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ON EXISTENCE CONDITIONS FOR COMPATIBLE TOLERANCES

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1. Conditions for the existence of compatible tolerances on various algebras which are not congruences were studied in many papers (see [1], [3]–[10]). The problem of finding necessary and sufficient conditions is still open, although in [10] one of such conditions was formulated (see Theorem 5 in [10]) for *WA*-lattices and lattices; however, this condition assumes the existence of a compatible tolerance which is not a congruence on a sublattice.

Some new conditions for the existence of compatible tolerances which are not congruences are established in this paper.

2. The symbol $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ will denote an algebra with the support A and with the set of fundamental operations $\mathscr{F}$. A tolerance relation on a set M is a reflexive and symmetric relation on M. In particular, each equivalence on M is a tolerance on M. A tolerance relation T on the set A is called compatible with $\mathfrak{A}$, if and only if for each n-ary operation $f \in \mathscr{F}$ (where n is a positive integer) and for any $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ of A which fulfil $x_i T y_i$ for $i = 1, \ldots, n$ we have $f(x_1, \ldots, x_n) \,.\, T f(y_1, \ldots, y_n)$.

3. Every equivalence relation is a tolerance relation. As is well-known, every equivalence on a set M determines a certain partition on M; the classes of this partition are called equivalence classes. Here we shall formulate an analogous result for tolerance relations.

Definition. Let M be a non-empty set. The family $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$, where Γ is a subscript set, is called *a covering of M by subsets*, if and only if each M_γ for $\gamma \in \Gamma$ is a subset of M and $\bigcup_{\gamma \in \Gamma} M_\gamma = M$. (We suppose $M_{\gamma_1} \neq M_{\gamma_2}$ for $\gamma_1 \in \Gamma, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2$.) A covering $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$ of a set M by subsets is called *a τ-covering of M*, if and only if $\mathfrak{M}$ fulfils the following two conditions:

(1) if $\gamma_0 \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, then

$$M_{\gamma_0} \subseteq \bigcup_{\gamma\in\Gamma_0} M_\gamma \Rightarrow \bigcap_{\gamma\in\Gamma_0} M_\gamma \subseteq M_{\gamma_0};$$

(2) if $N \subseteq M$ and N is not contained in any set from $\mathfrak{M}$, then N contains a two-element subset of the same property.

In particular, if $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$ is a τ-covering of M, then $M_{\gamma_1} \subseteq M_{\gamma_2}$ for $\gamma_1 \in \Gamma$, $\gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$. This can be proved by putting $\gamma_0 = \gamma_1$, $\Gamma_0 = \{\gamma_2\}$. This implies also that all the sets of $\mathfrak{M}$ are non-empty.

Theorem 1. *Let M be a non-empty set. Then there exists a one-to-one correspondence between tolerance relations on M and τ-coverings of $\mathfrak{M}$ such that if T is a tolerance relation on M and $\mathfrak{M}_T$ is the τ-covering of M corresponding to T, then any two elements of M are in the relation T if and only if there exists a set from $\mathfrak{M}_T$ which contains both of them.*

