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l_1-CONTINUOUS PARTITIONS OF UNITY ON NORMED SPACES

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If X is a uniform space, denote by $\mathscr{U} = \{U_\beta\}$ a uniform covering of X and by $\mathscr{A} = \{f_\alpha\}_{\alpha \in I}$ a partition of unity on X. A partition of unity $\mathscr{A}$ is subordinate to $\mathscr{U}$ (we write $\mathscr{A} \prec \mathscr{U}$) if the supports of f_α form a covering which refines $\mathscr{U}$. The following notion will play the basic role:

Definition. A partition of unity $\{f_\alpha\}_{\alpha \in I}$ is l_p-continuous $(1 \leqq p \leqq \infty)$ if the mapping

$$\{x \to \{f_\alpha(x)\}_{\alpha \in I}\} : X \to l_p(I)$$

is uniformly continuous.

The simplest assertion of the type studied here is the following (see [2]):

For every uniform covering $\mathscr{U}$ there exists a partition of unity $\mathscr{A} = \{f_\alpha\}_{\alpha \in I}$ subordinate to $\mathscr{U}$ such that the family $\{f_\alpha\}_{\alpha \in I}$ is equiuniformly continuous. (In other words, $\mathscr{A}$ is l_∞-continuous.)

We will show in section 1 that an analogous result holds for l_p-continuity, if $1 < p < \infty$. This is quite elementary. The case $p = 1$ is the most important, and it seems to be non trivial. Let us notice the assertion:

A partition of unity $\{f_\alpha\}_{\alpha \in I}$ is l_1-continuous if and only if $\{\sum_{I'} f_\alpha\}_{I' \subset I}$ is equiuniformly continuous (see [1]).

For $p = 1$, our preceding result (the existence of l_1-continuous partition of unity subordinate to the given $\mathscr{U}$) does not hold for any infinite-dimensional normed space, and this is the main result.

To obtain the main result, Theorem 1.4, we have to make a detailed study of partitions of unity in Euclidean spaces.

We define a useful notion, namely the integral partition of unity. This is done in Section 2, the main result being Theorem 2.9.

In Section 3, we refer to 2 and show how "the modulus of continuity" of an arbitrary partition of unity in E_n depends on the dimension n.

Finally, in Section 4, we use the preceding results (Corollary 3.9 and Lemma 3.10). This is immediate for Hilbert spaces. In the case of an arbitrary normed space, we use a theorem of Dvoretzky.

1.

In this section we deal with l_p-continuity for $p > 1$ and state the main theorem for $p = 1$.

1.1. Proposition. *Let $\mathcal{U}$ be a uniform covering of a uniform space X. Then there exists an l_∞-continuous partition of unity subordinate to $\mathcal{U}$.*

Proof. see [2], p. 62.

1.2. Proposition. *Let $\mathcal{U}$ be a uniform covering of X, let $1 < p < \infty$. Then there exists an l_p-continuous partition of unity subordinate to $\mathcal{U}$.*

Proof. By 1.1, there exists a partition of unity $\mathcal{A} = \{f_\alpha\}_{\alpha\in I}$ such that the family $\{f_\alpha\}_{\alpha\in I}$ is uniformly equicontinuous. Define the functions

$$f_{\alpha,n} = \left(\left(f_\alpha \wedge \frac{1}{n}\right) - \frac{1}{n+1}\right) \vee 0\,, \quad n = 1, 2, \ldots .$$

It is clear that $\{f_{\alpha,n}\}$ forms a partition of unity. Finally, define (for some $M(n)$)

$$g_{\alpha,n,i} = \frac{f_{\alpha,n}}{M(n)}\,, \quad i = 1, 2, \ldots, M(n)\,.$$

We shall show that the family $\{g_{\alpha,n,i}\}_{\alpha\in I, n\in N, i\leqq M}$ is the desired partition of unity, if $M(n)$ is suitably chosen.

