} \in B\,.$$

Hence, by [7] Lemma 2.3, we have $a^p b^q \in B$. Next we suppose that $a^u b^v \notin B$. Then, by [7] Lemma 2.3, $a^{up} b^{vp} \notin B$. Let C be the archimedean class containing the element $a^{up} b^{vp}$. Then, since

$$a^{up+vp} \leqq a^{up} b^{vp} \leqq b^{up+vp}\,,$$

and since $a^{up} b^{vp} \notin A$ and $a^{up} b^{vp} \notin B$, we have $A < C < B$. By [7] Lemma 5.6, we have $A\delta = B\delta = A\delta \wedge B\delta \leqq C\delta$, and, by Lemma 1, $C \notin B\delta$ and so $B \operatorname{non} \delta\, C$. Hence, by [7] Theorem 6.1, we have $CB \subseteq B$. Hence

$$a^{up} b^{uq} = (a^{up} b^{vp})\, b^{uq-vp} \in CB \subseteq B$$

and so $a^p b^q \in B$.

(9) can be proved in a similar way.

(10) By (1) and (7), we have $\sup L(a, b) \leqq \inf U(a, b)$. By way of contradiction, we assume $\sup L(a, b) < \inf U(a, b)$. We take positive rational numbers q/p and v/u such that

$$\sup L(a, b) < q/p < v/u < \inf U(a, b)\,.$$

Then $q/p \notin L(a, b)$ and, by (8), $a^u b^v \in B$. Hence $v/u \in U(a, b)$, contradicting $v/u <$ $< \inf U(a, b)$.

(11) Suppose $r \in L(a, b)$. Then there exists a positive rational number q/p such that $r \leqq q/p$ and $a^p b^q \in A$. Since $a^2 \in A$ and A is negative torsion-free, there exists a natural number $n > 1$ such that $(a^p b^q)^n < a^2$. First suppose that $ab \leqq ba$. Then

$a^{np}b^{nq} \leqq (a^p b^q)^n < a^2$ and so

$$a^{np-1+nq} \leqq a^{np-1}b^{nq} < a\,.$$

Hence $a^{np-1}b^{nq} \in A$. Next suppose that $ba \leqq ab$. Then $b^{nq}a^{np} \leqq (a^p b^q)^n < a^2$ and so

$$a^{np-1+nq} \leqq b^{nq}a^{np-1} < a\,.$$

Hence $b^{nq}a^{np-1} \in A$ and so, by [7] Lemma 2.3, we obtain the same result $a^{np-1}b^{nq} \in A$. Therefore always we have $nq/(np-1) \in L(a, b)$ with $r \leqq q/p < nq/(np-1)$. This proves the first assertion. The second assertion can be proved in a similar way.

(12) First suppose $a^p b^q \in A$. Then $q/p \in L(a, b)$, by definition. Next suppose $q/p \in L(a, b)$. Then, by (11), there exists $r \in L(a, b)$ such that $q/p < r$. Hence

$$q/p < r \leqq \sup L(a, b) = r(a, b)\,.$$

Finally suppose $q/p < r(a, b)$. Then there exists $r \in L(a, b)$ such that $q/p < r$. Hence, by (2), $a^p b^q \in A$.

(13) can be proved in a similar way.

(14) follows from (12) and (13).

Lemma 4. (1) *Let $a \in A$. Then the positive real number $r(a, b)\, w_2(b)$ is determined uniquely irrespective of the choice of $b \in B$.*

(2) *Let $b \in B$. Then the negative real number $r(a, b)/w_1(a)$ is determined uniquely irrespective of the choice of $a \in A$.*

Proof. (1) Let $a \in A$ and $b, b' \in B$. Let r be an arbitrary positive real number such that $r < (r(a, b)\, w_2(b))/w_2(b')$. Then there exist natural numbers p, q, u and v such that $r < qv/pu$, $q/p < r(a, b)$ and $v/u < w_2(b)/w_2(b')$. Hence

$$w_2(b'^v) = v\, w_2(b') < u\, w_2(b) = w_2(b^u)$$

and so $b'^v < b^u$. By Lemma 3 (12), we have $a^p b^q \in A$ and, by [7] Lemma 2.3, $a^{pu}b^{qu} \in A$. Hence

$$a^{pu+qv} \leqq a^{pu}b'^{qv} \leqq a^{pu}b^{qu} \in A$$

and so $a^{pu}b'^{qv} \in A$. Hence $qv/pu \in L(a, b')$ and $r \in L(a, b')$. Hence $r \leqq \sup L(a, b') = r(a, b')$. Therefore

$$(r(a, b)\, w_2(b))/w_2(b') \leqq r(a, b')$$

and so $r(a, b)\, w_2(b) \leqq r(a, b')\, w_2(b')$. The converse inequality can be proved in a similar way. Thus we have the assertion (1).

