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ON THE EXISTENCE OF PARALLEL NORMAL VECTOR
FIELDS OF SURFACES IN E^4

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In the last time, there have been achieved many results about submanifolds possessing a parallel normal vector field; see, p.ex., [1] and [2]. In what follows, I am going to show a simple condition which ensures the existence of such a vector field in the case of a surface of a 4-space.

Let $M \subset E^4$ be a surface of class C^∞ . Let $\{U_\alpha\}$ be its cover such that in each domain U_α there is a field of orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ such that $v_1, v_2 \in T(M)$, $v_3, v_4 \in N(M)$; $N(M)$ be the normal bundle of M . The fundamental equations of our surface are

$$\begin{aligned}
 (1) \quad dM &= \omega^1 v_1 + \omega^2 v_2, \\
 dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4, \\
 dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4, \\
 dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4, \\
 dv_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3;
 \end{aligned}$$

$$(2) \quad d\omega^i = \omega^j \wedge \omega_j^i, \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j; \quad \omega_i^i + \omega_j^j = 0;$$

$$(3) \quad \omega^3 = \omega^4 = 0.$$

From (3), we get (by means of exterior differentiation and the application of Cartan's lemma) the existence of functions $a_1, \dots, b_3, \alpha_1, \dots, \beta_4, A_1, \dots, B_5$ such that

$$(4) \quad \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0, \quad \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0;$$

$$\begin{aligned}
 (5) \quad \omega_1^3 &= a_1 \omega^1 + a_2 \omega^2, & \omega_2^3 &= a_2 \omega^1 + a_3 \omega^2, \\
 \omega_1^4 &= b_1 \omega^1 + b_2 \omega^2, & \omega_2^4 &= b_2 \omega^1 + b_3 \omega^2;
 \end{aligned}$$

$$(6) \quad \begin{aligned} &\{da_1 - 2a_2\omega_1^2 - b_1\omega_3^4\} \wedge \omega^1 + \{da_2 + (a_1 - a_3)\omega_1^2 - b_2\omega_3^4\} \wedge \omega^2 = 0, \\ &\{da_2 + (a_1 - a_3)\omega_1^2 - b_2\omega_3^4\} \wedge \omega^1 + \{da_3 + 2a_2\omega_1^2 - b_3\omega_3^4\} \wedge \omega^2 = 0, \\ &\{db_1 - 2b_2\omega_1^2 + a_1\omega_3^4\} \wedge \omega^1 + \{db_2 + (b_1 - b_3)\omega_1^2 + a_2\omega_3^4\} \wedge \omega^2 = 0, \\ &\{db_2 + (b_1 - b_3)\omega_1^2 + a_2\omega_3^4\} \wedge \omega^1 + \{db_3 + 2b_2\omega_1^2 + a_3\omega_3^4\} \wedge \omega^2 = 0; \end{aligned}$$

$$(7) \quad \begin{aligned} da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 &= \alpha_1\omega^1 + \alpha_2\omega^2, \\ da_2 + (a_1 - a_3)\omega_1^2 - b_2\omega_3^4 &= \alpha_2\omega^1 + \alpha_3\omega^2, \\ da_3 + 2a_2\omega_1^2 - b_3\omega_3^4 &= \alpha_3\omega^1 + \alpha_4\omega^2, \\ db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 &= \beta_1\omega^1 + \beta_2\omega^2, \\ db_2 + (b_1 - b_3)\omega_1^2 + a_2\omega_3^4 &= \beta_2\omega^1 + \beta_3\omega^2, \\ db_3 + 2b_2\omega_1^2 + a_3\omega_3^4 &= \beta_3\omega^1 + \beta_4\omega^2; \end{aligned}$$