For $x, y \in X$ put

$$\varrho(x, y) = \sup\{|f_\alpha(x) - f_\alpha(y)|\}\,.$$

Since $\{f_\alpha\}$ satisfies 1.1, ϱ is uniformly continuous. We have

$$\|g_{\alpha,n,i}(x) - g_{\alpha,n,i}(y)\|_{l_p} = \Big(\sum_{\alpha,n,i} |g_{\alpha,n,i}(x) - g_{\alpha,n,i}(y)|^p\Big)^{1/p} =$$
$$= \Big(\sum_n M^{1-p} \sum_\alpha |f_{\alpha,n}(x) - f_{\alpha,n}(y)|^p\Big)^{1/p} \leqq \Big(\sum_n M^{1-p}\, 2(n+1)\, \varrho(x, y)^p\Big)^{1/p}$$

(because $f_{\alpha,n} > 0$ for at most $n + 1$ indices $\alpha \in I$). Choose $M(n)$ such that $2M^{1-p}(n + 1) < 1/2^n$. We get

$$\|\{g_{\alpha,n,i}(x) - g_{\alpha,n,i}(y)\}\|_{l_p} \leqq \varrho(x, y)$$

and the l_p-continuity of $\{g_{\alpha,n,i}\}$ is proved, q.e.d.

Now we shall investigate the case $p = 1$.

1.3. Definition. Let $\mathscr{A} = \{f_\alpha\}_{\alpha\in I}$ be a partition of unity on a metric space (X, ϱ). Define a function ("modulus of continuity")

$$(1) \qquad H_{\mathscr{A}}(t) = \sup_{\varrho(x,y)\leqq t} \|\{f_\alpha(x) - f_\alpha(y)\}\|_{l_1}, \quad t \geqq 0 .$$

Notice that $H_{\mathscr{A}}$ is monotone, $H_{\mathscr{A}} \leqq 2$. Obviously $\mathscr{A}$ is l_1-continuous if and only if $H_{\mathscr{A}}$ is continuous at $t = 0$.

1.4. Theorem. *Let $\mathscr{U}$ denote the covering consisting of all open balls with radius 1 on an infinite dimensional normed space X. Then there is no l_1-continuous partition of unity subordinate to $\mathscr{U}$.*

Remark. This theorem answers the problem in [1], p. 107.

Proof is contained in Sections 2, 3, 4. Let us sketch the idea of proof. Suppose, for simplicity, that X is a Hilbert space. Let X_n be an n-dimensional subspace of X, let $\mathscr{A}_n$ be the restriction to X_n of some partition of unity $\mathscr{A}$ in X.

It is obvious that

$$(2) \qquad H_{\mathscr{A}}(t) \geqq H_{\mathscr{A}_n}(t) \quad \text{for all} \quad t \geqq 0 .$$

X_n is a Euclidean space. If $\mathscr{A} \prec \mathscr{U}$ we shall show in Section 3 that

$$\lim_{n\to\infty} H_{\mathscr{A}_n}(t) = 2 \quad \text{for all} \quad t > 0 .$$

Then we use (2) to show that $H_{\mathscr{A}}$ is not continuous at $t = 0$.

2.

In this section we investigate partitions of unity on Euclidean spaces.

2.1. Denote by $E = E_n$ the Euclidean space with the Euclidean norm $\| \ \|$. The group G of all isometries on E has the following properties:

(3) i) for all $x, y \in E$ there exists $T \in G$ such that $Tx = y$.

ii) for all $x, y, z \in E$ satisfying $\|y - z\| = \|x - z\|$ there exists $T \in G$ such that $Tz = z$, $Tx = y$. G is a locally compact group in the topology defined by all the pseudometrics of the type

$$\varrho_F(T_1, T_2) = \sup_{x\in F} \|T_1 x - T_2 x\|$$

where F is a finite subset of E.

On G there exists Haar unimodular (left and right) measure m (see for example [4], (2, 7, 16), example 7). For any $x \in E$ consider the mapping

$$(4) \qquad \{T \to Tx\} : G \to E .$$

The preimage of an arbitrary ball $\{z, \|z - y\| \leqq 1\}$ equals the set $\{T, \|Tx - T_y x\| \leqq 1\}$ where $T_y x = y$. This is a compact set in G. Thus the image of m with respect to $\{T \to Tx\}$ exists. Denote it by μ. Since m is right-invariant, μ does not depend on the choice of $x \in E$. Since m is left-invariant, μ is invariant with respect to G. Hence μ is, up to a constant, the Lebesgue measure.

2.2. Definition. Let A be a function on $E \times E$. We say that A is *an integral partition of unity* if the following holds:

(5) i) $A \geqq 0$;

ii) $A(\cdot, y) \in L_1(E, \mu)$ for almost all $y \in E$;

iii) $\int_E A(x, y)\,\mathrm{d}x(\mu) = 1$ for almost all $y \in E$.

