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DISCONTINUOUS LIAPUNOV FUNCTIONS FOR DIFFERENTIAL EQUATIONS WITH MEASURABLE RIGHT-HAND SIDES

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I. INTRODUCTION

The behaviour of solutions of the ordinary differential equation

$$\dot{x} = f(x)\,, \tag{1}$$

$f : R^n \to R^n$ being continuous, is frequently studied by means of Liapunov's direct method, i.e. through a scalar smooth function $V: R^n \to R$. The solution of the equation (1) can be defined even without the assumption that the right-hand side of (1) is continuous — it is well-known that measurability and local boundedness is sufficient (see Filippov [1]). For f piecewise continuous it seems to be natural to use piecewise continuous Liapunov functions instead of smooth ones. We shall investigate piecewise Lipschitzian Liapunov functions in connection with stability.

II. NOTATION

Let R^n be n-dimensional Euclidean space, o its zero element, $U(x, \delta)$ the open ball with a center x and a radius δ. The closed convex hull of a set A, $A \subset R^n$, will be denoted by conv A. For $f : R^n \to R^n$, f measurable, the set $\bigcap_{\delta > 0} \bigcap_{\substack{N \\ \mu N = 0}} \operatorname{conv} f(U(x, \delta) - N)$ will be denoted by $K\{f(x)\}$. By Lip A, $A \subset R^n$ we shall understand the set of all functions $f : A \to R^k$ locally Lipschitzian on A. Let τ be a real number, $A \subset R^n$. The set $\{y \in R^n \mid y = \tau . x,\ x \in A\}$ will be denoted by $\tau . A$.

Let $x_1, x_2, \ldots, x_n$ be linearly independent vectors in R^n. The set

$$\{z \in R^n \mid z = \sum_{i=1}^{n} \alpha_i x_i,\ \alpha_i \geqq 0 \text{ for } i = 1, 2, \ldots, n\}$$

is called a cone and its subset H,

$$H = \{z \in R^n,\ z = \sum_{i=1}^{n} \alpha_i x_i,\ \alpha_i \geqq 0 \text{ for } i = 1, 2, \ldots, n \text{ and } \alpha_{i_1} = \alpha_{i_2} = \ldots = \alpha_{i_k} = 0\},$$

$0 \leqq k \leqq n$ is called its $(n - k)$-dimensional face. The relative interior of H (i.e. the interior in the topology of R^{n-k}) is denoted by ri H.

Definition 1. Let $\mathscr{K} = \{K_1, K_2, \ldots, K_m\}$ be a set of cones such that

1) $\operatorname{ri} K_i \cap ri\, K_j = \emptyset$ for each i, j such that $1 \leqq i, j \leqq m$, $i \neq j$;

2) let $H_1, H_2, \ldots, H_r$ be faces of cones from $\mathscr{K}$. Then $\bigcap_{i=1}^{r} H_i$ is a face of each face H_j, $j = 1, 2, \ldots, r$ (i.e. $\bigcap_{i=1}^{r} H_i \subset H_j - ri\, H_j$ for each j, $1 \leqq j \leqq r$).

3) $\bigcup_{i=1}^{m} K_i = R^n$.

Then $\mathscr{K}$ is called a *decomposition of* R^n *into cones.* The set comprising all faces of all cones from $\mathscr{K}$ is denoted by $\mathscr{R}(\mathscr{K})$, its subset comprising all $(n - 1)$-dimensional faces is denoted by $\mathscr{R}_1(\mathscr{K})$.

