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ARCHIMEDEAN CLASSES IN AN ORDERED SEMIGROUP III

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The terminology and notation of our previous papers [3] and [4] are used throughout. In particular, we denote by S an ordered semigroup and by $\mathscr{C}$ the set of all archimedean classes of S. Also, for an archimedean class C of S, we denote by C_+ and C_- the set of all nonnegative elements of C and the set of all nonpositive elements of C, respectively.

In this paper we study the behavior of the set product AB of two archimedean classes A and B of S such that $A \,\delta\, B$ and the δ-class in $\mathscr{C}$ containing A and B is periodic. Thus, throughout this paper, we assume that $A, B \in \mathscr{C}$ such that $A < B$, $A \,\delta\, B$ and the δ-class in $\mathscr{C}$ containing A and B is periodic. We denote by e and f the idempotent of A and the idempotent of B, respectively.

Lemma 1. *For $a \in A$ and $b \in B$, $e \leqq ab \leqq f$ and $e \leqq ba \leqq f$.*

Proof. Since $A < B$, we have $a < f$ and $e < b$. Also, since e is the zero element of A and f is the zero element of B, we have

$$e = ae \leqq ab \leqq fb = f\,, \quad e = ea \leqq ba \leqq bf = f\,.$$

Theorem 2. *Suppose $A\delta = B\delta$ is of L-type [R-type].*

(1) *AB [BA] is contained in a single archimedean class if and only if $AB \subseteq A_-$ [$BA \subseteq A_-$];*

(2) *BA [AB] is contained in a single archimedean class if and only if $BA \subseteq B_+$ [$AB \subseteq B_+$].*

Proof. (1) Suppose AB is contained in a single archimedean class. By [3] Theorem 2.7, we have $eb = e \in A$ for every $b \in B$. Hence $AB \subseteq A$. Moreover, by Lemma 1,

$e \leqq ab$ for every $a \in A$ and $b \in B$. Hence $AB \subseteq A_-$. Conversely if $AB \subseteq A_-$, then clearly AB is contained in a single archimedean class. (2) can be proved in a similar way.

Theorem 3. *Suppose* $A\delta = B\delta$ *is of L-type [R-type].*

(1) *Suppose that* BA $[AB]$ *is contained in a single archimedean class but* AB $[BA]$ *is not contained in a single archimedean class. Then there exists an idempotent* g *such that* $e < g < f$, $g \mathscr{D}_E e$ *and* e *and* g *are consecutive in* $e\mathscr{D}_E$. *Also* $AB \subseteq A_- \cup C_+$ $[BA \subseteq A_- \cup C_+]$, *where* C *is the archimedean class containing the idempotent* g.

(2) *Suppose that* AB $[BA]$ *is contained in a single archimedean class but* BA $[AB]$ *is not contained in a single archimedean class. Then there exists an idempotent* g *such that* $e < g < f$, $g \mathscr{D}_E e$ *and* f *and* g *are consecutive in* $e\mathscr{D}_E$. *Also* $BA \subseteq B_+ \cup C_-$ $[AB \subseteq B_+ \cup C_-]$, *where* C *is the archimedean class containing the idempotent* g.

Proof. (1) Suppose that BA is contained in a single archimedean class but AB is not contained in a single archimedean class. By [3] Corollary 5.4, $B * A = B$ and so, by [3] Lemma 6.7, there exists an idempotent g such that $e < g < f$, $g \mathscr{D}_E e$ and e and g are consecutive in $e\mathscr{D}_E$. Now let $x \in A$ and $y \in B$. If $xy \in A$, then, by Lemma 1, $e \leqq xy$ and so $xy \in A_-$. Next suppose $xy \notin A$. Then, by [3] (6.7.5),

$$x \in A_- \setminus \{e\}\,, \tag{3.1}$$

and, by [3] (6.7.12)–(6.7.16),

$$xf = g\,. \tag{3.2}$$

First, let $y \in B_+$. Then, by Theorem 2, $yx \in BA \subseteq B_+$ and so $y_1 = \min(y, yx) \in B_+$. By [2] Lemma 1.13, the order of y_1 is at most 2. Hence $f = y_1^2 \leqq (yx)\,y \leqq f^2 = f$ and so $yxy = f$. Next let $y \in B_-$. Then, since $yx \in B_+$, we have $yx \leqq f \leqq y$. Since f is the zero element of B, we have $f = (yx)f \leqq yxy \leqq fy = f$ and so $yxy = f$. Thus, in both cases, we have

$$yxy = f\,. \tag{3.3}$$

Hence $(xy)^2 = x(yxy) = xf = g$. Also, since $e < g$, we have $A < C$ and so $x < g$. Further, since $C\,\delta\,A\,\delta\,B$, it follows from [3] Theorem 2.7 that $gy = g$. Hence $xy \leqq gy = g$. Hence

$$xy \in C_+\,. \tag{3.4}$$

Thus we obtain $AB \subseteq A_- \cup C_+$.