If further $\|x - y\| \geqq 1$ implies $A(x, y) = 0$ we say that A is subordinate to $\mathscr{U}$ ($\mathscr{U}$ will always denote the covering of all open balls with radius 1), and write $A \prec \mathscr{U}$. We put analogously as in (1)

$$H_A(t) = \sup_{\|x-y\| \leqq t} \int_E |A(z, x) - A(z, y)|\,\mathrm{d}z(\mu)\,. \tag{1'}$$

Our aim is to restrict ourselves to the partitions of the type (5), which are much more convenient for the forthcoming computations. This enables us to establish the following lemma which is the main step in the proof of 1.4.

2.3. Lemma. *Let $\mathscr{A} = \{f_\alpha\}_{\alpha \in I}$ be a partition of unity. Then for every $\varepsilon > 0$ there exists an integral partition of unity A such that*

$$H_A(t) \leqq (1 + \varepsilon)\, H_{\mathscr{A}}(t) \quad \text{for all} \quad t \geqq 0\,.$$

Further, if $\mathscr{A} \prec \mathscr{U}$, then $A \prec \mathscr{U}$.

Proof. First we introduce some preliminary definitions and constructions.

2.4. Definition. If $\mathscr{A} = \{f_\alpha\}_{\alpha \in I}$ is a partition of unity, $T \in G$, put

$$\mathscr{A}^T = \{f_\alpha \circ T\}_{\alpha \in I}\,.$$

Analogously, for integral partitions of unity, put

$$A^T(x, y) = A(Tx, Ty)\,.$$

2.5. Put

$$G_{xy} = \{T \in G,\ Tx = y\}\,.$$

G_{xy} equals the preimage of y with respect to the mapping $\{T \to Tx\}$. Using the desintegration theorem (see [3], VI, § 3,1) we obtain:

There exist Radon measures m_{xy} on G_{xy} such that $m_{xy} \geqq 0$ and $\{m_{xy}, y \in E\}$ form a desintegration of m with respect to the mapping $\{T \to Tx\} : (G, m) \to (E, \mu)$.

2.6. First we describe the construction of A in a simpler situation, namely when S (instead of E) is a sphere S in E_{n+1}, and G is a compact group of all isometries on S. The measures m, μ, m_{xy} are defined analogously as before. We suppose $\|m\| = 1$ and put

$$A(x, y) = \sum_I \int_{G_{x_\alpha x}} f_\alpha(T^{-1}y) \, \mathrm{d}T(m_{x_\alpha x})$$

where for each $\alpha \in I$ we choose $x_\alpha \in S$ such that

$$\|y - x_\alpha\| > 1 \quad \text{implies} \quad f_\alpha(y) = 0 \,. \tag{6}$$

2.7. In the case of E there is a minor technical complication (m is not finite). Choose, for each $\alpha \in I$, $x_\alpha \in E$ satisfying (6). Let i be an integer. Put

$$K_i = \{y \in E, \ \|y\| \leqq i\} \,,$$
$$I_i = \{\alpha \in I, \ x_\alpha \in K_i\} \,.$$

It is easy to show that

$$\sum_{I_i} f_\alpha(x) = 1 \quad \text{for all} \quad x \in K_{i-1} \,, \tag{7}$$
$$\sum_{I_i} f_\alpha(x) = 0 \quad \text{for all} \quad x \notin K_{i+1} \,.$$

Now we construct analogously as in 2.6

$$A_i(x, y) = \sum_{I_i} \int_{G_{x_\alpha x}} f_\alpha(T^{-1}(y)) \, \mathrm{d}T(m_{x_\alpha x}) \,.$$

Notice that $\mathscr{A} \prec \mathscr{U}$ implies $A_i \prec \mathscr{U}$. I_i is countable, $\int_{G_{x_\alpha(\cdot)}} f_\alpha(T^{-1}y) \, \mathrm{d}T(m_{x_\alpha(\cdot)})$ is a μ-integrable function (according to the desintegration theorem). Then we can write

$$\int_E \sum_{I_i} f_\alpha(x) \, \mathrm{d}x(\mu) = \int_G \sum_{I_i} f_\alpha(T^{-1}y) \, \mathrm{d}T(m) = \sum_{I_i} \int_G f_\alpha(T^{-1}y) \, \mathrm{d}T(m) = \tag{8}$$

$$= (\text{desintegration theorem}) = \sum_{I_i} \int_E \left(\int_{G_{x_\alpha x}} f_\alpha(T^{-1}y) \, \mathrm{d}T(m_{x_\alpha x}) \right) \mathrm{d}x(\mu) =$$

$$= \int_E \left(\sum_{I_i} \int_{G_{x_\alpha x}} f_\alpha(T^{-1}y) \, \mathrm{d}T(m_{x_\alpha x}) \right) \mathrm{d}x(\mu) = \int_E A_i(x, y) \, \mathrm{d}x(\mu) \,.$$