Proof. Let T be a tolerance relation on M. Let $\mathfrak{L}_T$ be the family of all subsets of M with the property that any two elements of the subset are in T. The family $\mathfrak{L}_T$ contains all one-element subsets of M, therefore it is a covering of M by subsets. Let $\mathfrak{M}_T$ be the family of all sets of $\mathfrak{L}_T$ which are maximal with respect to the set inclusion (according to Zorn's Lemma such elements exist). Each set from $\mathfrak{L}_T$ is contained in a set from $\mathfrak{M}_T$ and $\mathfrak{L}_T$ is a covering of M, therefore also $\mathfrak{M}_T$ is a covering of M. Let $\mathfrak{M}_T = \{M_\gamma, \gamma \in \Gamma\}$, where Γ is a subscript set. Now let $\gamma_0 \in \Gamma$, $\Gamma_0 \subseteq \Gamma$ and let $M_{\gamma_0} \subseteq \bigcup_{\gamma\in\Gamma_0} M_\gamma$. Let $P = \bigcap_{\gamma\in\Gamma_0} M_\gamma$ and suppose $P \subseteq M_{\gamma_0}$. Let $x \in P - M_{\gamma_0}$, $y \in M_{\gamma_0}$. This means $y \in \bigcup_{\gamma\in\Gamma_0} M_\gamma$ and thus there exists $\gamma_1 \in \Gamma_0$ such that $y \in M_{\gamma_1}$. As $x \in P - M_{\gamma_0}$, we have $x \in P = \bigcap_{\gamma\in\Gamma_0} M_\gamma$ and thus also $x \in M_{\gamma_1}$. We have xTy. As y was chosen arbitrarily, we have xTy for each $y \in M_{\gamma_0}$. Thus the set $M_{\gamma_0} \cup \{x\} \in \mathfrak{L}_T$ and M_{γ_0} is its proper subset; this means $M_{\gamma_0} \notin \mathfrak{M}_T$, which is a contradiction. We have necessarily $\bigcap_{\gamma\in\Gamma_0} M_\gamma \subseteq M_{\gamma_0}$ and (1) is fulfilled. Now if a subset N of M is not contained in any set from $\mathfrak{M}_T$, then $N \notin \mathfrak{L}_T$ and there exist two elements a, b of N which are not in the relation T. Thus the set $\{a, b\}$ is not contained in any set from $\mathfrak{M}_T$ and (2) is fulfilled. We have proved that $\mathfrak{M}_T$ is a τ-covering. Now let $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$ be a τ-covering of M and let T be a relation on M such that xTy for $x \in M$, $y \in M$ if and only if there exists $\gamma \in \Gamma$ such that $x \in M_\gamma$, $y \in M_\gamma$. The relation T is evidently a tolerance. Now it remains to prove that if $\mathfrak{M}_T$ is assigned to T according to the above rule, then $\mathfrak{M}_T = \mathfrak{M}$. This means to prove that each M_γ for $\gamma \in \Gamma$ is a maximal element in $\mathfrak{L}_T$ and each maximal element of $\mathfrak{L}_T$ is in $\mathfrak{M}$. Suppose that M_{γ_1} for some $\gamma_1 \in \Gamma$ is not a maximal element in $\mathfrak{L}_T$; this means that there exists $L \in \mathfrak{L}_T$ such that M_{γ_1} is a proper subset of L. Let $x \in L - M_{\gamma_1}$. As $L \in \mathfrak{L}_T$, $M_{\gamma_1} \subset L$, $x \in L$, we have xTy for each $y \in M_{\gamma_1}$. This means that to each $y \in M_{\gamma_1}$ there exists $\gamma(y) \in \Gamma$ so that

$y \in M_{\gamma(y)}$, $x \in M_{\gamma(y)}$. We have $M_{\gamma_1} \subseteq \bigcup_{y \in M_{\gamma_1}} M_{\gamma(y)}$. As $\mathfrak{M}_T$ is a τ-covering, it is necessarily $\bigcap_{y \in M_{\gamma_1}} M_{\gamma(y)} \subseteq M_{\gamma_1}$. But $x \in M_{\gamma(y)}$ for each $y \in M_{\gamma_1}$, thus $x \in \bigcap_{y \in M_{\gamma_1}} M_{\gamma(y)}$ and $x \in M_{\gamma_1}$, which is a contradiction. Now suppose that there exists a set $L' \in \mathfrak{M}_T - \mathfrak{M}$. As $\mathfrak{M} \subseteq \mathfrak{M}_T$, the set L' is not contained in any set from $\mathfrak{M}$. Thus there exist two elements c, d of L' such that the set $\{c, d\}$ is not contained in any set from $\mathfrak{M}$. This means that c, d are not in the relation T, thus $L' \notin \mathfrak{L}_T$ and also $L' \notin \mathfrak{M}_T$, which is a contradiction.

When T is an equivalence relation, the corresponding τ-covering $\mathfrak{M}_T$ is the partition of M into equivalence classes of T. This follows from the construction of $\mathfrak{M}_T$.