(2) can be proved in a similar way.

Lemma 5. (1) *For* $a, a' \in A$ *and* $b \in B$, $r(a, b) + r(a', b) = r(aa', b)$.

(2) *For* $a \in A$ *and* $b, b' \in B$, $(1/r(a, b)) + (1/r(a, b')) = 1/r(a, bb')$.

Proof. (1) By Lemma 4 (2), there exists a negative real number k such that $r(a, b) = k\, w_1(a)$, $r(a', b) = k\, w_1(a')$ and $r(aa', b) = k\, w_1(aa')$. Hence

$$r(aa', b) = k\, w_1(aa') = k(w_1(a) + w_1(a')) = k\, w_1(a) + k\, w_1(a') = \\ = r(a, b) + r(a', b)\,.$$

(2) can be proved in a similar way.

Lemma 6. *For* $a \in A$ *and* $b \in B$ *such that* $ab \in A$, $1 + r(ab, b) = r(a, b)$.

Proof. Let $r \in L(ab, b)$. Then there exists a positive rational number q/p such that $r \leqq q/p$ and $(ab)^p\, b^q \in A$. If $ab \leqq ba$, then $a^{2p+q} \leqq a^p b^{p+q} \leqq (ab)^p\, b^q$ with a^{2p+q}, $(ab)^p\, b^q \in A$ and so $a^p b^{p+q} \in A$. Also, if $ba \leqq ab$, then $a^{2p+q} \leqq b^{p+q} a^p \leqq (ab)^p\, b^q$ and so $b^{p+q} a^p \in A$, whence, by [7] Lemma 2.3, we obtain again $a^p b^{p+q} \in A$. Therefore

$$1 + r \leqq 1 + (q/p) = (p + q)/p \leqq \sup L(a, b) = r(a, b)$$

and so $1 + r(ab, b) = \sup(1 + L(ab, b)) \leqq r(a, b)$. By taking an arbitrary element in $U(ab, b)$ instead of an element in $L(ab, b)$, we obtain in a similar way that $1 + r(ab, b) \geqq r(a, b)$. Hence we have the assertion.

In a similar way, we can prove

Lemma 7. *For* $a \in A$ *and* $b \in B$ *such that* $ab \in B$, $1 + (1/r(a, ab)) = 1/r(a, b)$.

Lemma 8. (1) *Let* $x \in T \setminus (A \cup B)$ *and* $y \in B$. *Then* $xy, yx \in B$ *and the pairs* $\{xy, y\}$ *and* $\{yx, y\}$ *form anomalous pairs.*

(2) *Let* $x \in T \setminus (A \cup B)$ *and* $y \in A$. *Then* $xy, yx \in A$ *and the pairs* $\{xy, y\}$ *and* $\{yx, y\}$ *form anomalous pairs.*

Proof. (1) Let X be the archimedean class containing the element x. Then, since $x \in T \setminus (A \cup B)$, we have $A < X < B$. By assumption $A\, \delta\, B$ and so, by [7] Lemma 4.3, we have $B\, \gamma\, X$. Also, by Lemma 1, we have B non $\delta\, X$. Hence, by [7] Theorem 6.1, $xy \in XB \subseteq B$ and $yx \in BX \subseteq B$. Let n be an arbitrary natural number. Since $x^{2n} \in A$ and $y \in B$, we have $x^{2n} < y$. First suppose $xy \leqq yx$. Then

$$(yx)^{2n} \leqq y^{2n} x^{2n} \leqq y^{2n+1} < y^{2n+2}$$

and so $(xy)^n \leqq (yx)^n < y^{n+1}$. By way of contradiction, we suppose $(xy)^{n+1} \leqq y^n$. Then

$$(x^{n+1} y)\, y^n = x^{n+1} y^{n+1} \leqq (xy)^{n+1} \leqq y^n$$

with $x^{n+1} y \in XB \subseteq B$ and $y^n \in B$. This contradicts [5] Theorem 6. Hence $y^n <$

$< (xy)^{n+1} \leqq (yx)^{n+1}$. Hence $\{xy, y\}$ and $\{yx, y\}$ form anomalous pairs. In the case when $yx \leqq xy$, we obtain the same conclusion in a similar way.

