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Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 2, 259–269

Persistent URL: `http://dml.cz/dmlcz/101397`

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

WEAKLY ASSOCIATIVE LATTICES AND TOLERANCE RELATIONS

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(Received October 16, 1974)

The investigation of tolerance relations has been rather expansive in a few last years. A number of results in this theory show its principal role in various branches of algebra and its applications (for example tolerance spaces, graph theory, topology etc.). In the paper [10] some results on the existence of non-trivial compatible tolerance relations on lattices were derived. Some of them can be generalized to weakly associative lattices and these "generalized" lattices offer a new view of these problems. A weakly associative lattice is obtained, roughly speaking, if the transitivity of the lattice ordering is omitted. These algebraic structures have very interesting properties and many of their applications play a principal role in algebra as is shown in the papers [1], [2], [3], [4], [5]. The purpose of this paper is to establish some results on the existence and basic properties of tolerance relations compatible with weakly associative lattices and tournaments.

1. PRELIMINARIES

Definition 1. A non-empty set A with two binary operations denoted by the symbols $\vee$ and $\wedge$ is called a *weakly associative lattice* (briefly WA-lattice), if for arbitrary a, b, c of A the following identities are fulfilled:

1° $a \vee a = a,\ a \wedge a = a$ (idempotency);

2° $a \vee b = b \vee a,\ a \wedge b = b \wedge a$ (commutativity);

3° $a \vee (b \wedge a) = a,\ a \wedge (b \vee a) = a$ (absorption);

4° $[(a \wedge c) \vee (b \wedge c)] \vee c = c,\ [(a \vee c) \wedge (b \vee c)] \wedge c = c$ (weak associativity).

Further, if for arbitrary a, b of A either $a \vee b = a$ or $a \vee b = b$, then $(A, \vee, \wedge)$ is called a tournament.

In the papers [1] and [3] a relation $\leqq$ on a WA-lattice A is introduced so that $a \leqq b$ if and only if $a \vee b$. This relation is reflexive and antisymmetric and evidently it is uniquely determined by the operation $\vee$. It is also uniquely determined by the operation $\wedge$; we have $a \precsim b$ if and only if $a \wedge b = a$. Conversely, the operations $\vee$ and $\wedge$ are uniquely determined by the relation $\precsim$. For any two elements a, b of A there exists a unique element c such that $c \succsim a$, $c \succsim b$ and $c \precsim c'$ for each $c' \in A$ such that $c' \succsim a$, $c' \succsim b$; this element $c = a \vee b$. There exists also a unique element d such $d \precsim a$, $d \precsim b$ and $d \succsim d'$ for each $d' \in A$ such that $d' \precsim a$, $d' \precsim b$; this element $d = a \wedge b$. For any a, b of A the equality $a \vee b = b$ is equivalent to $a \wedge b = a$. If A is a tournament, then $\precsim$ is a complete relation, i.e. for any a, b we have either $a \precsim b$ or $b \precsim a$ (and not both simultaneously). If $\precsim$ is also transitive, then a WA-lattice A is an ordinary lattice and a tournament A is a chain. (Note that in general WA-lattices are not lattices and $\precsim$ need not be an ordering.)

WA-lattices which are subdirectly irreducible and satisfy Congruence Extension Property are very important for investigating the structure of WA-lattices. In [4] it is proved that the class of all WA-lattices which are subdirectly irreducible and satisfy Congruence Extension Property is equal to the class of all WA-lattices with Unique Bound Property (briefly UBP), i.e. with the property that to any two elements of A there is exactly one element greater and exactly one element less than they both. In [4] it is proved that the class of WA-lattices satisfying UBP is decomposed into two (non-disjoint) subclasses, the so-called singular and regular WU-systems (Theorem 1 in [4]) and that there exists exactly one WA-lattice satisfying UBP which is simultaneously regular and singular.

We can give an exact definition of these concepts.

Definition 2. A WA-lattice W_A is called a *singular WU-system*, if $W_A = A \cup \{0, 1\}$ for $A \neq \emptyset$, $A \cap \{0, 1\} = \emptyset$ and the WA-lattice ordering is defined by the relations

$$1 \prec 0 , \quad 0 \prec a \prec 1 \quad \text{for all} \quad a \in A ,$$

where $x \precsim y$ if and only if $x \prec y$ or $x = y$.

