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ON EXISTENCE CONDITIONS FOR COMPATIBLE TOLERANCES

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1. Conditions for the existence of compatible tolerances on various algebras which are not congruences were studied in many papers (see [1], [3]—[10]). The problem of finding necessary and sufficient conditions is still open, although in [10] one of such conditions was formulated (see Theorem 5 in [10]) for WA-lattices and lattices; however, this condition assumes the existence of a compatible tolerance which is not a congruence on a sublattice.

Some new conditions for the existence of compatible tolerances which are not congruences are established in this paper.

2. The symbol $\mathfrak{A} = \langle A, \mathcal{F} \rangle$ will denote an algebra with the support $A$ and with the set of fundamental operations $\mathcal{F}$. A tolerance relation on a set $M$ is a reflexive and symmetric relation on $M$. In particular, each equivalence on $M$ is a tolerance on $M$. A tolerance relation $T$ on the set $A$ is called compatible with $\mathfrak{A}$, if and only if for each $n$-ary operation $f \in \mathcal{F}$ (where $n$ is a positive integer) and for any $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ of $A$ which fulfil $x_iTy_i$ for $i = 1, \ldots, n$ we have $f(x_1, \ldots, x_n) \cdot T f(y_1, \ldots, y_n)$.

3. Every equivalence relation is a tolerance relation. As is well-known, every equivalence on a set $M$ determines a certain partition on $M$; the classes of this partition are called equivalence classes. Here we shall formulate an analogous result for tolerance relations.

Definition. Let $M$ be a non-empty set. The family $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$, where $\Gamma$ is a subscript set, is called a covering of $M$ by subsets, if and only if each $M_\gamma$ for $\gamma \in \Gamma$ is a subset of $M$ and $\bigcup_{\gamma \in \Gamma} M_\gamma = M$. (We suppose $M_{\gamma_1} \neq M_{\gamma_2}$ for $\gamma_1 \in \Gamma, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2$.)

A covering $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$ of a set $M$ by subsets is called a $\tau$-covering of $M$, if and only if $\mathfrak{M}$ fulfills the following two conditions:
(1) if \( \gamma_0 \in \Gamma \) and \( \Gamma_0 \subseteq \Gamma \), then
\[
M_{\gamma_0} \subseteq \bigcup_{\gamma \in \Gamma_0} M_{\gamma} = \bigcap_{\gamma \in \Gamma_0} M_{\gamma} \leq M_{\gamma_0};
\]

(2) if \( N \subseteq M \) and \( N \) is not contained in any set from \( \mathcal{M} \), then \( N \) contains a two-element subset of the same property.

In particular, if \( \mathcal{M} = \{M_{\gamma}, \gamma \in \Gamma\} \) is a \( \tau \)-covering of \( M \), then \( M_{\gamma_1} \subseteq M_{\gamma_2} \) for \( \gamma_1 \in \Gamma, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2 \). This can be proved by putting \( \gamma_0 = \gamma_1, \Gamma_0 = \{\gamma_2\} \). This implies also that all the sets of \( \mathcal{M} \) are non-empty.

Theorem 1. Let \( M \) be a non-empty set. Then there exists a one-to-one correspondence between tolerance relations on \( M \) and \( \tau \)-coverings of \( \mathcal{M} \) such that if \( T \) is a tolerance relation on \( M \) and \( \mathcal{M}_T \) is the \( \tau \)-covering of \( M \) corresponding to \( T \), then any two elements of \( M \) are in the relation \( T \) if and only if there exists a set from \( \mathcal{M}_T \) which contains both of them.