(2) can be proved in a similar way.

Theorem 9. *There exists an o-homomorphism v of T into the additive ordered group of real numbers such that*

if $x \in A$, then $v(x) < 0$;
if $x \in T \setminus (A \cup B)$, then $v(x) = 0$;
if $x \in B$, then $v(x) > 0$,

and, for $x, y \in T$, $v(x) = v(y)$ if and only if either x and y form an anomalous pair or $x, y \in T \setminus (A \cup B)$.

Proof. We define the mapping v of T into the set of real numbers by:

if $x \in A$, then $v(x) = -r(x, b)\, w_2(b)$ where $b \in B$;
if $x \in T \setminus (A \cup B)$, then $v(x) = 0$;
if $x \in B$, then $v(x) = w_2(x)$.

We remark that it follows from Lemma 4 that, for $x \in A$, $v(x)$ is determined uniquely irrespective of the choice of $b \in B$. Now we show that, for $x, y \in T$, $v(xy) = v(x) + v(y)$ by dividing into the following cases.

(a) The case when $x, y \in A$:

In this case $xy \in A$. We take $b \in B$ arbitrarily. Then, by Lemma 5 (1),

$$v(xy) = -r(xy, b)\, w_2(b) = -(r(x, b) + r(y, b))\, w_2(b) = \\ = -r(x, b)\, w_2(b) - r(y, b)\, w_2(b) = v(x) + v(y)\,.$$

(b) The case when $x, y \in B$:

In this case $xy \in B$ and, by Lemma 2,

$$v(xy) = w_2(xy) = w_2(x) + w_2(y) = v(x) + v(y)\,.$$

(c) The case when $x \in A$, $y \in B$ and $xy \in A$:

By Lemma 6, we have

$$v(xy) = -r(xy, y)\, w_2(y) = -r(x, y)\, w_2(y) + w_2(y) = v(x) + v(y)\,.$$

(d) The case when $x \in B$, $y \in A$ and $xy \in A$:

By [7] Lemma 2.3, we have $yx \in A$ and, by (c), $v(yx) = v(y) + v(x)$. Also, by (a),

$$v(y) + v(xy) = v(yxy) = v(yx) + v(y)\,.$$

Hence $v(xy) = v(yx) = v(x) + v(y)$.

(e) The case when $x \in A$, $y \in B$ and $xy \in B$:

By Lemmas 4 (1) and 7, we have

$$v(xy) = w_2(xy = (r(x, y)/r(x, xy))\, w_2(y) = (1 - r(x, y))\, w_2(y) =$$
$$= -r(x, y)\, w_2(y) + w_2(y) = v(x) + v(y)\,.$$

(f) The case when $x \in B$, $y \in A$ and $xy \in B$:

We have $yx \in B$ and, by (e), $v(yx) = v(y) + v(x)$. Also, by (b), $v(xy) + v(x) = v(xyx) = v(x) + v(yx)$. Hence $v(xy) = v(yx) = v(x) + v(y)$.

(g) The case when $x \in A$, $y \in B$ and $xy \in T \setminus (A \cup B)$:

By way of contradiction, we assume $r(x, y) > 1$. We take a real number r such that $1 < r < r(x, y)$. Then $r \in L(x, y)$ and so there exists a rational number q/p such that $r \leqq q/p$ and $x^p y^q \in A$. Since $1 < r \leqq q/p$, we have $p < q$ and so

$$x^q y^q = x^{q-p}(x^p y^q) \in A\,.$$

Hence, by [7] Lemma 2.3, we have $xy \in A$, which is a contradiction. Similarly we can prove that $r(x, y) < 1$ implies a contradiction. Hence $r(x, y) = 1$ and so

$$v(x) + v(y) = -r(x, y)\, w_2(y) + w_2(y) = -w_2(y) + w_2(y) = 0 = v(xy)\,.$$

(h) The case when $x \in B$, $y \in A$ and $xy \in T \setminus (A \cup B)$:

By [7] Lemma 2.3, $yx \in T \setminus (A \cup B)$ and, by (g),

$$v(x) + v(y) = v(y) + v(x) = v(yx) = 0 = v(xy)\,.$$

(i) The case when either $x \in T \setminus (A \cup B)$ and $y \in B$ or $x \in B$ and $y \in T \setminus (A \cup B)$:

It follows from Lemmas 2 and 8 (1) that, if $x \in T \setminus (A \cup B)$ and $y \in B$, then

$$v(xy) = w_2(xy) = w_2(y) = 0 + w_2(y) = v(x) + v(y)\,,$$

and, if $x \in B$ and $y \in T \setminus (A \cup B)$, then

$$v(xy) = w_2(xy) = w_2(x) = w_2(x) + 0 = v(x) + v(y)\,.$$

(j) The case when either $x \in A$ and $y \in T \setminus (A \cup B)$ or $x \in T \setminus (A \cup B)$ and $y \in A$:

Suppose $x \in A$ and $y \in T \setminus (A \cup B)$. Then, by Lemmas 2 and 8 (2), we have $w_1(x) = w_1(xy)$. Let $b \in B$. By Lemma 4 (2), we have $r(x, b)/w_1(x) = r(xy, b)/w_1(xy)$. Hence $r(x, b) = r(xy, b)$ and so

$$v(xy) = -r(xy, b)\, w_2(b) = -r(x, b)\, w_2(b) + 0 = v(x) + v(y)\,.$$

The case when $x \in T \setminus (A \cup B)$ and $y \in A$ can be treated in a similar way.

(k) The case when $x, y \in T \setminus (A \cup B)$:

Let X, Y and Z be the archimedean classes containing x, y and xy, respectively. Then $A < X < B$ and $A < Y < B$. Since xy lies between x^2 and y^2, Z lies between X and Y. Hence $A < Z < B$ and so $xy \in T \setminus (A \cup B)$. Therefore

$$v(xy) = 0 = 0 + 0 = v(x) + v(y) .$$

This proves that v is a homomorphism of T into the additive ordered group of real numbers. By the definition of v, $v(x) < 0$ if $x \in A$, $v(x) = 0$ if $x \in T \setminus (A \cup B)$ and $v(x) > 0$ if $x \in B$. Also it follows from Lemma 2 that v is order-preserving and $v(x) = v(y)$ if and only if either x and y form an anomalous pair or $x, y \in T \setminus (A \cup B)$.

Corollary 10. *The set $T \setminus (A \cup B)$ is a convex subsemigroup of S, if it is nonvoid.*

Corollary 11. *The following conditions are equivalent:*

(1) $AB \subseteq A \cup B$;

(2) $BA \subseteq A \cup B$;

(3) $r(a, b) \neq 1$ *for every* $a \in A$ *and* $b \in B$;

(4) $r(a, b)$ *is irrational for every* $a \in A$ *and* $b \in B$.

Proof. (1) ⇔ (2) follows from [7] Lemma 2.3. (1) ⇒ (4). By way of contradiction, we assume $r(a, b)$ is equal to a rational number n/m. Then, by Lemma 5, we obtain $r(a^m, b^n) = 1$ and so

$$v(a^m b^n) = v(a^m) + v(b^n) = -r(a^m, b^n)\, w_2(b^n) + w_2(b^n) = 0 .$$

Hence $a^m b^n \in T \setminus (A \cup B)$, contradicting Condition (1). (4) ⇒ (3) is clear. (3) ⇒ (1). Let $a \in A$ and $b \in B$. Then, by Condition (3), $r(a, b) > 1$ or $r(a, b) < 1$. If $r(a, b) > 1$, then

$$v(ab) = v(a) + v(b) = -r(a, b)\, w_2(b) + w_2(b) < 0$$

and so $ab \in A$. If $r(a, b) < 1$, then

$$v(ab) = -r(a, b)\, w_2(b) + w_2(b) > 0$$

and so $ab \in B$.

Finally we give an example which shows that there is no restriction for the structure of the ordered semigroup $T \setminus (A \cup B)$.

Example. Let U be an arbitrary ordered semigroup and let $S = R \times U$ be the lexicographic product of the ordered additive group R of real numbers and U. Then, since R is cancellative, it follows from [6] Corollary 8 that S is an ordered semigroup. Put

$$A = \{(r, u) \in R \times U;\ r < 0\}, \quad B = \{(r, u) \in R \times U;\ r > 0\}.$$

Then A is the least and B is the greatest archimedean class on S. It can be easily checked that both A and B are torsion-free, $A\ \delta\ B$ in $\mathscr{C}$, and the ordered semigroup $S \setminus (A \cup B)$ is o-isomorphic to U.

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