Let W be a WA-lattice, $a \in W$ and $U(a) = \{x \in W \mid a \precsim x\}$, $L(a) = \{y \in W \mid y \precsim a\}$. If card $U(x) =$ card $U(y) =$ card $L(y)$ for arbitrary two elements x, y of W, then W is called a regular WU-system.

For singular WU-systems the non-existence of a non-trivial compatible tolerance will be proved.

Definition 3. Let S be a set and T a binary relation on S. The relation T is said to be *a tolerance*, if it is reflexive and symmetric. Let W be a WA-lattice and T a tolerance on W. The tolerance T is called *compatible with* W, if the following implication is true:

$$a_1, a_2, b_1, b_2 \in W , \quad a_1 T b_1, a_2 T b_2 \Rightarrow a_1 \vee a_2 T b_1 \vee b_2 , \quad a_1 \wedge a_2 T b_1 \wedge b_2 .$$

This is a special case of the definition of a tolerance compatible with an algebra; this definition can be found in [9]. Tolerances on algebras are studied in [8], [9], [10].

For the sake of brevity we shall introduce the following concepts.

Definition 4. Let W be a WA-lattice, $a \in W$, $b \in W$. The set $\{x \in W \mid a \lesseqq x \lesseqq b$ or $b \lesseqq x \lesseqq a\}$ is said to be *a segment of W* and is denoted by $S(a, b)$.

Definition 5. Let S be a set. By the *identical relation on S* we mean the relation I fulfilling aIb if and only if $a = b$ for all $a \in S$, $b \in S$. By the *universal relation on S* we mean the binary relation U fulfilling aUb for arbitrary two elements a, b of S. Let R be a binary relation on S. We say that R is *complete*, if for arbitrary two elements a, b of S either aRb or bRa is true.

2. SIMPLE CHARACTERISTICS OF TOLERANCES ON WA-LATTICES

Let S be a set and let ϱ be a binary relation on S. We say that a binary relation ϱ_S on S is the symmetric hull of ϱ, if for arbitrary a, b of S we have $a\ \varrho_S\ b$ if and only if $a\ \varrho\ b$ or $b\ \varrho\ a$. It is clear that for a reflexive relation ϱ the symmetric hull ϱ_S is a tolerance.

Proposition 1. *Let S be a non-empty set with an antisymmetric binary relation ϱ. Then (S, ϱ) is a tournament, if and only if the symmetric hull ϱ_S of ϱ is the universal relation on S.*

Proof. If (S, ϱ) is a tournament, then for any a, b of S we have $a\ \varrho\ b$ or $b\ \varrho\ a$, therefore $a\ \varrho_S\ b$. On the other hand, if $a\ \varrho_S\ b$ for any two elements a, b of S, then for any two elements a, b of S either $a\ \varrho\ b$, or $b\ \varrho\ a$. As ϱ is antisymmetric, for $a \neq b$ only one of these two possibilities can occur, thus (S, ϱ) is a tournament.

Proposition 2. *Let S be a non-empty set with an antisymmetric acyclic binary relation ϱ. Then (S, ϱ) is a chain, if and only if the symmetric hull ϱ_S of ϱ is the universal relation on S.*

Proof follows from Proposition 1, because a chain is an acyclic tournament.

Theorem 1. *Let W be a WA-lattice and let T be a tolerance compatible with W. Then for arbitrary two elements a, b of W*

$$aTb \Rightarrow xTy \quad \textit{for arbitrary} \quad x, y \in S(a \wedge b,\ a \vee b)\,.$$

Proof. Let aTb. From the reflexivity of T we have bTb and by the compatibility of T we obtain: $aTb, bTb \Rightarrow a \wedge bTb, a \vee bTb$. Analogously $a \wedge bTa$, $a \vee bTa$. Further, $a \wedge bTa$, $a \wedge bTb \Rightarrow (a \wedge b) \vee (a \wedge b)\ Ta \vee b$, i.e. $a \wedge bTa \vee b$. Let x and y be in $S(a \wedge b,\ a \vee b)$. From $a \wedge bTa \vee b$, xTx we obtain $(a \wedge b) \vee \vee\ xT(a \vee b) \vee x$, $(a \wedge b) \wedge xT(a \vee b) \wedge x$. If $a \wedge b \lesseqq x \lesseqq a \vee b$, then $(a \wedge b) \wedge x = x$, $(a \vee b) \vee x = a \vee b$, thus $xTa \vee b$. If $a \vee b \lesseqq x \lesseqq a \wedge b$, then $(a \wedge b) \vee x = x$, $(a \vee b) \wedge x = a \vee b$, thus also $xTa \vee b$. Analogously