Theorem 2. *Let M be a non-empty set, let T_1 and T_2 be tolerances on M. Let $T = T_1 \cap T_2$. Let $\mathfrak{M}_{T_1}, \mathfrak{M}_{T_2}, \mathfrak{M}_T$ be the τ-coverings of M corresponding to T_1, T_2, T respectively. Then each set of $\mathfrak{M}_T$ is the intersection of a set from $\mathfrak{M}_{T_1}$ and a set from $\mathfrak{M}_{T_2}$. Any intersection of a set from $\mathfrak{M}_{T_1}$ and a set from $\mathfrak{M}_{T_2}$ is a subset of some set from $\mathfrak{M}_T$.*

Proof. Let $M_0 \in \mathfrak{M}_T$. Then, as we have seen in the proof of Theorem 1, any two elements of M_0 are in T, this means simultaneously in T_1 and T_2. Thus $M_0 \in \mathfrak{L}_{T_1}$, $M_0 \in \mathfrak{L}_{T_2}$ and there exist sets $M_1 \in \mathfrak{M}_{T_1}$, $M_2 \in \mathfrak{M}_{T_2}$ such that $M_0 \subseteq M_1$, $M_0 \subseteq M_2$, this means $M_0 \subseteq M_1 \cap M_2$. On the other hand, any two elements of $M \cap M_2$ are in T, thus $M_1 \cap M_2 \in \mathfrak{L}_T$ and there exists $M'_0 \in \mathfrak{M}_T$ such that $M_1 \cap M_2 \subseteq M'_0$. We have $M_0 \subseteq M'_0$; as no set from $\mathfrak{M}_T$ is a proper subset of another, we have $M_0 = M'_0$ and then also $M_0 = M_1 \cap M_2$. Now let $N_1 \in \mathfrak{M}_{T_1}$, $N_2 \in \mathfrak{M}_{T_2}$. If $N_1 \cap N_2 = \emptyset$, then this set is a subset of every set. Thus let $N_1 \cap N_2 \neq \emptyset$. Any two elements of $N_1 \cap N_2$ are simultaneously in T_1 and T_2, thus they are in T and $N_1 \cap N_2 \in \mathfrak{L}_T$. Thus there exists $N_0 \in \mathfrak{M}_T$ such that $N_1 \cap N_2 \subseteq N_0$.

This theorem cannot be strengthened so that any intersection of a set from $\mathfrak{M}_{T_1}$ ad a set from $\mathfrak{M}_{T_2}$ be a set from $\mathfrak{M}_T$. Let $M = \{a, b, c, d, e, f\}$, let T_1 consist of all pairs (x, x) for $x \in M$ and of the pairs (a, c), (c, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c), let T_2 consist of all pairs (x, x) for $x \in M$ and of the pairs (c, d), (d, c), (c, e), (e, c), (c, f), (f, c), (d, f), (f, d). Then $T = T_1 \cap T_2$ consists of all pairs (x, x) for $x \in M$ and of the pairs $(c, d), (d, c)$. We have $\mathfrak{M}_{T_1} = \{\{a, c\}, \{b, c, d\}, \{e\}, \{f\}\}$, $\mathfrak{M}_{T_2} = \{\{a\}, \{b\}, \{c, d, f\}, \{c, e\}\}$, $\mathfrak{M}_T = \{\{a\}, \{b\}, \{c, d\}, \{e\}, \{f\}\}$. The sets $\{a, c\} \in \mathfrak{M}_{T_1}$, $\{c, d\} \in \mathfrak{M}_{T_2}$ have the intersection $\{c\} \notin \mathfrak{M}_T$.