III. DIFFERENTIAL EQUATIONS AND SOLUTIONS IN THE SENSE OF FILIPPOV

Let $f : R^n \to R^n$ be measurable and bounded on R^n, i.e. there is a constant B such that $\|f(x)\| \leqq B$ a.e. in R^n. We shall consider a differential equation of the form (1). According to Filippov we say that a function $x(\cdot)$ is a solution of the equation (1) on an interval $I = \langle t_1, t_2 \rangle$, $t_1 < t_2$ if $x(\cdot)$ is absolutely continuous on I and $\dot{x}(t) \in K\{f(x(t))\}$ a.e. in I (see Filippov [1]). The symbol $x(\cdot, x_0)$ will denote a solution of (1) which satisfies the initial condition $x(0, x_0) = x_0$. We shall suppose $o \in K\{f(o)\}$, i.e. the function $x(\cdot, o)$ with the property $x(t, o) = o$ for every t is a solution of (1) on $\langle 0, +\infty)$. This solution will be called the trivial solution.

IV. DISCONTINUOUS LIAPUNOV FUNCTIONS AND STABILITY

Let $\mathscr{K}$ be a decomposition of R^n into cones. We should try to investigate Liapunov stability of the trivial solution by means of scalar functions $V_H(\cdot)$ defined and Lipschitzian on every $H \in \mathscr{R}(\mathscr{K})$ and with the usual properties concerning their derivatives along solutions. To insure stability, some additional conditions concerning the values $V_L(x)$ and $V_H(x)$ for x in $L \cap H$, $L, H \in \mathscr{R}(\mathscr{K})$ have to be fulfilled. To express these conditions and eventually to define Liapunov functions, certain subsets of $\mathscr{R}(\mathscr{K})$ are introduced.

Let $x \in R^n$ and let $y(\cdot, x)$ be a solution of the equation (1). We adopt the following notation:

$$\mathfrak{N}^+(y(\cdot, x)) = \bigcap_{\delta>0} \{H \in \mathscr{R}(\mathscr{K}) \mid \exists_{\tau} \tau \in (0, \delta),\ y(\tau, x) \in \text{ri}\, H\}\,,$$

$$\mathfrak{N}^-(y(\cdot, x)) = \bigcap_{\delta>0} \{H \in \mathscr{R}(\mathscr{K}) \mid \exists_{\tau}\, \tau \in (-\delta, 0),\ y(\tau, x) \in \text{ri}\, H\}\,,$$

$$\mathfrak{M}^+(x) = \bigcup \mathfrak{N}^+(y(\cdot, x))\,,$$

$$\mathfrak{M}^-(x) = \bigcup \mathfrak{N}^-(y(\cdot, x))\,.$$

Here the symbol $\bigcup$ denotes the union over all solutions $y(\cdot, x)$ of the equations (1) for which $y(0, x) = x$. The following lemma concerning the set $\mathfrak{N}^+(y(\cdot, x))$ is valid:

Lemma 1. *Let $x \in R^n$ and let $y(\cdot, x)$ be a solution of (1) on an interval $\langle 0, T\rangle$, $T > 0$. Then there exists a δ, $\delta > 0$, such that if there exist τ and H, $\tau \in (0, \delta)$, $H \in \mathscr{R}(\mathscr{K})$ such that $y(\tau, x) \in \text{ri}\, H$, then $H \in \mathfrak{N}^+(y(\cdot, x))$.*

Proof. Let us suppose this lemma to be false. Then a vector x in R^n and a solution $y(\cdot, x)$ exist such that for every positive integer n there exist a face H_n in $\mathscr{R}(\mathscr{K})$ and a number τ_n in $(0, 1/n)$ such that simultaneously $y(\tau_n, x) \in \text{ri}\, H_n$ and $H_n \notin \mathfrak{N}^+(y(\cdot, x))$. This and the finiteness of the set $\mathscr{R}(\mathscr{K})$ implies that there exists a face H in $\mathscr{R}(\mathscr{K})$ such that $H \in \mathfrak{N}^+(y(\cdot, x))$ and simultaneously $H \notin \mathfrak{N}^+(y(\cdot, x))$. This contradiction proves the assertion of the lemma.

Note. An analogous lemma concerning the set $\mathfrak{N}^-(y(\cdot, x))$ is valid.