$yTa \vee b$, $yTa \wedge b$, $xTa \wedge b$. Let $a \wedge b \leqq x \leqq a \vee b$, $a \wedge b \leqq y \leqq a \vee b$; then $xTa \vee b$ and $a \vee bTy$ imply $x \wedge (a \vee b) T(a \vee b) \wedge y$, i.e. xTy. Dually $a \vee b \leqq \leqq x \leqq a \wedge b$, $a \vee b \leqq y \leqq a \wedge b$ imply xTy. If $a \vee b \leqq x \leqq a \wedge b$, $a \wedge b \leqq \leqq y \leqq a \vee b$, then $a \wedge bTx$, $yTa \vee b$ yields $(a \wedge b) \vee yTx \vee (a \wedge b)$, i.e. again xTy.

3. EXISTENCE OF COMPATIBLE TOLERANCES ON *WA*-LATTICES

In this item we shall study compatible tolerances on *WA*-lattices. Evidently, on any *WA*-lattice (with more than one element) there exist at least two compatible tolerances, namely the identical relation and the universal relation. Also each congruence on a *WA*-lattice is a compatible tolerance on it. We are interested mainly in compatible tolerances which are not congruences.

Definition 6. Let W be a *WA*-lattice, let A be a subset of W. The set A is called *a cycle in W*, if $A = S(a, b)$ for arbitrary two distinct elements a, b of A.

Lemma 1. *Let W be a WA-lattice. Then each cycle of W is a tournament with at most three elements.*

Proof. If a, b are two distinct elements of A, then $a \in A = S(a, b)$, thus either $a \leqq a \leqq b$ or $b \leqq a \leqq a$. As a, b are distinct, we have either $a \prec b$ or $b \prec a$ and A is a tournament. Suppose that there exists a cycle A of W with more than three elements. Let a, b, c, d be some four of them. We have $A = S(a, b)$, therefore either $a \prec c \prec b$ or $b \prec c \prec a$; without loss of generality let $a \prec c \prec d$. If $a \prec b$, then $a \notin S(b, c)$ and $A \neq S(b, c)$, which is a contradiction. Thus $b \prec a$. The element d must satisfy either $a \prec d \prec b$ or $b \prec d \prec a$. In the first case $a \prec c$, $a \prec d$, thus $a \notin S(c, d)$ and $A \neq S(c, d)$; in the second case $b \prec a$, $b \prec d$ and thus $b \notin S(a, d)$; both these cases lead to contradictions.

Theorem 2. *Let W be a WA-lattice, let A be a cycle of W and let T be a tolerance compatible with W. Then the restriction T′ of T onto A is either the identical relation or the universal relation on A.*

Proof. According to Lemma 1 A cannot have more than three elements. If it has less than three elements, then any tolerance on A is either the identical relation or the universal relation. Let A have three elements a, b, c and $a \prec b \prec c \prec a$. If T' is not the identical relation, then there exist two distinct elements of A which are in T'; without loss of generality let $aT'b$. From $aT'b$ and $cT'c$ we have $a = a \vee cT'b \vee \vee c = c$, $b = b \wedge cT'a \wedge c = c$ and T' is the universal relation.

Lemma 2. *Each three-element WA-lattice is a cycle or a chain.*

This assertion is evident.

Lemma 3. *Each tolerance compatible with a three-element or a four-element singular WU-system is either the identical relation I or the universal relation U on the system.*

Proof. Let W_A be a three-element singular WU-system. Then W_A is a cycle and according to Theorem 2 each tolerance compatible with it is I or U. If W_A is a four-element singular WU-system, then $A = \{a, b\}$ and $\{a, 0, 1\}$ is a cycle of W_A. This means that the restriction T' of T onto $\{a, 0, 1\}$ is either the identical relation or the universal relation. Analogously the restriction T'' onto $\{b, 0, 1\}$ is either the identical relation or the universal relation. Let both T' and T'' be identical relations. Then either $T = I$, or aTb. If aTb, then aTb, bTb imply $a \vee bTb \vee b$, which means $1\ Tb$ and $1\ T''b$, which is a contradiction with the assumption that T'' is the identical relation. Now let T' be the universal relation on $\{a, 0, 1\}$. Then $0T1$ and according to Theorem 1 we have xTy for any two elements x, y of W_A and $T = U$. Analogously if T'' is the universal relation on $\{b, 0, 1\}$.