Proof. Let \( T \) be a tolerance relation on \( M \). Let \( \mathcal{L}_T \) be the family of all subsets of \( M \) with the property that any two elements of the subset are in \( T \). The family \( \mathcal{L}_T \) contains all one-element subsets of \( M \), therefore it is a covering of \( M \) by subsets. Let \( \mathcal{M}_T \) be the family of all sets of \( \mathcal{L}_T \) which are maximal with respect to the set inclusion (according to Zorn's Lemma such elements exist). Each set from \( \mathcal{L}_T \) is contained in a set from \( \mathcal{M}_T \) and \( \mathcal{L}_T \) is a covering of \( M \), therefore also \( \mathcal{M}_T \) is a covering of \( M \). Let \( \mathcal{M}_T = \{M_{\gamma}, \gamma \in \Gamma\} \), where \( \Gamma \) is a subscript set. Now let \( \gamma_0 \in \Gamma, \Gamma_0 \subseteq \Gamma \) and let \( M_{\gamma_0} \subseteq \bigcup_{\gamma \in \Gamma_0} M_{\gamma} \). Let \( P = \bigcap_{\gamma \in \Gamma_0} M_{\gamma} \) and suppose \( P \subseteq M_{\gamma_0} \). Let \( x \in P - M_{\gamma_0}, y \in M_{\gamma_0} \).

This means \( y \in \bigcup_{\gamma \in \Gamma_0} M_{\gamma} \) and thus there exists \( \gamma_1 \in \Gamma_0 \) such that \( y \in M_{\gamma_1} \). As \( x \in P - M_{\gamma_0} \), we have \( x \in \bigcap_{\gamma \in \Gamma_0} M_{\gamma} \) and thus also \( x \in M_{\gamma_1} \). We have \( xTy \). As \( y \) was chosen arbitrarily, we have \( xTy \) for each \( y \in M_{\gamma_0} \). Thus the set \( M_{\gamma_0} \cup \{x\} \in \mathcal{L}_T \) and \( M_{\gamma_0} \) is its proper subset; this means \( M_{\gamma_0} \neq \mathcal{M}_T \), which is a contradiction. We have necessarily \( \bigcap_{\gamma \in \Gamma_0} M_{\gamma} \subseteq M_{\gamma_0} \) and (1) is fulfilled. Now if a subset \( N \) of \( M \) is not contained in any set from \( \mathcal{M}_T \), then \( N \not\in \mathcal{L}_T \) and there exist two elements \( a, b \) of \( N \) which are not in the relation \( T \). Thus the set \( \{a, b\} \) is not contained in any set from \( \mathcal{M}_T \) and (2) is fulfilled. We have proved that \( \mathcal{M}_T \) is a \( \tau \)-covering. Now let \( \mathcal{M} = \{M_{\gamma}, \gamma \in \Gamma\} \), be a \( \tau \)-covering of \( M \) and let \( T \) be a relation on \( M \) such that \( xTy \) for \( x \in M, y \in M \) if and only if there exists \( \gamma \in \Gamma \) such that \( x \in M_{\gamma}, y \in M_{\gamma} \). The relation \( T \) is evidently a tolerance. Now it remains to prove that if \( \mathcal{M}_T \) is assigned to \( T \) according to the above rule, then \( \mathcal{M}_T = \mathcal{M} \). This means to prove that each \( M_{\gamma} \) for \( \gamma \in \Gamma \) is a maximal element in \( \mathcal{L}_T \) and each maximal element of \( \mathcal{L}_T \) is in \( \mathcal{M} \). Suppose that \( M_{\gamma_1} \) for some \( \gamma_1 \in \Gamma \) is not a maximal element in \( \mathcal{L}_T \); this means that there exists \( L \in \mathcal{L}_T \) such that \( M_{\gamma_1} \) is a proper subset of \( L \). Let \( x \in L - M_{\gamma_1} \). As \( L \in \mathcal{L}_T, M_{\gamma_1} \subseteq L, x \in L \), we have \( xTy \) for each \( y \in M_{\gamma_1} \). This means that to each \( y \in M_{\gamma_1} \) there exists \( \gamma(y) \in \Gamma \) so that