Nor does an analogous assertion for $T' = T_1 \cup T_2$ hold. A set from $\mathfrak{M}_{T'}$ need not be the union of sets from $\mathfrak{M}_{T_1}$ and sets from $\mathfrak{M}_{T_2}$. (It is always contained in such a union, but this is a trivial assertion, because the whole M is such a union as well.) Let M consist of the elements 1, 2, 3, 4, 12, 13, 14, 23, 24, 34. Let $\mathfrak{M}_{T_1} = \{\{1, 2, 12\}, \{2, 4, 24\}, \{3, 4, 34\}, \{13\}, \{14\}, \{23\}\}$, $\mathfrak{M}_{T_2} = \{\{1, 4, 14\}, \{1, 3, 13\}, \{2, 3, 23\}, \{12\}, \{24\}, \{34\}\}$. The proof that $\mathfrak{M}_{T_1}$ and $\mathfrak{M}_{T_2}$ are τ-coverings of M is left to the reader. Let T_1, T_2 be tolerances on M corresponding to $\mathfrak{M}_{T_1}$ and $\mathfrak{M}_{T_2}$. The τ-

covering $\mathfrak{M}_{T'}$ corresponding to $T' = T_1 \cup T_2$ is $\mathfrak{M}_{T'} = \{\{1, 2, 3, 4\}, \{1, 2, 12\}, \{1, 3, 13\}, \{1, 4, 14\}, \{2, 3, 23\}, \{2, 4, 24\}, \{3, 4, 34\}\}$. The set $\{1, 2, 3, 4\} \in \mathfrak{M}_T$ is not the union of sets from $\mathfrak{M}_{T_1}$ and $\mathfrak{M}_{T_2}$.

4. Now let us study compatible tolerances on algebras.

Theorem 3. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let T be a tolerance on A. Let $\mathfrak{M}_T$ be the τ-covering of A corresponding to T. The tolerance T is compatible with $\mathfrak{A}$, if and only if there exists an algebra $\mathfrak{B} = \langle B, \mathscr{G} \rangle$ with these properties:*

(i) *there exists a one-to-one mapping $\varphi : \mathscr{F} \to \mathscr{G}$ such that for any positive integer n and for each $f \in \mathscr{F}$ the operation φf is n-ary if and only if f is n-ary;*

(ii) *there exists a one-to-one mapping $\chi : \mathfrak{M}_T \to B$ such that for each n-ary operation $f \in \mathscr{F}$, where n is a positive integer, and for any $n + 1$ elements $M_0, M_1, \ldots$ $\ldots, M_n$ from $\mathfrak{M}_T$ the equality $\varphi f(\chi(M_1), \ldots, \chi(M_n)) = \chi(M_0)$ implies that for any n elements $a_1, \ldots, a_n$ of A such that $a_i \in M_i$ for $i = 1, \ldots, n$ the element $f(a_1, \ldots, a_n) \in M_0$.*

Proof. Let T be compatible with $\mathfrak{A}$. Construct the τ-covering $\mathfrak{M}_T$. Let $M_1 \ldots, M_n$ be n elements of $\mathfrak{M}_T$, let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of A such that $a_i \in M_i$, $b_i \in M_i$ for $i = 1, \ldots, n$. Let $f \in \mathscr{F}$ be an n-ary operation. We have $a_i T b_i$ for $i = 1, \ldots, n$, thus from the compatibility $f(a_1, \ldots, a_n)\, T f(b_1, \ldots, b_n)$. The elements a_i, b_i were chosen arbitrarily, therefore the set of all elements $f(x_1, \ldots, x_n)$, where $x_i \in M_i$ for $i = 1, \ldots, n$, has the property that any two of its elements are in T and is contained in set $M_0 \in \mathfrak{M}_T$. Thus we may put $B = \mathfrak{M}_T$. The mapping χ will be the identical mapping on $\mathfrak{M}_T$. For any $f \in \mathscr{F}$ the operation φf is defined so that $\varphi f(\chi(M_1), \ldots, \chi(M_n)) = \chi(M_0)$ if and only if $f(a_1, \ldots, a_n) \in M_0$, where $a_i \in M_i$ for $i = 1, \ldots, n$. Now suppose that the conditions (i) and (ii) are fulfilled. If $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ are elements of A such that $x_i T y_i$ for $i = 1, \ldots, n$, then for every i both the elements x_i, y_i belong to some set M_i from $\mathfrak{M}_T$. Now let $f \in \mathscr{F}$ and let M_0 be the set of $\mathfrak{M}_T$ such that $\varphi f(\chi(M_1), \ldots, \chi(M_n)) = \chi(M_0)$. According to the assumption $f(x_1, \ldots, x_n) \in M_0$, $f(y_1, \ldots, y_n) \in M_0$, thus $f(x_1, \ldots, x_n)\, T f(y_1, \ldots, y_n)$.