Definition 2. A mapping $V : R^n \to R$ is called *a Liapunov function for the equation* (1) *and the decomposition $\mathscr{K}$* if the following conditions are satisfied:

1) $V(\cdot)$ is continuous at o and $V(o) = 0$;

2) for every H in $\mathscr{R}(\mathscr{K})$ there exists a $V_H \in \text{Lip}\, H$ such that

   α) $V(x) = V_H(x)$ whenever $x \in \text{ri}\, H$,

   β) $V_H(x) > 0$ for every x in $H - \{o\}$;

3) if $R \in \mathfrak{M}^+(x)$, $S \in \mathfrak{M}^-(x)$, $x \in \text{ri}\, H$ then

$$V_R(x) \leqq V_H(x) \leqq V_S(x)\,;$$

4) for each $x \in R^n$ and $H \in \mathfrak{M}^+(x)$,

$$\overline{\lim_{\tau \searrow 0}} \frac{V_H(x + \xi \,.\, \tau) - V_H(x)}{\tau} \leqq 0$$

holds provided ξ satisfies the following conditions:

α) there is τ_0 positive such that

$$x + \xi . \tau \in H \cap \{x + \tau . K\{f(x)\}\}$$

for every τ in $(0, \tau_0)$;

β) there is a solution $y(\cdot, x)$ of the equation (1) with the property $\dot{y}(0, x) = \xi$.

Note. We shall say briefly "*Liapunov function*" instead of the rather long term "*Liapunov function for the system* (1) *and decomposition* $\mathscr{K}$".

Our goal is to prove that the existence of a Liapunov function implies the stability of the trivial solution. We shall need the following two lemmas:

Lemma 2. *Let $x(\cdot, x_0)$ be a solution of the equation* (1) *on an interval $I = \langle 0, T \rangle$, $0 < T < \infty$ and let $V(\cdot)$ be a Liapunov function. Then there exists a positive constant M such that for every t in I there exists a δ positive such that*

(2) $$V(x(\tau, x_0)) - V(x(t, x_0)) \leqq M(\tau - t) \quad \textit{whenever} \quad \tau \in \langle t, t + \delta \rangle \cap I ,$$

and

(3) $$V(x(t, x_0)) - V(x(\tau, x_0)) \leqq M(t - \tau) \quad \textit{whenever} \quad \tau \in \langle t - \delta, t \rangle \cap I .$$

Proof. We prove the part concerning the inequality (2). The inequality (3) can be proved in a similar way.

Let $t \in I$, $x(t, x_0) \in \operatorname{ri} P$, $P \in \mathscr{R}(\mathscr{K})$ and let $R \in \mathfrak{M}^+(x)$. Since the interval I is compact and $x(\cdot, x_0)$ is a solution of the equation (1) with a bounded right-hand side, it follows that there is a constant B_1 such that

$$\|x(t, x_0)\| \leqq B_1 \quad \text{whenever} \quad t \in I .$$

The assumption $R \in \mathfrak{M}^+(x)$ implies $V_R(x) \leqq V_P(x)$, and to the constant B_1 there is L such that for each face H, $H \in \mathscr{R}(\mathscr{K})$ the inequality

$$V_H(x_1) - V_H(x_2) \leqq L\|x_1 - x_2\|$$

holds provided $x_i \in H$, $\|x_i\| \leqq B_1$, $i = 1, 2$. We have

$$V(x(\tau, x_0)) - V(x(t, x_0)) = V_R(x(\tau, x_0)) - V_P(x(t, x_0)) \leqq$$
$$\leqq V_R(x(\tau, x_0) - V_R(x(t, x_0) \leqq L\|x(\tau, x_0) - x(t, x_0)\|$$

whenever $x(\tau, x_0) \in \operatorname{ri} R$ and $\tau \in I$. This and Lemma 1 implies that there exists a $\delta > 0$ such that

(4) $$V(x(\tau, x_0)) - V(x(t, x_0)) \leqq L\|x(\tau, x_0) - x(t, x_0)\|$$
$$\text{whenever} \quad \tau \in I \cap \langle t, t \in \delta \rangle .$$

Since $x(\cdot, x_0)$ is a solution of the equation (1) with a bounded right-hand side, it follows that (see part III)

$$\|x(\tau, x_0) - x(t, x_0)\| \leqq B(\tau - t) \quad \text{whenever} \quad 0 \leqq t \leqq \tau \leqq T.$$

This and the inequality (4) yields

$$V(x(\tau, x_0)) - V(x(t, x_0)) \leqq L \,.\, B(\tau - t) \quad \text{whenever} \quad \tau \in I \cap \langle t, t + \delta \rangle,$$

and the lemma is proved.