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\begin{align*}
y & \in M_{y(y)}, \ x \in M_{y(y)}. \text{ We have } M_{y_1} \subseteq \bigcup_{y \in M_{y_1}} M_{y(y)}. \text{ As } \mathfrak{M}_T \text{ is a } \tau\text{-covering, it is necessarily } \bigcap_{y \in M_{y_1}} M_{y(y)} \subseteq M_{y_1}. \text{ But } x \in M_{y(y)} \text{ for each } y \in M_{y_1}, \text{ thus } x \in \bigcap_{y \in M_{y_1}} M_{y(y)} \text{ and } x \in M_{y_1}, \text{ which is a contradiction. Now suppose that there exists a set } L \in \mathfrak{M}_T - \mathfrak{M}.
\end{align*}

As \mathfrak{M}_T is a \tau\text{-covering, it is necessarily } M \subseteq \mathfrak{M}_T. \text{ The set } L' \text{ is not contained in any set from } \mathfrak{M}. \text{ Thus there exist two elements } c, d \text{ of } L' \text{ such that the set } \{c, d\} \text{ is not contained in any set from } \mathfrak{M}. \text{ This means that } c, d \text{ are not in the relation } \tau, \text{ thus } L' \notin \mathfrak{M}_T \text{ and also } L' \notin \mathfrak{M}_T, \text{ which is a contradiction.}

When \tau \text{ is an equivalence relation, the corresponding } \tau\text{-covering } \mathfrak{M}_T \text{ is the partition of } M \text{ into equivalence classes of } T. \text{ This follows from the construction of } \mathfrak{M}_T.

**Theorem 2.** Let \( M \) be a non-empty set, let \( T_1 \) and \( T_2 \) be tolerances on \( M \). Let \( T = T_1 \cap T_2 \). Let \( \mathfrak{M}_{T_1}, \mathfrak{M}_{T_2}, \mathfrak{M}_T \) be the \( \tau\)-coverings of \( M \) corresponding to \( T_1, T_2, T \) respectively. Then each set of \( \mathfrak{M}_T \) is the intersection of a set from \( \mathfrak{M}_{T_1} \) and a set from \( \mathfrak{M}_{T_2} \). Any intersection of a set from \( \mathfrak{M}_{T_1} \) and a set from \( \mathfrak{M}_{T_2} \) is a subset of some set from \( \mathfrak{M}_T \).

**Proof.** Let \( M_0 \in \mathfrak{M}_T. \) Then, as we have seen in the proof of Theorem 1, any two elements of \( M_0 \) are in \( T \), this means simultaneously in \( T_1 \) and \( T_2 \). Thus \( M_0 \in \mathfrak{M}_{T_1}, \mathfrak{M}_{T_2} \) and there exist sets \( M_1 \in \mathfrak{M}_{T_1}, M_2 \in \mathfrak{M}_{T_2} \) such that \( M_0 \subseteq M_1, M_0 \subseteq M_2 \), this means \( M_0 \subseteq M_1 \cap M_2 \). On the other hand, any two elements of \( M \cap M_2 \) are in \( T \), thus \( M_1 \cap M_2 \in \mathfrak{M}_T \) and there exists \( M'_0 \in \mathfrak{M}_T \) such that \( M_1 \cap M_2 \subseteq M'_0 \). We have \( M_0 \subseteq M'_0 \); as no set from \( \mathfrak{M}_T \) is a proper subset of another, we have \( M_0 = M'_0 \) and then also \( M_0 = M_1 \cap M_2 \). Now let \( N_1 \in \mathfrak{M}_{T_1}, N_2 \in \mathfrak{M}_{T_2}. \) If \( N_1 \cap N_2 = \emptyset \), then this set is a subset of every set. Thus let \( N_1 \cap N_2 \neq \emptyset \). Any two elements of \( N_1 \cap N_2 \) are simultaneously in \( T_1 \) and \( T_2 \), thus they are in \( T \) and \( N_1 \cap N_2 \subseteq \mathfrak{M}_T. \) Thus there exists \( N_0 \in \mathfrak{M}_T \) such that \( N_1 \cap N_2 \subseteq N_0 \).