If T is a congruence, then $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$ in the homormophism σ corresponding to the congruence T. To each element x of A exactly one set $M(x)$ from $\mathfrak{M}_T$ exists which contains it; thus σ is determined by χ so that $\sigma(x) = \chi(M(x))$. If T is not a congruence, then this is not so, because there exist elements which are contained in more than one set from $\mathfrak{M}_T$.

5. Definition. *An algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ is called idempotent, if for each element $a \in A$ and for each n-ary operation $f \in \mathscr{F}$ the equality $f(\underbrace{a, \ldots, a}_{n \text{ times}}) = a$ holds.*

Lemma 1. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an idempotent algebra and let T be a tolerance compatible with $\mathfrak{A}$. Denote* $\operatorname{Tol}(x) = \{y \in A \mid yTx\}$. *Then* $\operatorname{Tol}(x)$ *is a subalgebra of $\mathfrak{A}$ for each $x \in A$.*

Proof. Let $a_1, \ldots, a_n$ be in Tol (x), let $f \in \mathscr{F}$ be an n-ary operation. Then $a_i T x$ for $i = 1, \ldots, n$; from the compatibility of T we obtain $f(a_1, \ldots, a_n)\, T f(\underbrace{x, \ldots, x}_{n \text{ times}})$, therefore $f(a_1, \ldots, a_n) \in \operatorname{Tol}(x)$.

Theorem 4. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an idempotent algebra, let T be a tolerance compatible with $\mathfrak{A}$. Then all the sets of the τ-covering $\mathfrak{M}_T$ corresponding to T are subalgebras of $\mathfrak{A}$.*

Proof. Let $M_0 \in \mathfrak{M}_T$, let $N = \bigcap_{x \in M_0} \operatorname{Tol}(x)$. For each x and each y from M_0 we have xTy, therefore $y \in \operatorname{Tol}(x)$; as this holds for each $x \in M_0$, we have $y \in N$ for each $y \in M_0$ and thus $M_0 \subseteq N$. Suppose that $N - M_0 \neq \emptyset$ and let $z \in N - M_0$. Then $z \in \operatorname{Tol}(x)$ for each $x \in M_0$, this means zTx. The set $M_0 \cup \{z\} \in \mathfrak{L}_T$ and M_0 is its proper subset, therefore $M_0 \notin \mathfrak{M}_T$, which is a contradiction. We have $N - M_0 = \emptyset$, this means $M_0 = N = \bigcap_{x \in M_0} \operatorname{Tol}(x)$. The set $\operatorname{Tol}(x)$ for each $x \in M_0$ is a subalgebra of $\mathfrak{A}$ according to Lemma 1, thus M_0 is a non-empty intersection of some subalgebras of $\mathfrak{A}$ and is a subalgebra of $\mathfrak{A}$.

The above proved theorems imply Theorem 10 from [10].

Theorem. *Let L be a lattice and let there exist a proper ideal J and a proper filter F of L such that $J \cup F = L$, $J \cap F \neq \emptyset$. Then there exists a compatible tolerance on L which is not a congruence.*

Proof. The pair $\{J, F\}$ is a τ-covering of L. For $\mathfrak{B}$ we may take a two-element lattice L_0 consisting of the elements O, I, where $O < I$. The join (or meet) in L is assigned the join (or meet, respectively) in L_0 by φ. Further $\chi(J) = O$, $\chi(F) = I$. We can verify that all assumptions of Theorem 2 are fulfilled, therefore the assertion is true.