$$(8) \quad \begin{aligned} \{d\alpha_1 - 3\alpha_2\omega_1^2 - \beta_1\omega_3^4\} \wedge \omega^1 + \{d\alpha_2 + (\alpha_1 - 2\alpha_3)\omega_1^2 - \beta_2\omega_3^4\} \wedge \omega^2 &= \\ &= (2a_2K + b_1k)\omega^1 \wedge \omega^2, \\ \{d\alpha_2 + (\alpha_1 - 2\alpha_3)\omega_1^2 - \beta_2\omega_3^4\} \wedge \omega^1 + \{d\alpha_3 + (2\alpha_2 - \alpha_4)\omega_1^2 - \beta_3\omega_3^4\} \wedge \omega^2 &= \\ &= (a_3K - a_1K + b_2k)\omega^1 \wedge \omega^2, \\ \{d\alpha_3 + (2\alpha_2 - \alpha_4)\omega_1^2 - \beta_3\omega_3^4\} \wedge \omega^1 + \{d\alpha_4 + 3\alpha_3\omega_1^2 - \beta_4\omega_3^4\} \wedge \omega^2 &= \\ &= (-2a_2K + b_3k)\omega^1 \wedge \omega^2, \\ \{d\beta_1 - 3\beta_2\omega_1^2 + \alpha_1\omega_3^4\} \wedge \omega^1 + \{d\beta_2 + (\beta_1 - 2\beta_3)\omega_1^2 + \alpha_2\omega_3^4\} \wedge \omega^2 &= \\ &= (2b_2K - a_1k)\omega^1 \wedge \omega^2, \\ \{d\beta_2 + (\beta_1 - 2\beta_3)\omega_1^2 + \alpha_2\omega_3^4\} \wedge \omega^1 + \{d\beta_3 + (2\beta_2 - \beta_4)\omega_1^2 + \alpha_3\omega_3^4\} \wedge \omega^2 &= \\ &= (b_3K - b_1K - a_2k)\omega^1 \wedge \omega^2, \\ \{d\beta_3 + (2\beta_2 - \beta_4)\omega_1^2 + \alpha_3\omega_3^4\} \wedge \omega^1 + \{d\beta_4 + 3\beta_3\omega_1^2 + \alpha_4\omega_3^4\} \wedge \omega^2 &= \\ &= (-2b_2K - a_3k)\omega^1 \wedge \omega^2; \end{aligned}$$

$$(9) \quad \begin{aligned} d\alpha_1 - 3\alpha_2\omega_1^2 - \beta_1\omega_3^4 &= A_1\omega^1 + (A_2 - a_2K - \frac{1}{2}b_1k)\omega^2, \\ d\alpha_2 + (\alpha_1 - 2\alpha_3)\omega_1^2 - \beta_2\omega_3^4 &= (A_2 + a_2K + \frac{1}{2}b_1k)\omega^1 + (A_3 + a_1K - \frac{1}{2}b_2k)\omega^2, \\ d\alpha_3 + (2\alpha_2 - \alpha_4)\omega_1^2 - \beta_3\omega_3^4 &= (A_3 + a_3K + \frac{1}{2}b_2k)\omega^1 + (A_4 + a_2K - \frac{1}{2}b_3k)\omega^2, \\ d\alpha_4 + 3\alpha_3\omega_1^2 - \beta_4\omega_3^4 &= (A_4 - a_2K + \frac{1}{2}b_3k)\omega^1 + A_5\omega^2, \\ d\beta_1 - 3\beta_2\omega_1^2 + \alpha_1\omega_3^4 &= B_1\omega^1 + (B_2 - b_2K + \frac{1}{2}a_1k)\omega^2, \end{aligned}$$

$$\begin{aligned}d\beta_2 + (\beta_1 - 2\beta_3)\omega_1^2 + \alpha_2\omega_3^4 &= (B_2 + b_2K - \frac{1}{2}a_1k)\omega^1 + (B_3 + b_1K + \frac{1}{2}a_2k)\omega^2, \\d\beta_3 + (2\beta_2 - \beta_4)\omega_1^2 + \alpha_3\omega_3^4 &= (B_3 + b_3K - \frac{1}{2}a_2k)\omega^1 + (B_4 + b_2K + \frac{1}{2}a_3k)\omega^2, \\d\beta_4 + 3\beta_3\omega_1^2 + \alpha_4\omega_3^4 &= (B_4 - b_2K - \frac{1}{2}a_3k)\omega^1 + B_5\omega^2,\end{aligned}$$

where

$$(10) \quad K = a_1a_3 - a_2^2 + b_1b_3 - b_2^2, \quad k = (a_1 - a_3)b_2 - (b_1 - b_3)a_2,$$

i.e.,

$$(11) \quad d\omega_1^2 = -K\omega^1 \wedge \omega^2, \quad d\omega_3^4 = -k\omega^1 \wedge \omega^2.$$