This theorem cannot be strengthened so that any intersection of a set from \( \mathfrak{M}_{T_1} \), ad a set from \( \mathfrak{M}_{T_2} \) be a set from \( \mathfrak{M}_T \). Let \( M = \{a, b, c, d, e, f\} \), let \( T_1 \) consist of all pairs \((x, x)\) for \( x \in M \) and of the pairs \((a, c), (c, a), (b, c), (b, d), (d, b), (c, d), (d, c), (e, c), (c, f), (f, c), (d, f), (f, d)\). Then \( T = T_1 \cap T_2 \) consists of all pairs \((x, x)\) for \( x \in M \) and of the pairs \((c, d), (d, c)\). We have \( \mathfrak{M}_{T_1} = \{\{a, c\}, \{b, c, d\}, \{e\}, \{f\}\}, \mathfrak{M}_{T_2} = \{\{a\}, \{b\}, \{c, d\}, \{e\}, \{f\}\}\). The sets \( \{a, c\} \in \mathfrak{M}_{T_1}, \{c, d\} \in \mathfrak{M}_{T_2} \) have the intersection \( \{c\} \notin \mathfrak{M}_T. \)

Nor does an analogous assertion for \( T' = T_1 \cup T_2 \) hold. A set from \( \mathfrak{M}_T \) need not be the union of sets from \( \mathfrak{M}_{T_1} \) and sets from \( \mathfrak{M}_{T_2} \). (It is always contained in such a union, but this is a trivial assertion, because the whole \( M \) is such a union as well.) Let \( M \) consist of the elements 1, 2, 3, 4, 12, 13, 14, 23, 24, 34. Let \( \mathfrak{M}_{T_1} = \{\{1, 2, 12\}, \{2, 4, 24\}, \{3, 4, 34\}, \{13\}, \{14\}, \{23\}\}, \mathfrak{M}_{T_2} = \{\{1, 4, 14\}, \{1, 3, 13\}, \{2, 3, 23\}, \{12\}, \{24\}, \{34\}\}. \) The proof that \( \mathfrak{M}_{T_1} \) and \( \mathfrak{M}_{T_2} \) are \( \tau\)-coverings of \( M \) is left to the reader. Let \( T_1, T_2 \) be tolerances on \( M \) corresponding to \( \mathfrak{M}_{T_1}, \mathfrak{M}_{T_2} \). The \( \tau\)-
covering \( \mathfrak{M}_T \), corresponding to \( T' = T_1 \cup T_2 \) is \( \mathfrak{M}_{T'} = \{ \{1, 2, 3, 4\}, \{1, 2, 12\}, \{1, 3, 13\}, \{1, 4, 14\}, \{2, 3, 23\}, \{2, 4, 24\}, \{3, 4, 34\} \}. \) The set \( \{1, 2, 3, 4\} \in \mathfrak{M}_T \) is not the union of sets from \( \mathfrak{M}_{T_1} \) and \( \mathfrak{M}_{T_2} \).

4. Now let us study compatible tolerances on algebras.

**Theorem 3.** Let \( \mathfrak{A} = \langle A, \mathcal{F} \rangle \) be an algebra, let \( T \) be a tolerance on \( A \). Let \( \mathfrak{M}_T \) be the \( \tau \)-covering of \( A \) corresponding to \( T \). The tolerance \( T \) is compatible with \( \mathfrak{A} \), if and only if there exists an algebra \( \mathfrak{B} = \langle B, \mathcal{F} \rangle \) with these properties:

(i) there exists a one-to-one mapping \( \varphi : \mathfrak{M}_T \rightarrow B \) such that for any positive integer \( n \) and for each \( f \in \mathcal{F} \) the operation \( \varphi f \) is \( n \)-ary if and only if \( f \) is \( n \)-ary;

(ii) there exists a one-to-one mapping \( \chi : \mathfrak{M}_T \rightarrow B \) such that for each \( n \)-ary operation \( f \in \mathcal{F} \), where \( n \) is a positive integer, and for any \( n + 1 \) elements \( M_0, M_1, \ldots, M_n \) from \( \mathfrak{M}_T \) the equality \( \varphi f(\chi(M_1), \ldots, \chi(M_n)) = \chi(M_0) \) implies that for any \( n \) elements \( a_1, \ldots, a_n \) of \( A \) such that \( a_i \in M_i \) for \( i = 1, \ldots, n \) the element \( f(a_1, \ldots, a_n) \in M_0 \).