Lemma 4. *Let W be an at least five-element singular WU-system. Then each tolerance compatible with W is either the identical relation I, or the universal relation U.*

Proof. Let W be an at least five-element singular WU-system and let T be a tolerance compatible with W. Further let $T \neq I$. Then there exist two distinct elements a, b or W such that aTb. If $a = 0$, $b = 1$, then $T = U$ according to Theorem 1. If $a \neq 0$, $a \neq 1$, $b \neq 0$, $b \neq 1$, then $a \wedge bTa \vee b$ according to Theorem 1, thus $0T1$ and $T = U$. Let $a = 0$, $b \neq 1$. As W has at least five elements, there exist $c \in W$, $d \in W$ which are pairwise distinct and distinct from 0, 1 and b. As T is reflexive, cTc, dTd. From aTb and $a = 0$ we have $0Tb$. Then cTc, $0Tb \Rightarrow c = c \vee 0Tc \vee b = 1$. Thus $cT1$ and analogously $dT1$. This implies $0 = c \wedge dT1 \wedge 1 = 1$, thus $0T1$ and the situation is the same as in the preceding case. If $a = 1$, $b \neq 0$, the proof is dual.

Theorem 3. *On each singular WU-system there exist only two compatible tolerances, namely the identical relation and the universal relation.*

Proof follows directly from Lemmas 3 and 4.

4. SUB-WA-LATTICES AND COMPATIBLE TOLERANCES

If can we prove the existence of a compatible tolerance which is not a congruence on a special sub-WA-lattice of a WA-lattice, we can extend this result onto the whole WA-lattice. We formulate this exactly in the following theorem.

Theorem 4. *Let W be a WA-lattice. A necessary and sufficient condition for the existence of a tolerance T compatible with W which is not a congruence is the* *following: there exists a sub-WA-lattice W_0 of W, a tolerance T_0 compatible with W_0 which is not a congruence and a homomorphism φ of W onto a WA-lattice W_1 such that $\varphi(x) = \varphi(y)$ if and only if either $x = y$ or both x and y belong to W_0.*

Proof. The necessity is obvious; if the required tolerance T exists, we may put $W_0 = W$, $T_0 = T$ and φ equal to the mapping of W onto a one-element WA-lattice. Let us prove the sufficiency. First we prove that if $a \in W_0$, $b \in W - W_0$, $a \vee b \in W_0$, then $a \vee b = a$. Suppose that it is not so. Then $a \wedge b \neq b$. But $\varphi(a) = \varphi(a \vee b)$ in W_1, thus $\varphi(a) \wedge \varphi(b) = \varphi(a \vee b) \wedge \varphi(b) \neq \varphi(b)$ in W_1, because $b \in W - W_0$ and thus $\varphi(b)$ is the image of only one element of W in φ. But $(a \vee b) \wedge b = b$ in W, thus $\varphi(a \vee b) \wedge \varphi(b) = \varphi(b)$ in W_1, because φ is a homomorphism. We have a contradiction. Dually we can prove that if $a \in W_0$, $b \in W - W_0$, $a \wedge b \in W_0$, then $a \wedge b = a$. Now let T be a tolerance on W defined so that xTy if and only if either $x = y$, or $x \in W_0$, $y \in W_0$, xT_0y. Let x_1, x_2, y_1, y_2 be elements of W and x_1Ty_1, x_2Ty_2. If $x_1 = y_1$, $x_2 = y_2$, then $x_1 \vee x_2 = y_1 \vee y_2$, $x_1 \wedge x_2 = y_1 \wedge y_2$ and thus $x_1 \vee x_2Ty_1 \vee y_2$, $x_1 \wedge x_2Ty_1 \wedge y_2$. If all the elements x_1, x_2, y_1, y_2 are in W_0 and $x_1T_0y_1, x_2T_0y_2$, then $x_1 \vee x_2$, $y_1 \vee y_2$, $x_1 \wedge x_2$, $y_1 \wedge y_2$ are all in W_0 and $x_1 \vee x_2T_0y_1 \vee y_2$, $x_1 \wedge x_2T_0y_1 \wedge y_2$, which means $x_1 \vee x_2Ty_1 \vee y_2$, $x_1 \wedge$ $\wedge$ $x_2Ty_1 \wedge y_2$. Now if $x_1 = y_1 \in W - W_0$ and x_2, y_2 are in W_0 and $x_2T_0y_2$, we have $\varphi(x_1) \vee \varphi(x_2) = \varphi(y_1) \vee \varphi(y_2)$ in W_1. This means that either $x_1 \vee x_2 = y_1 \vee y_2$ or $x_1 \vee x_2$, $y_1 \vee y_2$ are both in W_0. In the first case evidently $x_1 \vee x_2Ty_1 \vee y_2$. In the second case $x_1 \vee x_2 = x_2$, $y_1 \vee y_2 = y_2$ (as proved above), thus again $x_1 \vee x_2Ty_1 \vee y_2$. Dually we prove $x_1 \wedge x_2Ty_1 \wedge y_2$.