Let $\{M; v_1^*, v_2^*, v_3^*, v_4^*\}$ be another field of frames over U_a ; let

$$(12) \quad \begin{aligned}v_1 &= \varepsilon_1 \cos \varrho \cdot v_1^* - \varepsilon_1 \sin \varrho \cdot v_2^*, & v_2 &= \sin \varrho \cdot v_1^* + \cos \varrho \cdot v_2^*, \\v_3 &= \varepsilon_2 \cos \sigma \cdot v_3^* - \varepsilon_2 \sin \sigma \cdot v_4^*, & v_4 &= \sin \sigma \cdot v_3^* + \cos \sigma \cdot v_4^*; \\ \varepsilon_1^2 &= \varepsilon_2^2 = 1.\end{aligned}$$

Write $dM = \Omega^i v_i^*$, $dv_i^* = \Omega_j^i v_j^*$; the corresponding functions be denoted by $*$. From (1) and (1*),

$$(13) \quad \Omega^1 = \varepsilon_1 \cos \varrho \cdot \omega^1 + \sin \varrho \cdot \omega^2; \quad \Omega^2 = -\varepsilon_1 \sin \varrho \cdot \omega^1 + \cos \varrho \cdot \omega^2;$$

$$(14) \quad \Omega_1^2 = \varepsilon_1 \omega_1^2 + d\varrho, \quad \Omega_3^4 = \varepsilon_2 \omega_3^4 + d\sigma;$$

$$(15) \quad \begin{aligned}\Omega_1^3 &= \varepsilon_1 \varepsilon_2 \cos \varrho \cos \sigma \cdot \omega_1^3 + \varepsilon_2 \sin \varrho \cos \sigma \cdot \omega_2^3 + \varepsilon_1 \cos \varrho \sin \sigma \cdot \omega_1^4 + \\ &+ \sin \varrho \sin \sigma \cdot \omega_2^4,\end{aligned}$$

$$\begin{aligned}\Omega_2^3 &= -\varepsilon_1 \varepsilon_2 \sin \varrho \cos \sigma \cdot \omega_1^3 + \varepsilon_2 \cos \varrho \cos \sigma \cdot \omega_2^3 - \varepsilon_1 \sin \varrho \sin \sigma \cdot \omega_1^4 + \\ &+ \cos \varrho \sin \sigma \cdot \omega_2^4,\end{aligned}$$

$$\begin{aligned}\Omega_1^4 &= -\varepsilon_1 \varepsilon_2 \cos \varrho \sin \sigma \cdot \omega_1^3 - \varepsilon_2 \sin \varrho \sin \sigma \cdot \omega_2^3 + \varepsilon_1 \cos \varrho \cos \sigma \cdot \omega_1^4 + \\ &+ \sin \varrho \cos \sigma \cdot \omega_2^4,\end{aligned}$$

$$\begin{aligned}\Omega_2^4 &= \varepsilon_1 \varepsilon_2 \sin \varrho \sin \sigma \cdot \omega_1^3 - \varepsilon_2 \cos \varrho \sin \sigma \cdot \omega_2^3 - \varepsilon_1 \sin \varrho \cos \sigma \cdot \omega_1^4 + \\ &+ \cos \varrho \cos \sigma \cdot \omega_2^3\end{aligned}$$

and

$$(16) \quad \begin{aligned}a_1^* &= \varepsilon_2 \cos \sigma \cdot (\cos^2 \varrho \cdot a_1 + \varepsilon_1 \sin 2\varrho \cdot a_2 + \sin^2 \varrho \cdot a_3) + \\ &+ \sin \sigma \cdot (\cos^2 \varrho \cdot b_1 + \varepsilon_1 \sin 2\varrho \cdot b_2 + \sin^2 \varrho \cdot b_3),\end{aligned}$$