**Proof.** Let \( T \) be compatible with \( \mathfrak{A} \). Construct the \( \tau \)-covering \( \mathfrak{M}_T \). Let \( M_1, \ldots, M_n \) be elements of \( \mathfrak{M}_T \), let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be elements of \( A \) such that \( a_i \in M_i, b_i \in M_i \) for \( i = 1, \ldots, n \). Let \( f \in \mathcal{F} \) be an \( n \)-ary operation. We have \( a_i T b_i \) for \( i = 1, \ldots, n \), thus from the compatibility \( f(a_1, \ldots, a_n) T f(b_1, \ldots, b_n) \). The elements \( a_i, b_i \) were chosen arbitrarily, therefore the set of all elements \( f(x_1, \ldots, x_n) \), where \( x_i \in M_i \) for \( i = 1, \ldots, n \), has the property that any two of its elements are in \( T \) and is contained in set \( M_0 \in \mathfrak{M}_T \). Thus we may put \( B = \mathfrak{M}_T \). The mapping \( \chi \) will be the identical mapping on \( \mathfrak{M}_T \). For any \( f \in \mathcal{F} \) the operation \( \varphi f \) is defined so that \( \varphi f(\chi(M_1), \ldots, \chi(M_n)) = \chi(M_0) \) if and only if \( f(a_1, \ldots, a_n) \in M_0 \), where \( a_i \in M_i \) for \( i = 1, \ldots, n \). Now suppose that the conditions (i) and (ii) are fulfilled. If \( x_1, \ldots, x_n, y_1, \ldots, y_n \) are elements of \( A \) such that \( x_i T y_i \) for \( i = 1, \ldots, n \), then for every \( i \) both the elements \( x_i, y_i \) belong to some set \( M_i \) from \( \mathfrak{M}_T \). Now let \( f \in \mathcal{F} \) and let \( M_0 \) be the set of \( \mathfrak{M}_T \) such that \( \varphi f(\chi(M_1), \ldots, \chi(M_n)) = \chi(M_0) \). According to the assumption \( f(x_1, \ldots, x_n) \in M_0, f(y_1, \ldots, y_n) \in M_0 \), thus \( f(x_1, \ldots, x_n) T f(y_1, \ldots, y_n) \).

If \( T \) is a congruence, then \( \mathfrak{B} \) is a homomorphic image of \( \mathfrak{A} \) in the homomorphism \( \sigma \) corresponding to the congruence \( T \). To each element \( x \) of \( A \) exactly one set \( M(x) \) from \( \mathfrak{M}_T \) exists which contains it; thus \( \sigma \) is determined by \( \chi \) so that \( \sigma(x) = \chi(M(x)) \). If \( T \) is not a congruence, then this is not so, because there exist elements which are contained in more than one set from \( \mathfrak{M}_T \).

5. **Definition.** An algebra \( \mathfrak{A} = \langle A, \mathcal{F} \rangle \) is called idempotent, if for each element \( a \in A \) and for each \( n \)-ary operation \( f \in \mathcal{F} \) the equality \( f(a, \ldots, a) = a \) holds.

**Lemma 1.** Let \( \mathfrak{A} = \langle A, \mathcal{F} \rangle \) be an idempotent algebra and let \( T \) be a tolerance compatible with \( \mathfrak{A} \). Denote \( \text{Tol}(x) = \{ y \in A \mid y T x \} \). Then \( \text{Tol}(x) \) is a subalgebra of \( \mathfrak{A} \) for each \( x \in A \).
Proof. Let \( a_1, \ldots, a_n \) be in \( \text{Tol}(x) \), let \( f \in \mathcal{F} \) be an \( n \)-ary operation. Then \( a_i T x \)
for \( i = 1, \ldots, n \); from the compatibility of \( T \) we obtain \( f(a_1, \ldots, a_n) T f(x, \ldots, x) \),
therefore \( f(a_1, \ldots, a_n) \in \text{Tol}(x) \).