Corollary 1. *Let L be a lattice with at least three elements. Then there exists a sublattice L_0 of L on which a compatible tolerance exists which is not a congruence.*

Proof. As L contains at least three elements, there exists an element $a \in L$ which is neither the greatest nor the least element of L. Let L_0 be the set of all elements of L which are comparable with a. The set L_0 is evidently a sublattice of L. Now let J (or F) be the set of all elements of L_0 which are less (or greater, respectively) then or equal to a. Evidently J is a proper ideal of L_0, F is a proper filter if L_0, $J \cup F = L_0$ and $J \cap F = \{a\} \neq \emptyset$. Thus the assertion is true.

6. Now we shall prove some theorems for concrete types of lattices.

Theorem 5. *Let L be a relatively complementary lattice. Then every compatible tolerance on L is a congruence.*

Proof. Let T be a compatible tolerance on L, let a, b, c be three elements of L such that aTb, bTc. Denote $\bar{a} = a \wedge b \wedge c$, $\bar{c} = a \vee b \vee c$. Let d be a relative complement of b in the interval $\langle \bar{a}, \bar{c} \rangle$. Then $\bar{a} = (a \wedge b \wedge c)\, T(b \wedge b \wedge b) = b$, $\bar{c} = (a \vee b \vee c)\, T(b \vee b \vee b) = b$. This implies $\bar{a} = b \wedge (d \vee \bar{a})\, T\bar{c} \wedge (d \vee b) = \bar{c}$. Thus, according to [9], Theorem 1, any two elements of $\langle \bar{a}, \bar{c} \rangle$ are in T, in particular aTc and T is transitive, i.e. it is a congruence.

Lemma 2. *Let a, b, c be elements of a complete infinitely distributive lattice L such that $a < b < c$ and b has no relative complement in the interval $\langle a, c \rangle$. Then the ideal J generated by the set $M = \{b\} \cup \{x \in L \mid x \wedge b = a\}$ does not contain c.*

Proof. Suppose that J contains c. Then there exists a subset S of L such that $x \wedge b = a$ for each $x \in S$ and $c \leqq b \vee \bigvee_{x \in S} x$. Then

$$b \vee (c \wedge \bigvee_{x \in S} x) = (b \vee c) \wedge (b \vee \bigvee_{x \in S} x) = c\,,$$

$$b \wedge (c \wedge \bigvee_{x \in S} x) = b \wedge \bigvee_{x \in S} x = \bigvee_{x \in S} (b \wedge x) = a\,,$$

therefore $c \wedge \bigvee_{x \in S} x$ is a relative complement to b in the interval $\langle a, c \rangle$, which is a contradiction.

The union of all elements of a chain of ideals is an ideal and according to Zorn's Lemma there exists a maximal ideal J in L containing M and not containing c.

Lemma 3. *Let a, b, c be three elements of a distributive lattice L such that $a < b < c$ and b has no relative complement in the interval $\langle a, c \rangle$. Let J be the maximal ideal containing the set $M = \{b\} \cup \{x \in L \mid x \wedge b = a\}$ and not containing c. Then $E = L - J$ is a filter of L and the filter F of L generated by the set $E \cup \{b\}$ does not contain a.*

Proof. Evidently $E \neq \emptyset$, because $c \in E$. Let $x \in E$, let y be an arbitrary element of L. Then $x \vee y \in E$; otherwise it would be $x \vee y \in J$ and $x = x \wedge (x \vee y) \in J$, which would be a contradiction. Now let $x \in E$, $y \in E$. To the element x there exists an element $x' \in J$ such that $x \vee x' \geqq c$; otherwise by adding x and all elements less than x to J we should obtain an ideal containing M and not containing c and J would not be the maximal ideal with this property. Analogously there exists an element $y' \in J$ such that $y \vee y' \geqq c$. Let $z = x' \vee y'$; we have $x \vee z \geqq c$, $y \vee z \geqq c$, $z \in J$. Then $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) \geqq c$, thus $x \wedge y \notin J$ and $x \wedge y \in E$. We have proved that E is a filter of L. Now let F be the filter of L generated by the set $E \cup \{b\}$. If $a \in F$, then $a \geqq b \wedge y$, where y is an element of E. But then $b \wedge (y \vee a) = (b \wedge y) \vee (b \wedge a) = a$, which means that $y \vee a \in M \subseteq J$. As $y \leqq y \vee a$, we have also $y \in J$, which is a contradiction. We have proved that $a \notin F$.