$$\begin{aligned}
a_2^* &= \varepsilon_2 \cos \sigma \cdot \left(-\frac{1}{2} \sin 2\varrho \cdot a_1 + \varepsilon_1 \cos 2\varrho \cdot a_2 + \frac{1}{2} \sin 2\varrho \cdot a_3\right) + \\
&\quad + \sin \sigma \cdot \left(-\frac{1}{2} \sin 2\varrho \cdot b_1 + \varepsilon_1 \cos 2\varrho \cdot b_2 + \frac{1}{2} \sin 2\varrho \cdot b_3\right), \\
a_3^* &= \varepsilon_2 \cos \sigma \cdot \left(\sin^2 \varrho \cdot a_1 - \varepsilon_1 \sin 2\varrho \cdot a_2 + \cos^2 \varrho \cdot a_3\right) + \\
&\quad + \sin \sigma \cdot \left(\sin^2 \varrho \cdot b_1 - \varepsilon_1 \sin 2\varrho \cdot b_2 + \cos^2 \varrho \cdot b_3\right), \\
b_1^* &= -\varepsilon_2 \sin \sigma \cdot \left(\cos^2 \varrho \cdot a_1 + \varepsilon_1 \sin 2\varrho \cdot a_2 + \sin^2 \varrho \cdot a_3\right) + \\
&\quad + \cos \sigma \cdot \left(\cos^2 \varrho \cdot b_1 + \varepsilon_1 \sin 2\varrho \cdot b_2 + \sin^2 \varrho \cdot b_3\right), \\
b_2^* &= \varepsilon_2 \sin \sigma \cdot \left(\frac{1}{2} \sin 2\varrho \cdot a_1 - \varepsilon_1 \cos 2\varrho \cdot a_2 - \frac{1}{2} \sin 2\varrho \cdot a_3\right) - \\
&\quad - \cos \sigma \cdot \left(\frac{1}{2} \sin 2\varrho \cdot b_1 - \varepsilon_1 \cos 2\varrho \cdot b_2 - \frac{1}{2} \sin 2\varrho \cdot b_3\right), \\
b_3^* &= -\varepsilon_2 \sin \sigma \cdot \left(\sin^2 \varrho \cdot a_1 - \varepsilon_1 \sin 2\varrho \cdot a_2 + \cos^2 \varrho \cdot a_3\right) + \\
&\quad + \cos \sigma \cdot \left(\sin^2 \varrho \cdot b_1 - \varepsilon_1 \sin 2\varrho \cdot b_2 + \cos^2 \varrho \cdot b_3\right).
\end{aligned}$$

Thus

$$\begin{aligned}
(17) \quad a_1^* + a_3^* &= \varepsilon_2 \cos \sigma \cdot (a_1 + a_3) + \sin \sigma \cdot (b_1 + b_3), \\
b_1^* + b_3^* &= -\varepsilon_2 \sin \sigma \cdot (a_1 + a_3) + \cos \sigma \cdot (b_1 + b_3),
\end{aligned}$$

and we get from (7) and (17),

$$\begin{aligned}
(18) \quad \alpha_1^* + \alpha_3^* &= \varepsilon_1 \varepsilon_2 \cos \varrho \cos \sigma \cdot (\alpha_1 + \alpha_3) + \varepsilon_2 \sin \varrho \cos \sigma \cdot (\alpha_2 + \alpha_4) + \\
&\quad + \varepsilon_1 \cos \varrho \sin \sigma \cdot (\beta_1 + \beta_3) + \sin \varrho \sin \sigma \cdot (\beta_2 + \beta_4), \\
\alpha_2^* + \alpha_4^* &= -\varepsilon_1 \varepsilon_2 \sin \varrho \cos \sigma \cdot (\alpha_1 + \alpha_3) + \varepsilon_2 \cos \varrho \cos \sigma \cdot (\alpha_2 + \alpha_4) - \\
&\quad - \varepsilon_1 \sin \varrho \sin \sigma \cdot (\beta_1 + \beta_3) + \cos \varrho \sin \sigma \cdot (\beta_2 + \beta_4), \\
\beta_1^* + \beta_3^* &= -\varepsilon_1 \varepsilon_2 \cos \varrho \sin \sigma \cdot (\alpha_1 + \alpha_3) - \varepsilon_2 \sin \varrho \sin \sigma \cdot (\alpha_2 + \alpha_4) + \\
&\quad + \varepsilon_1 \cos \varrho \cos \sigma \cdot (\beta_1 + \beta_3) + \sin \varrho \cos \sigma \cdot (\beta_2 + \beta_4), \\
\beta_2^* + \beta_4^* &= \varepsilon_1 \varepsilon_2 \sin \varrho \sin \sigma \cdot (\alpha_1 + \alpha_3) - \varepsilon_2 \cos \varrho \sin \sigma \cdot (\alpha_2 + \alpha_4) - \\
&\quad - \varepsilon_1 \sin \varrho \cos \sigma \cdot (\beta_1 + \beta_3) + \cos \varrho \cos \sigma \cdot (\beta_2 + \beta_4).
\end{aligned}$$