Theorem 4. Let \( \mathfrak{A} = \langle A, \mathcal{F} \rangle \) be an idempotent algebra, let \( T \) be a tolerance compatible with \( \mathfrak{A} \). Then all the sets of the \( \tau \)-covering \( \mathfrak{M}_T \) corresponding to \( T \) are subalgebras of \( \mathfrak{A} \).

Proof. Let \( M_0 \in \mathfrak{M}_T \), let \( N = \bigcap_{x \in M_0} \text{Tol}(x) \). For each \( x \) and each \( y \) from \( M_0 \) we have \( x T y \), therefore \( y \in \text{Tol}(x) \); as this holds for each \( x \in M_0 \), we have \( y \in N \) for each \( y \in M_0 \) and thus \( M_0 \subseteq N \). Suppose that \( N - M_0 \neq \emptyset \) and let \( z \in N - M_0 \). Then \( z \in \text{Tol}(x) \) for each \( x \in M_0 \), this means \( z T x \). The set \( M_0 \cup \{z\} \in \mathfrak{O}_T \) and \( M_0 \) is its proper subset, therefore \( M_0 \notin \mathfrak{M}_T \), which is a contradiction. We have \( N - M_0 = \emptyset \), this means \( M_0 = N = \bigcap_{x \in M_0} \text{Tol}(x) \). The set \( \text{Tol}(x) \) for each \( x \in M_0 \) is a subalgebra of \( \mathfrak{A} \) according to Lemma 1, thus \( M_0 \) is a non-empty intersection of some subalgebras of \( \mathfrak{A} \) and is a subalgebra of \( \mathfrak{A} \).

The above proved theorems imply Theorem 10 from [10].

Theorem. Let \( L \) be a lattice and let there exist a proper ideal \( J \) and a proper filter \( F \) of \( L \) such that \( J \cup F = L \), \( J \cap F \neq \emptyset \). Then there exists a compatible tolerance on \( L \) which is not a congruence.

Proof. The pair \( \{J, F\} \) is a \( \tau \)-covering of \( L \). For \( \mathfrak{B} \) we may take a two-element lattice \( L_0 \) consisting of the elements \( O, I \), where \( O < I \). The join (or meet) in \( L \) is assigned the join (or meet, respectively) in \( L_0 \) by \( \varphi \). Further \( \chi(J) = O \), \( \chi(F) = I \). We can verify that all assumptions of Theorem 2 are fulfilled, therefore the assertion is true.

Corollary 1. Let \( L \) be a lattice with at least three elements. Then there exists a sublattice \( L_0 \) of \( L \) on which a compatible tolerance exists which is not a congruence.

Proof. As \( L \) contains at least three elements, there exists an element \( a \in L \) which is neither the greatest nor the least element of \( L \). Let \( L_0 \) be the set of all elements of \( L \) which are comparable with \( a \). The set \( L_0 \) is evidently a sublattice of \( L \). Now let \( J \) (or \( F \)) be the set of all elements of \( L_0 \) which are less (or greater, respectively) then or equal to \( a \). Evidently \( J \) is a proper ideal of \( L_0 \), \( F \) is a proper filter if \( L_0 \), \( J \cup F = L_0 \) and \( J \cap F = \{a\} \neq \emptyset \). Thus the assertion is true.