(2) can be proved in a similar way.

Example 1. Let S be the ordered semigroup consisting of six elements ordered by

$$e < a < c < g < b < f$$

with the multiplication table

	e	a	c	g	b	f
e	e	e	e	e	e	e
a	e	e	e	e	c	g
c	g	g	g	g	g	g
g	g	g	g	g	g	g
b	f	f	f	f	f	f
f	f	f	f	f	f	f

This example shows that in ordered semigroups S mentioned in Theorem 3 (1), the set product AB may contain an element of C different from g (Cf. [2] Theorem 6.8).

Theorem 4. *Suppose that $A\delta = B\delta$ is of L-type [R-type] and that neither AB nor BA is contained in a single archimedean class. Then there exist idempotents g and h such that $e < g \leqq h < f$, $e\mathcal{D}_E\, g\, \mathcal{D}_E\, h\, \mathcal{D}_E f$ and both $\{e, g\}$ and $\{h, f\}$ are consecutive in $e\mathcal{D}_E$. Moreover $AB \subseteq [e, g]$ and $BA \subseteq [h, f]$ $[BA \subseteq [e, g]$ and $AB \subseteq [h, f]]$.*

Proof. By [3] Corollary 5.4, $A * B = A$ and $B * A = B$. Hence, by [3] Lemma 6.7, there exists idempotents g and h such that $e\mathcal{D}_E\, g\, \mathcal{D}_E h\, \mathcal{D}_E f$, $e < g < f$, $e < h < f$ and both $\{e, g\}$ and $\{h, f\}$ are consecutive in $e\, \mathcal{D}_E$. This implies $e < g \leqq h < f$. Moreover for $a \in A$ and $b \in B$, $ab \leqq gb = (gf)\, b = g(fb) = gf = g$ and, by Lemma 1, $e \leqq ab$. Hence $AB \subseteq [e, g]$. In a similar way we can prove that $BA \subseteq [h, f]$.

It is easily seen that in an ordered semigroup S satifying the assumption of Theorem 4, $[e, g] \setminus (A_- \cup C_+)$ is a convex subsemigroup of S, if it is nonempty, where C is the archimedean class containing the element g. The ordered semigroups constructed in [1] show that $[e, g] \setminus (A_- \cup C_+)$ may be nonempty. The following example shows that the subsemigroup $[e, g] \setminus (A_- \cup C_+)$ may carry a much more general character.

Example 2. Let T be an arbitrary ordered semigroup. Let S be the ordered semigroup consisting of elements

$$\{a(t);\ t \in T\} \cup \{u(t);\ t \in T\} \cup \{v(t);\ t \in T\} \cup \{b(t);\ t \in T\} \cup \{e, f, g, h\}$$

ordered by

$$e < a(s) < a(t) < u(s) < u(t) < g < h < v(s) < v(t) < b(s) < b(t) < f$$

for $s, t \in T$ such that $s < t$ and with the multiplication table:

	e	$a(t)$	$u(t)$	g	h	$v(t)$	$b(t)$	f
e	e	e	e	e	e	e	e	e
$a(s)$	e	e	e	e	e	$a(st)$	$u(st)$	g
$u(s)$	e	$a(st)$	$u(st)$	g	g	g	g	g
g	g	g	g	g	g	g	g	g
h	h	h	h	h	h	h	h	h
$v(s)$	h	h	h	h	h	$v(st)$	$b(st)$	f
$b(s)$	h	$v(st)$	$b(st)$	f	f	f	f	f
f	f	f	f	f	f	f	f	f

where $s, t \in T$. It can be seen that $A = \{e\} \cup \{a(t); t \in T\}$ is the least and $B = \{b(t); t \in T\} \cup \{f\}$ is the greatest archimedean class of S which satisfy the assumption of Theorem 4, and $[e, g] \setminus (A_- \cup C_+)$ is equal to $\{u(t); t \in T\}$, which is o-isomorphic to T. Moreover, if T satisfies the condition that $T^2 = T$, then $[e, g] \setminus (A_- \cup C_+) \subseteq AB$.

References

[1] *T. Saitô,* Elements of finite order in an ordered semigroup whose product is of infinite order, Proc. Japan Acad. *50* (1974), 268–270.

[2] *T. Saitô,* Archimedean classes in a nonnegatively ordered semigroup, to appear.

[3] *T. Saitô,* Archimedean classes in an ordered semigroup I, Czechoslovak Math. J. *26(101)* (1976), 218–238.

[4] *T. Saitô,* Archimedean classes in an ordered semigroup II, Czechoslovak Math. J. *26(101)* (1976), 239–247.

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