From (10) and (11),

$$(19) \quad K^* = K, \quad k^* = \varepsilon_1 \varepsilon_2 k.$$

The mean curvature vector being

$$(20) \quad \xi = (a_1 + a_3) v_3 + (b_1 + b_3) v_4,$$

we have

$$(21) \quad \xi^* = \xi ;$$

for the *mean curvature*

$$(22) \quad H = \|\ddot{\xi}\|^2 = (a_1 + a_3)^2 + (b_1 + b_3)^2 ,$$

we get

$$(23) \quad H^* = H .$$

Let $m \in M$ be a fixed point. Introduce the linear maps

$$(24) \quad C_m^T : T_m(M) \rightarrow T_m(M) , \quad C_m^N : T_m(M) \rightarrow N_m(M)$$

by

$$(25) \quad C_m^T(v) = \pi_m^N(v\xi) , \quad C_m^N(v) = \pi_m^N(v\xi) \quad \text{for } v \in T_m(M) ,$$

π_m^T or π_m^N being the orthogonal projection onto $T_m(M)$ or $N_m(M)$ resp. Because of

$$(26) \quad d\xi = - \{ (a_1 + a_3)(a_1\omega^1 + a_2\omega^2) + (b_1 + b_3)(b_1\omega^1 + b_2\omega^2) \} v_1 - \\ - \{ (a_1 + a_3)(a_2\omega^1 + a_3\omega^2) + (b_1 + b_3)(b_2\omega^1 + b_3\omega^2) \} v_2 + \\ + \{ (\alpha_1 + \alpha_3)\omega^1 + (\alpha_2 + \alpha_4)\omega^2 \} v_3 + \\ + \{ (\beta_1 + \beta_3)\omega^1 + (\beta_2 + \beta_4)\omega^2 \} v_4 ,$$

we get

$$(27) \quad C^T(v_1) = - \{ (a_1 + a_3)a_1 + (b_1 + b_3)b_1 \} v_1 - \\ - \{ (a_1 + a_3)a_2 + (b_1 + b_3)b_2 \} v_2 ,$$

$$C^T(v_2) = - \{ (a_1 + a_3)a_2 + (b_1 + b_3)b_2 \} v_1 - \\ - \{ (a_1 + a_3)a_3 + (b_1 + b_3)b_3 \} v_2 ;$$

$$(28) \quad C^N(v_1) = (\alpha_1 + \alpha_3)v_3 + (\beta_1 + \beta_3)v_4 ,$$

$$C^N(v_2) = (\alpha_2 + \alpha_4)v_3 + (\beta_2 + \beta_4)v_4 .$$

From (18),

$$(29) \quad \begin{vmatrix} \alpha_1^* + \alpha_3^* & \beta_1^* + \beta_3^* \\ \alpha_2^* + \alpha_4^* & \beta_2^* + \beta_4^* \end{vmatrix} = \varepsilon_1 \varepsilon_2 \begin{vmatrix} \alpha_1 + \alpha_3 & \beta_1 + \beta_3 \\ \alpha_2 + \alpha_4 & \beta_2 + \beta_4 \end{vmatrix} .$$

Let us prove the following auxiliary results: (i) If $C^N(v) = 0$ for each $v \in T(M)$ and $H \neq 0$ on M , then $k = 0$. (ii) Let the normal bundle $N(M)$ be trivial. Then $k = 0$ if and only if there is, in $N(M)$, a non-zero parallel vector field.