Lemma 3. *Let $x(\cdot, x_0)$ be a solution of the equation (1) on an interval I and let $V(\cdot)$ be a Liapunov function. If*

(5) $$A = \{t \in I \mid \dot{x}(t, x_0) \notin K\{f(x(t, x_0))\} \text{ or } \dot{x}(\cdot, x_0) \text{ does not exist at } t\}$$

then $\mu(A) = 0$ and for every t in $I - A$ and for every ε positive there exists γ such that $0 < \gamma < \varepsilon$ and

(6) $$V(x(\tau, x_0)) - V(x(t, x_0)) \leqq (\tau - t)\,\varepsilon \quad \text{whenever} \quad \tau \in \langle t, t + \gamma \rangle \cap I.$$

Proof. Since $x(\cdot, x_0)$ is a solution of (1) the result $\mu(A) = 0$ is obvious. Let $t \in I - A$ and let $x(t, x_0) \in \operatorname{ri} P$, $P \in \mathscr{R}(\mathscr{K})$. For $R \in \mathfrak{N}^+(x(\cdot, x(t, x_0))$ let us denote $M_R = \{\tau \in I \mid \tau \geqq t,\, x(\tau, x_0) \in \operatorname{ri} R\}$. Then t is a cluster point of the set M_R and it is possible to investigate the derivative of the function $V(x(\cdot, x_0))$ at t with respect to M_R. Since $t \in I - A$ we obtain

(7) $$\lim_{\tau \to t} \frac{x(\tau, x_0) - x(t, x_0)}{\tau - t} = \xi \in K\{f(x(t, x_0))\}.$$

We shall prove

(8) $$\overline{\lim_{\substack{\tau \searrow t \\ \tau \in M_R}}} \frac{V_R(x(\tau, x_0) - V_R(x(t, x_0)}{\tau - t} \leqq 0.$$

Since $V_R(x(\tau, x_0)) = V(x(\tau, x_0))$ for $\tau \in M_R$ and $V_R(x(t, x_0)) \leqq V_P(x(t, x_0)) = V(x(t, x_0))$ it follows from Lemma 1 and the inequality (8) that for every ε positive there exists a γ positive such that the inequality (6) holds.

To prove (8) we show first $x(t, x_0) + \xi(\tau - t) \in R$ whenever $\tau \in \langle t, t + \gamma \rangle$ and γ is sufficiently small. It is easy to show that there exists a γ positive such that the vector

$$x(t, x_0) + \varkappa \frac{x(\tau, x_0) - x(t, x_0)}{\tau - t}$$

belongs to R whenever $\tau \in M_R$ and $\varkappa \in \langle 0, \gamma \rangle$. Since the face R is closed it follows from (7) that

$$x(t, x_0) + \varkappa \xi \in R \quad \text{whenever} \quad \varkappa \in \langle 0, \gamma \rangle.$$

Hence, we have for τ in $\langle t, t + \gamma\rangle$

$$\begin{aligned} V_R(x(\tau, x_0)) - V_R(x(t, x_0)) = \\ = V_R(x(t, x_0) + \xi(\tau - t) + o(\tau - t)) - V_R(x(t, x_0) + \xi(\tau - t)) + \\ + V_R(x(t, x_0) + \xi(\tau - t)) - V_R(x(t, x_0)) \leqq \\ \leqq L\|o(\tau - t)\| + V_R(x(t, x_0) + \xi(\tau - t)) - V_R(x(t, x_0)) \end{aligned}$$

which yields

$$\overline{\lim_{\substack{\tau \searrow t \\ \tau \in M_R}}} \frac{V_R(x(\tau, x_0)) - V_R(x(t, x_0))}{\tau - t} \leqq \overline{\lim_{\tau \searrow t}} \frac{V_R(x(t, x_0) + \xi(\tau - t)) - V_R(x(t, x_0))}{\tau - t} \leqq 0$$

and the lemma is proved.