6. Now we shall prove some theorems for concrete types of lattices.

Theorem 5. Let \( L \) be a relatively complementary lattice. Then every compatible tolerance on \( L \) is a congruence.
Proof. Let $T$ be a compatible tolerance on $L$, let $a, b, c$ be three elements of $L$ such that $aTb, bTc$. Denote $\bar{a} = a \land b \land c$, $\bar{c} = a \lor b \lor c$. Let $d$ be a relative complement of $b$ in the interval $\langle \bar{a}, \bar{c} \rangle$. Then $\bar{a} = (a \land b \land c) T (b \land b \land b) = b$, $\bar{c} = (a \lor b \lor c) T (b \lor b \lor b) = b$. This implies $\bar{a} = b \land (d \lor \bar{a}) T \bar{c} \land (d \lor \bar{b}) = \bar{c}$. Thus, according to [9], Theorem 1, any two elements of $\langle \bar{a}, \bar{c} \rangle$ are in $T$, in particular $aTc$ and $T$ is transitive, i.e. it is a congruence.

Lemma 2. Let $a, b, c$ be elements of a complete infinitely distributive lattice $L$ such that $a < b < c$ and $b$ has no relative complement in the interval $\langle a, c \rangle$. Then the ideal $J$ generated by the set $M = \{ b \} \cup \{ x \in L \mid x \land b = a \}$ does not contain $c$.

Proof. Suppose that $J$ contains $c$. Then there exists a subset $S$ of $L$ such that $x \land b = a$ for each $x \in S$ and $c \leq b \lor \bigvee x$. Then

$$\begin{align*}
\bigvee_{x \in S} (c \land \bigvee x) &= (b \lor c) \land (b \lor \bigvee x) = c, \\
\bigvee_{x \in S} (b \land c) &= b \land \bigvee x = \bigvee (b \land x) = a,
\end{align*}$$

therefore $c \land \bigvee_{x \in S} x$ is a relative complement to $b$ in the interval $\langle a, c \rangle$, which is a contradiction.

The union of all elements of a chain of ideals is an ideal and according to Zorn's Lemma there exists a maximal ideal $J$ in $L$ containing $M$ and not containing $c$.

Lemma 3. Let $a, b, c$ be three elements of a distributive lattice $L$ such that $a < b < c$ and $b$ has no relative complement in the interval $\langle a, c \rangle$. Let $J$ be the maximal ideal containing the set $M = \{ b \} \cup \{ x \in L \mid x \land b = a \}$ and not containing $c$. Then $E = L - J$ is a filter of $L$ and the filter $F$ of $L$ generated by the set $E \cup \{ b \}$ does not contain $a$.

Proof. Evidently $E \neq \emptyset$, because $c \in E$. Let $x \in E$, let $y$ be an arbitrary element of $L$. Then $x \lor y \in E$; otherwise it would be $x \lor y \in J$ and $x = x \land (x \lor y) \in J$, which would be a contradiction. Now let $x \in E$, $y \in E$. To the element $x$ there exists an element $x' \in J$ such that $x \lor x' \geq c$; otherwise by adding $x$ and all elements less than $x$ to $J$ we should obtain an ideal containing $M$ and not containing $c$ and $J$ would not be the maximal ideal with this property. Analogously there exists an element $y' \in J$ such that $y \lor y' \geq c$. Let $z = x' \lor y'$; we have $x \lor z \geq c$, $y \lor z \geq c$, $z \in J$. Then $(x \land y) \lor z = (x \lor z) \land (y \lor z) \geq c$, $z \in J$. Thus $(x \land y) \lor z = (x \lor z) \land (y \lor z) \geq c$, $z \in J$. As $x \lor y \in E$. We have proved that $E$ is a filter of $L$. Now let $F$ be the filter of $L$ generated by the set $E \cup \{ b \}$. If $a \in F$, then $a \geq b \land y$, where $y$ is an element of $E$. But then $b \land (y \lor a) = (b \land y) \lor (b \land a) = a$, which means that $y \lor a \in M \subseteq J$. As $y \geq y \lor a$, we have also $y \in J$, which is a contradiction. We have proved that $a \notin F$. 309
Theorem 6. Let $L$ be a distributive lattice which is not relatively complementary. Then in $L$ a proper ideal $J$ and a proper filter $F$ exist so that $J \cup F = L$, $J \cap F \neq \emptyset$.

Proof follows from Lemma 2 and Lemma 3.