Theorem 6. *Let L be a distributive lattice which is not relatively complementary. Then in L a proper ideal J and a proper filter F exist so that $J \cup F = L$, $J \cap F \neq \emptyset$.*

Proof follows from Lemma 2 and Lemma 3.

Corollary 2. *For a distributive lattice L the following three assertions are equivalent:*

(a) *L is relatively complementary.*

(b) *Each compatible tolerance on L is a congruence.*

(c) *If J is a proper ideal of L and F is a proper filter of L such that $J \cup F = L$, then $J \cap F = \emptyset$.*

7. In the end we shall prove other two theorems concerning tolerance relations on algebras in general.

Theorem 7. *Let $\mathfrak{A}_1 = \langle A_1, \mathscr{F}_1 \rangle$, $\mathfrak{A}_2 = \langle A_2, \mathscr{F}_2 \rangle$ be two algebras of the same type, let there exist a homomorphism ψ of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. Let there exist a tolerance T on A_2 which is not a congruence and is compatible with $\mathfrak{A}_2$. Then there exists a tolerance T' on A_1 which is not a congruence and is compatible with $\mathfrak{A}_1$.*

Proof. We construct T' so that for any two elements x, y of A_1 we have $xT'y$ if and only if $\psi(x)\, T\, \psi(y)$. The relation T' thus constructed is evidently a tolerance. Let $f_1 \in \mathscr{F}_1$ be an n-ary relation, let $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ be elements of A_1 such that $x_i T' y_i$ for $i = 1, \ldots, n$. Then $\psi(x_i)\, T\, \psi(y_i)$ for $i = 1, \ldots, n$. Let f_2 be the operation from $\mathscr{F}_2$ which corresponds to f_1 in the homomorphism ψ. As T is a tolerance compatible with $\mathfrak{A}_2$, we have $f_2(\psi(x_1), \ldots, \psi(x_n))\; T\; f_2(\psi(y_1), \ldots, \psi(y_n))$. As $f_2(\psi(x_1), \ldots, \psi(x_n)) = \psi(f_1(x_1, \ldots, x_n))$, $f_2(\psi(y_1), \ldots, \psi(y_n)) = \psi(f_1(y_1, \ldots, y_n))$, we have $f_1(x_1, \ldots, x_n)\; T'\; f_1(y_1, \ldots, y_n)$ and T' is a tolerance compatible with $\mathfrak{A}_1$. As T is not a congruence, there exist elements a, b, c of A_2 such that aTb, bTc, but not aTc. Let a' (or b', or c') be an element of A_1 such that $\psi(a') = a$ (or $\psi(b') = b$, or $\psi(c') = c$, respectively). Then $a'T'b'$, $b'T'c'$, but not $a'T'c'$ and T' is not a congruence.

Corollary 3. *Let an algebra $\mathfrak{A}$ be the direct product of the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. On at least one of the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ let there exist a tolerance compatible with this algebra which is not a congruence. Then there exists a tolerance compatible with $\mathfrak{A}$ which is not a congruence.*

Theorem 8. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, $|A| \geqq 3$. Let there exist an element $a \in A$ which cannot be obtained as a result of an operation from $\mathscr{F}$. Then there exists a tolerance T compatible with $\mathfrak{A}$ which is not a congruence.*

Proof. Let b be an element of A distinct from a. Consider the tolerance T consisting of the pairs (a, a), (a, b), (b, a) and of all pairs (x, y) for $x \in A - \{a\}$, $y \in \in A - \{a\}$. If $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ are elements of A and $f \in F$ is an n-ary operation, then $f(x_1, \ldots, x_n) \in A - \{a\}$, $f(y_1, \ldots, y_n) \in A - \{a\}$, thus $f(x_1, \ldots, x_n)\, T f(y_1, \ldots \ldots, y_n)$. Thus T is a tolerance compatible with $\mathfrak{A}$. It is not a congruence, because aTb, bTc, but not aTc, where c is an arbitrary element of $A - \{a, b\}$.

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