Lemma 5. *Let W be a WA-lattice and let W_0 be a sub-WA-lattice of W. The necessary and sufficient condition for the existence of a homomorphism φ of W into a WA-lattice W_1 such that $\varphi(x) = \varphi(y)$ for $x \neq y$ if and only if $x \in W_0$, $y \in W_0$ is the following: for each $x \in W - W_0$ either $x \prec y$ for each $y \in W_0$ or $y \prec x$ for each $y \in W_0$ or none of the cases $x \prec y$, $y \prec x$ occurs for any $y \in W_0$.*

Proof. Necessity. Let $w \in W_0$, $x \in W - W_0$. In W_1 we have either $\varphi(x) \prec \varphi(w)$ of $\varphi(w) \succ \varphi(x)$, or none of the cases $\varphi(x) \lesssim \varphi(w)$, $\varphi(w) \gtrsim \varphi(x)$ occurs. In the first case $\varphi(x) \wedge \varphi(w) = \varphi(x)$ in W_1 and we must have $x \wedge y = x$ for any $y \in W_0$, which means $x \prec y$ for each $y \in W_0$. In the second case dually $y \prec x$ for each $y \in W_0$. In the third case $\varphi(x) \vee \varphi(w)$ and $\varphi(x) \wedge \varphi(w)$ are both distinct from both $\varphi(x)$, $\varphi(w)$. As they are distinct from $\varphi(w)$, the elements $u = \varphi^{-1}(\varphi(x) \vee \varphi(w))$, $v = \varphi^{-1}(\varphi(x) \wedge \varphi(w))$ are determined uniquely and are in $W - W_0$. But $\varphi(y) = \varphi(w)$ for each $y \in W_0$, thus $\varphi(u) = \varphi(x) \vee \varphi(y)$, $\varphi(v) = \varphi(x) \wedge \varphi(y)$ for each $y \in W_0$, which means $u = x \vee y$, $v = x \wedge y$ for each $y \in W_0$. As u, v are distinct from x, none of the cases $x \prec y$, $y \prec x$ occurs for any $y \in W_0$.

Sufficiency. If $x \in W - W_0$, $y \prec x$ for each $y \in W_0$, then $x \vee y = x$, $x \wedge y =$ $= y \in W_0$ for each $y \in W_0$. If $x \in W - W_0$, $x \prec y$ for each $y \in W_0$, then $x \vee y =$ $= y \in W_0$, $x \wedge y = x$. Let $x \in W - W_0$ and neither $x \prec y$ nor $y \prec x$ for any

$y \in W_0$. Choose $w \in W_0$. Then $x \vee w \succ w$ and thus $x \vee w \succ y$ for each $y \in W_0$. Suppose that for some $y_0 \in W_0$ there exists $z \prec x \vee w$ such that $x \leqq z$, $y_0 \leqq z$. Then $y_0 \prec z$ and $y \prec z$ for each $y \in W_0$, in particular $w \prec z$ and thus we have $w \prec z$, $x \prec z$, $z \prec x \vee w$, which is a contradiction. We have proved that $x \vee y = x \vee w$ for each $y \in W_0$. Dually we can prove $x \wedge y = x \wedge w$ for each $y \in W_0$. Thus we have proved that the mapping φ of W into W_1 such that $\varphi(x) = \varphi(y)$ for $x \neq y$ if and only if $x \in W_0$, $y \in W_0$ is a homomorphism.