Indeed, suppose $C^N = 0$, i.e.,

$$(30) \quad \alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = \beta_1 + \beta_3 = \beta_2 + \beta_4 = 0.$$

Then (9) implies

$$(31) \quad \begin{aligned} A_1 + A_3 + a_3K + \frac{1}{2}b_2k &= 0, & A_2 + A_4 - \frac{1}{2}(b_1 + b_3)k &= 0, \\ A_2 + A_4 + \frac{1}{2}(b_1 + b_3)k &= 0, & A_3 + A_5 + a_1K - \frac{1}{2}b_2k &= 0, \\ B_1 + B_3 + b_3K - \frac{1}{2}a_2k &= 0, & B_2 + B_4 + \frac{1}{2}(a_1 + a_3)k &= 0, \\ B_2 + B_4 - \frac{1}{2}(a_1 + a_3)k &= 0, & B_3 + B_5 + b_1K + \frac{1}{2}a_2k &= 0; \end{aligned}$$

from (31_{2,3,6,7}), we get $(a_1 + a_3)k = (b_1 + b_3)k = 0$ and $k = 0$ because of $H \neq 0$. Let $v = xv_3 + yv_4$ be a normal vector field. Then

$$(32) \quad \begin{aligned} dv &= -(x\omega_1^3 + y\omega_1^4)v_1 - (x\omega_2^3 + y\omega_2^4)v_2 + \\ &\quad + (dx - y\omega_3^4)v_3 + (dy + x\omega_3^4)v_4 \end{aligned}$$

and v is parallel if and only if

$$(33) \quad dx - y\omega_3^4 = 0, \quad dy + x\omega_3^4 = 0.$$

The integrability conditions of (33) are $x d\omega_3^4 = y d\omega_3^4 = 0$, i.e., $xk = yk = 0$.

Suppose the space E^4 to be oriented and the moving frames to be positively oriented. Then

$$(34) \quad \varepsilon_1\varepsilon_2 = 1.$$

Further, suppose that the bundle $N(M)$ is trivial and $\xi \neq 0$ on M . The frames may be chosen in such a way that v_3 and ξ are linearly dependent, i.e.,

$$(35) \quad b_1 + b_3 = 0;$$

from (17₂),

$$(36) \quad \sin \sigma = 0.$$

Introduce the form

$$(37) \quad \tau = (a_1 + a_3)^2 \omega_3^4;$$

we get

$$(38) \quad \tau^* = \varepsilon_2 \tau$$

and

$$(39) \quad d\tau = \{2(\alpha_1 + \alpha_3)(\beta_2 + \beta_4) - 2(\alpha_2 + \alpha_4)(\beta_1 + \beta_3) - (a_1 + a_3)^2 k\} \omega^1 \wedge \omega^2.$$

From (7),

$$(40) \quad \tau = (a_1 + a_3) \{(\beta_1 + \beta_3) \omega^1 + (\beta_2 + \beta_4) \omega^2\}.$$

The integral formula

$$(41) \quad \int_{\partial M} (a_1 + a_3) \{(\beta_1 + \beta_3) \omega^1 + (\beta_2 + \beta_4) \omega^2\} = \int_M (2 \det C^N - Hk) \omega^1 \wedge \omega^2$$

and (28) imply the following

Theorem. *Let $M \subset E^4$ be a surface of class C^∞ , and suppose: (i) the normal bundle $N(M)$ is trivial; (ii) $H \neq 0$ on M ; (iii) on M , $\det C^N \geq 0$ and $k \leq 0$ or $\det C^N \leq 0$ and $k \geq 0$ resp.; (iv) $C^N(T_m(M)) \subset \xi_m$ for each point $m \in \partial M$. Then $k = 0$ on M , i.e., M admits a parallel normal vector field.*

Notice that $H = \text{const.}$ implies $\det C^N = 0$. Indeed: from $(7_{1,3})$, we get $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = 0$.

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