Corollary 2. For a distributive lattice $L$ the following three assertions are equivalent:

(a) $L$ is relatively complementary.

(b) Each compatible tolerance on $L$ is a congruence.

(c) If $J$ is a proper ideal of $L$ and $F$ is a proper filter of $L$ such that $J \cup F = L$, then $J \cap F = \emptyset$.

7. In the end we shall prove other two theorems concerning tolerance relations on algebras in general.

Theorem 7. Let $\mathfrak{A}_1 = \langle A_1, \mathcal{F}_1 \rangle$, $\mathfrak{A}_2 = \langle A_2, \mathcal{F}_2 \rangle$ be two algebras of the same type, let there exist a homomorphism $\psi$ of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. Let there exist a tolerance $T$ on $A_2$ which is not a congruence and is compatible with $\mathfrak{A}_2$. Then there exists a tolerance $T'$ on $A_1$ which is not a congruence and is compatible with $\mathfrak{A}_1$.

Proof. We construct $T'$ so that for any two elements $x, y$ of $A_1$ we have $x T' y$ if and only if $\psi(x) T \psi(y)$. The relation $T'$ thus constructed is evidently a tolerance. Let $f_1 \in \mathcal{F}_1$ be an $n$-ary relation, let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be elements of $A_1$ such that $x_i T' y_i$ for $i = 1, \ldots, n$. Then $\psi(x_i) T \psi(y_i)$ for $i = 1, \ldots, n$. Let $f_2$ be the operation from $\mathcal{F}_2$ which corresponds to $f_1$ in the homomorphism $\psi$. As $T$ is a tolerance compatible with $\mathfrak{A}_2$, we have $f_2(\psi(x_1), \ldots, \psi(x_n)) T f_2(\psi(y_1), \ldots, \psi(y_n))$. As $f_2(\psi(x_1), \ldots, \psi(x_n)) = \psi(f_1(x_1, \ldots, x_n))$, $f_2(\psi(y_1), \ldots, \psi(y_n)) = \psi(f_1(y_1, \ldots, y_n))$, we have $f_1(x_1, \ldots, x_n) T f_1(y_1, \ldots, y_n)$ and $T'$ is a tolerance compatible with $\mathfrak{A}_1$. As $T$ is not a congruence, there exist elements $a, b, c$ of $A_2$ such that $a T b$, $b T c$, but not $a T c$. Let $a'$ (or $b'$, or $c'$) be an element of $A_1$ such that $\psi(a') = a$ (or $\psi(b') = b$, or $\psi(c') = c$, respectively). Then $a' T' b'$, $b' T' c'$, but not $a' T' c'$ and $T'$ is not a congruence.

Corollary 3. Let an algebra $\mathfrak{A}$ be the direct product of the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. On at least one of the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ let there exist a tolerance compatible with this algebra which is not a congruence. Then there exists a tolerance compatible with $\mathfrak{A}$ which is not a congruence.

Theorem 8. Let $\mathfrak{A} = \langle A, \mathcal{F} \rangle$ be an algebra, $|A| \geq 3$. Let there exist an element $a \in A$ which cannot be obtained as a result of an operation from $\mathcal{F}$. Then there exists a tolerance $T$ compatible with $\mathfrak{A}$ which is not a congruence.
Proof. Let $b$ be an element of $A$ distinct from $a$. Consider the tolerance $T$ consisting of the pairs $(a, a), (a, b), (b, a)$ and of all pairs $(x, y)$ for $x \in A - \{a\}, y \in A - \{a\}$. If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are elements of $A$ and $f \in F$ is an $n$-ary operation, then $f(x_1, \ldots, x_n) \in A - \{a\}, f(y_1, \ldots, y_n) \in A - \{a\}$, thus $f(x_1, \ldots, x_n) T f(y_1, \ldots, y_n)$. Thus $T$ is a tolerance compatible with $\mathfrak{A}$. It is not a congruence, because $a T b, b T c$, but not $a T c$, where $c$ is an arbitrary element of $A - \{a, b\}$.

References


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