Theorem 5. *Let W be a WA-lattice and let C be a sub-WA-lattice of W which is a chain with at least three elements. Let any element $x \in W - C$ be either greater than all elements of C or less than all elements of C, or such that neither $x \prec y$ nor $y \prec x$ for any element $y \in C$. Then there exists a tolerance compatible with W which is not a congruence.*

Proof. By Theorem 4 from [10] there exists a compatible tolerance which is not a congruence on each chain with at least three elements. By Lemma 5 the assumptions of Theorem 4 are fulfilled, thus according to Theorem 4 the assertion holds.

5. TOURNAMENTS

The algebraic definition of a tournament was given in § 1. Nonetheless as is well-known, a tournament can be defined also graph-theoretically.

A tournament is a directed graph without loops in which any two distinct vertices are joined exactly by one directed edge.

The two definitions of a tournament represent two different view-points from which this concept can be considered. Substantially they express the same thing. We can take a tournament W according to the algebraic definition and for any two its distinct elements a, b for which $a \vee b = b$ we join a and b by a directed edge outgoing from a and coming into b. Then we obtain a tournament according to the graph-theoretical definition. For any two distinct elements a, b either $a \vee b = b$ or $a \vee b = a$ and not both simultaneously; thus any two distinct elements are joined exactly by one directed edge and the set of elements of W can be viewed as the set of vertices of a tournament according to the graph-theoretical definition. On the other hand, let W be a tournament according to the graph-theoretical definition. For any two distinct elements a and b we put $a \vee b$, $a \wedge b = a$, if and only if there exists a directed edge from a into b; we obtain a tournament according to the algebraic definition.

This enables us to consider tournaments from the two view-points. We shall always use the view-point which will be more convenient for our considerations.

Now we shall prove some theorems concerning tolerances on tournaments.

Theorem 6. *Let W be a tournament which is not strongly connected [6] and which has at least three vertices. Then there exists a tolerance T compatible with W which is neither the identical relation nor the universal relation.*

Proof. As W is not strongly connected, it has at least two quasicomponents. For two quasicomponents C_1, C_2 of W let $C_1 < C_2$ if and only if $C_1 \neq C_2$ and each edge joining a vertex from C_1 with a vertex of C_2 has its terminal vertex in C_2. As is well-known from the graph theory, this ordering is complete. Let C be a quasicomponent of W which is not minimal in this ordering. Let W_1 (or W_2) be a subtournament of W induced by the vertices of all quasicomponents C' for which $C' < C$ (or $C' > C$, respectively). The vertex sets of W_1 and W_2 are disjoint, non-empty and each edge joining a vertex of W_1 with a vertex of W_2 has its terminal vertex in W_2. Now let T be a tolerance on W such that two elements are in T if and only if they are both in W_1 or both in W_2. Evidently T is neither the identical relation, nor the universal relation. Now let $x_1 T y_1$, $x_2 T y_2$. If all the elements x_1, y_1, x_2, y_2 are in W_1, then also $x_1 \vee x_2$, $x_1 \wedge x_2$, $y_1 \vee y_2$, $y_1 \wedge y_2$ are all in W_1 and $x_1 \vee x_2 T y_1 \vee y_2$, $x_1 \wedge x_2 T y_1 \wedge y_2$. Analogously if all the elements x_1, y_1, x_2, y_2 are in W_2. Now let x_1, y_1 be in W_1 and let x_2, y_2 be in W_2. We have $x_1 \vee x_2 = x_2$, $x_1 \wedge x_2 = x_1$, $y_1 \vee y_2 = y_2$, $y_1 \wedge y_2 = y_1$ and thus $x_1 \vee x_2 = x_2 T y_2 = y_1 \vee y_2$, $x_1 \wedge x_2 = x_1 T y_1 = y_1 \wedge y_2$. Analogously if x_1, y_1 are in W_2 and x_2, y_2 are in W_1.