The main result is the following

Theorem. *Let f be measurable and bounded and let V be a Liapunov function for the decomposition $\mathscr{K}$ and the equation*

$$\dot{x} = f(x).$$

Then the trivial solution is stable.

Proof. Let $x(\cdot, x_0)$ be a solution. We prove that the function $V(x(\cdot, x_0))$ is non-increasing. This and the well-known prolongability theorem (see Filippov [1] p. 112) yields stability by means of the standard procedure (see e.g. Zubov [2] p. 47).

Let ε be an arbitrary positive number and let real numbers t_1, t_2 such that $0 \leqq \leqq t_1 \leqq t_2$ be given. It follows from the Vitali covering theorem and from Lemma 3 that there exists a finite system of closed disjoint intervals I_j, $I_j = \langle \tau_1^{(j)}, \tau_2^{(j)}\rangle$ such that

$$\langle t_1, t_2\rangle = \bigcup_{j=1}^{k} I_j \cup \Omega, \quad \mu(\Omega) < \varepsilon$$

and

$$\sum_{j=1}^{k} (V(x(\tau_2^{(j)}, x_0)) - V(x(\tau_1^{(j)}, x_0))) \leqq \varepsilon(t_2 - t_1).$$

Since $\Omega = \langle t_1, t_2\rangle - \bigcup_{j=1}^{k} I_j$ it follows that $\Omega = \bigcup_{i=1}^{q} J_i$, where J_i, $i = 1, 2, \ldots, q$ are disjoint intervals. Let a J_r, $r \in \{1, \ldots, q\}$ be fixed. Then $\bar{J}_r = \langle \tau_1^{(r)}, \tau_2^{(r)}\rangle$ and it follows from Lemma 2 that there exists a positive constant M which is independent of ε such that for every t in J_r there exists an open interval $U(t, \delta)$,

$$U(t, \delta) = (t - \delta, t + \delta)$$

such that

$$V(x(\tau, x_0)) - V(x(t, x_0)) \leqq M(\tau - t)$$

whenever $\tau \in \langle t, t + \delta \rangle$. The neighbourhoods $U(t, \delta)$ form an open covering of the interval $\bar{J}_r$ and there exists a finite system of intervals $U(t^{(i)}, \delta)$, $i = 1, 2, \ldots, s$ which still covers the interval $\bar{J}_r$. Using these intervals $U(t^{(i)}, \delta)$ it is easy to prove that there exist intervals L_i, $L_i = \langle \eta_i, \eta_{i+1} \rangle$, $i = 1, 2, \ldots, s_1$ which still cover the interval $\bar{J}_r$ and have disjoint interiors such that

$$V(x(\eta_{i+1}, x_0)) - V(x(\eta_i, x_0)) \leqq M(\eta_{i+1} - \eta_i) .$$

This results in the inequality

$$V(x(\tau_2^{(r)}, x_0)) - V(x(\tau_1^{(r)}, x_0)) \leqq M(\tau_2^{(r)} - \tau_1^{(r)}) .$$

Since the intervals I_i, $i = 1, 2, \ldots, k$ and J_i, $i = 1, 2, \ldots, q$ cover the interval $\langle t_1, t_2 \rangle$ and have disjoint interiors, we obtain

$$V(x(t_2, x_0)) - V(x(t_1, x_0)) \leqq \varepsilon(t_2 - t_1) + M\mu(\bigcup_{i=1}^{q} J_i) \leqq \varepsilon(t_2 - t_1) + \varepsilon M$$

where ε is an arbitrary positive number and the proof is complete.

Literature

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