Theorem 7. *Let W be a tournament having at least three quasicomponents. Then there exists a tolerance T compatible with W which is not a congruence.*

Proof. Let C_0 be a quasicomponent of W which is neither minimal nor maximal in the ordering described in the proof of Theorem 6. Let W_1 (or W_2) be a subtournament of W induced by the vertices of all quasicomponents C' of W for which $C' < C_0$ (or $C' > C_0$, respectively). The subtournaments W_1, W_2, C_0 are pairwise disjoint and all non-empty. Now let T be a tolerance on W such that two elements are in T if and only if they are both in $W_1 \cup C_0$ or both in $W_2 \cup C_0$. Any edge joining a vertex of $W_1 \cup C_0$ with a vertex of $W_2 \cup C_0$ has its terminal vertex in $W_2 \cup C_0$. Thus the proof of the compatibility of T is analogous to the proof of Theorem 6. Now if a is in W_1, b in C_0 and c in W_2, we have aTb, because both a, b are in $W_1 \cup C_0$, bTc, because both b, c are in $W_2 \cup C_0$, but a and c are not in T, because a is not in $W_2 \cup C_0$ and c is not in $W_1 \cup C_0$. The tolerance T is not transitive, therefore it is not a congruence.

Remark. We speak about vertices rather than about elements, because elements of a tournament in the graph theory are also edges.

Theorem 8. *If W is a strongly connected tournament with three vertices, then any tolerance compatible with it is either the identical relation or the universal relation. If W is a strongly connected tournament with four vertices, then there exists exactly one tolerance compatible with it which is neither the identical relation nor the universal relation.*

Proof. There exists only one (up to isomorphism) strongly connected tournament with three vertices, namely the cycle with three vertices. For it the assertion follows from Theorem 2. Now let W be a strongly connected tournament with four vertices. The tournament W cannot be acyclic. As is well-known, a tournament which is not acyclic contains at least one cycle with three vertices. Thus let $\{a, b, c\}$ be such a cycle, $a \prec b \prec c \prec a$. The fourth element d is neither greater than all a, b, c nor less than all a, b, c; otherwise W would not be strongly connected. Without loss of generality two cases may occur; either $d \succ a$, $d \succ b$, $d \prec c$ or $d \prec a$, $d \prec b$, $d \succ c$. The latter case is obtained from the former by the isomorphism induced by the permutation $(abcd) \to (cabd)$. Therefore there exists only one strongly connected tournament W with four vertices up to isomorphism; this tournament has vertices a, b, c, d and $a \prec b$, $a \succ c$, $a \prec d$, $b \prec c$, $b \prec d$, $c \succ d$. Let T be a tolerance on W consisting of the pairs (b, d), (d, b) and all pairs (x, x) for all vertices x of W. This tolerance is compatible with W; this can be easily proved. This tolerance T is a congruence and is neither the identical relation nor the universal relation. Let T' be a tolerance compatible with W which is neither the identical relation nor equal to T. Then T' must contain a pair of distinct elements other than (b, d) or (d, b). Both elements of such a pair either belong to $\{a, b, c\}$ or to $\{a, d, c\}$. Both these sets are cycles. If such a pair belongs to $\{a, b, c\}$, then the restriction of T' onto $\{a, b, c\}$ must be the universal relation according to Theorem 2. But then $aT'c$, both the elements a, c belong to the cycle $\{a, d, c\}$ and the restriction of T' onto $\{a, d, c\}$ must be the universal relation. Now from $aT'b, dT'a$ we obtain $d = a \vee dT'b \vee a = b$ and from the symmetry $bT'd$. Thus T' is the universal relation on W. Analogously if the pair belongs to $\{a, d, c\}$.

Definition 7. Let W be a tournament, let x and y be two of its vertices such that $x \prec z \prec y \prec x$ for each $z \in W - \{x, y\}$. Then we say that W is reducible and can be reduced onto $W_0 = W - \{x, y\}$ by deleting x and y.

This concept was defined in [7].

Lemma 6. *Let W be a reducible tournament which can be reduced onto a tournament W_0 by deleting its vertices x, y. Let T_0 be a tolerance compatible with W_0 which is not a congruence. Then there exists a tolerance compatible with W which is not a congruence.*

Proof follows from Lemma 5 and Theorem 4.

Theorem 9. *For each $n \geqq 5$ there exists a strongly connected tournament W with n vertices on which there exists a tolerance compatible with it which is not a congruence.*

Proof. If $n \geqq 5$, then $n - 2 \geqq 3$ and there exists a tournament W_0 with $n - 2$ vertices on which a compatible tolerance exists which is not a congruence; for example, a chain [10]. Let W be a tournament which can be reduced onto W_0 (in the sense of Definition 7). Then W is evidently strongly connected and has n vertices. According to Lemma 6 the assertion holds.

Here we shall give two theorems concerning tolerance relations on lattices. This is an addition to the paper [10].

Theorem 10. *Let L be a lattice. Let there exist a proper ideal J of L and a proper filter F of L such that $J \cup F = L$, $J \cap F \neq \emptyset$. Then there exists a tolerance T compatible with L which is not a congruence.*

Proof. Let T be a tolerance on L such that xTy if and only if x and y either both belong to J, or both belong to F. We shall prove that T is compatible with L. Let p, q, r, s be elements of L and pTq, rTs. This means that at least one of these cases occurs:

(i) $p \in J,\ q \in J,\ r \in J,\ s \in J$;

(ii) $p \in J,\ q \in J,\ r \in F,\ s \in F$;

(iii) $p \in F,\ q \in F,\ r \in J,\ s \in J$;

(iv) $p \in F,\ q \in F,\ r \in F,\ s \in F$.

In the case (i) the elements $p \wedge r$, $q \wedge s$, $p \vee r$, $q \vee s$ are all in J, because J is a sublattice of L. Thus $p \wedge rTq \wedge s$, $p \vee rTq \vee s$. In the case (ii) we have $p \wedge r \in J$, $q \wedge s \in J$, because J is an ideal, and $p \vee r \in F$, $q \vee s \in F$, because F is a filter; thus again $p \wedge rTq \wedge s$, $p \vee rTq \vee s$. The case (iii) is dual to the case (ii), the case (iv) is dual to (i). As J is a proper ideal of L, we have an element $a \in L - J$; as $J \vee F = L$, we have $a \in F$. As F is a proper filter of L, we have an element $b \in L - F$; as $J \cup F = L$, we have $b \in J$. As $J \cap F \neq \emptyset$, we have an element $c \in J \cap F$. Now aTc, because both a and c are in F, and bTc, because both b and c are in J. But the elements a and b are not in T, because $a \notin J$, $b \notin F$. The tolerance T is not transitive and is not a congruence.

Theorem 11. *Let L be a complete infinitely distributive non-complementary lattice. Let L be atomic and dually atomic. Then there exists a tolerance T compatible with L which is not a congruence.*

Proof. As L is non-complementary, there exists an element $a \in L$ to which no complement exists. This means that $a \wedge x = O$ implies $a \vee x \prec I$ for each $x \in L$. (The symbols O and I denote the least and the greatest element of L, respectively. These elements exist, because L is atomic and dually atomic.) Let $B = \{x \in L \mid a \wedge x = 0\}$. Denote $b = \bigvee_{x \in B} x$; this element exists, because L is complete. We have $a \wedge b = a \wedge \bigvee_{x \in B} x = \bigvee_{x \in B} a \wedge x = O$, because L is infinitely distributive. Let $c = a \vee b$; we have $c \prec I$. Further, let d be a dual atom of L such that $d \geqq c$. Denote $J = \langle O, d \rangle$; this is a proper ideal of L. As d is a dual atom, we have either $x \leqq d$

or $x \vee d = I$ for each $x \in L$. Let $E = \{x \in L \mid x \vee d = I\}$. Denote $e = \bigwedge_{x \in E} x$; we have $d \vee e = d \vee \bigwedge_{x \in E} x = \bigwedge_{x \in E} d \vee x = I$. Thus $e \in E$. From the definition of E we see that $E = \langle e, I \rangle$ and further $J \cap E = \emptyset$, $J \cup E = L$. Let f be an atom of L such that $f \leqq e$. As f is an atom, we have either $x \geqq f$ or $x \wedge f = O$ for each $x \in L$. Both these cases mean that $f \in J$. Let $F = \langle f, I \rangle$; this is a proper filter in L. As $E \subset F$ and $J \cup E = L$, we have $J \cup F = L$. Now according to Theorem 10 there exists a compatible tolerance on L which is not a congruence.

7. PROBLEMS

Problem 1. For which regular *WU*-systems is any compatible tolerance equal either to the identical relation or to the universal relation? For which regular *WU*-systems is any compatible tolerance a congruence?

Problem 2. According to Theorem 10 in [10] the set of all compatible tolerances of an algebra forms a lattice. For which *WA*-lattices is this lattice distributive (or modular, complementary etc.)?

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