Putting $\lambda_i = \int_E \sum_{I_i} f_\alpha(x)\, \mathrm{d}x(\mu)$ (notice that $\lambda_i \gtreqqless \mu(K_{i-1})$) we conclude:

(9) $$\frac{A_i}{\lambda_i} \text{ is an integral partition of unity .}$$

Now estimate H_{A_i}. Write

(10) $$\int_E |A_i(z, x) - A_i(z, y)|\, \mathrm{d}z(\mu) \leqq$$

$$\leqq \sum_{I_i} \int_E \left(\int_{G_{x_\alpha z}} |f_\alpha(T^{-1}x) - f_\alpha(T^{-1}y)|\, \mathrm{d}T(m_{x_\alpha z}) \right) \mathrm{d}z(\mu) =$$

$$= \sum_{I_i} \int_G |f_\alpha(T^{-1}x) - f_\alpha(T^{-1}y)\, \mathrm{d}T(m)$$

according to the desintegration theorem. Put

$$G_i = \{T \in G,\ T^{-1}x \in K_i\} .$$

Suppose $\|x - y\| \leqq t \leqq 2$. (7) implies

$$T^{-1}x \notin K_{i+3} \Rightarrow T^{-1}y \notin K_{i+1} \Rightarrow f_\alpha(T^{-1}x) = f_\alpha(T^{-1}y) = 0 .$$

Returning to (10), we obtain

(11) $$\int_E |A_i(z, x) - A_i(z, y)|\, \mathrm{d}z(\mu) \leqq$$

$$\leqq \sum_{I_i} \int_{G_{i+3}} |f_\alpha(T^{-1}x) - f_\alpha(T^{-1}y)|\, \mathrm{d}T(m) \leqq$$

$$\leqq \int_{G_{i+3}} H_{\mathscr{A}T^{-1}}(t)\, \mathrm{d}T(m) = H_{\mathscr{A}}(t)\, \mu(K_{i+3}) .$$

Combining (11) with (9), we conclude

$$H_{A_i/\lambda_i}(t) \leq \frac{\mu(K_{i+3})}{\mu(K_{i-1})} H_{\mathscr{A}}(t) .$$

Obviously $\lim_{i \to \infty} (\mu(K_{i+3}/\mu(K_{i-1}) = 1$, thus for sufficiently large i, A_i/λ_i is the desired integral partition of unity, q.e.d.

Remark. It is possible to show that the family $\{A_i/\lambda_i\}$ is equicontinuous. Passing to limits we can prove the lemma also for $\varepsilon = 0$.

2.8. Lemma. *A_i from the preceding lemma satisfies the property $A_i^T = A_i$ for each $T \in G$ (A_i is "symmetric").*

Proof. Remind that

$$A_i^T(x, y) = \sum_{I_i} \int_{G_{x_\alpha Tx}} f_\alpha(U^{-1} Ty) \, \mathrm{d}U(m_{x_\alpha Tx}) .$$

Consider the diagram

$$(G, m) \xrightarrow{\{U \to T \circ U\}} (G, m) \xrightarrow{\{U \to Ux_\alpha\}} (E, \mu) \xrightarrow{\{x \to T^{-1} x\}} (E, \mu)$$

(the mappings will be shortly denoted by $\varphi_1, \varphi_2, \varphi_3$). Then, if $\{m_{x_\alpha x}\}$ is the desintegration of m with respect to $\varphi_3 \circ \varphi_2 \circ \varphi_1$, then $\{\varphi_1(m_{x_\alpha x})\}$ is the desintegration of m with respect to $\varphi_3 \circ \varphi_2$. But $\{m_{x_\alpha Tx}\}$ is also the desintegration of m with respect to $\varphi_3 \circ \varphi_2$. Then we have for almost all $x \in (E, \mu)$ $m_{x_\alpha Tx} = \varphi_1(m_{x_\alpha x})$ and

$$A_i^T(x, y) = \sum_{I_i} \int_{G_{x_\alpha Tx}} f_\alpha(U^{-1} Ty) \, \mathrm{d}U(m_{x_\alpha Tx}) = \sum_{I_i} \int_{G_{x_\alpha Tx}} f_\alpha(U^{-1} y) \, \mathrm{d}U(\varphi_1(m_{x_\alpha x})) =$$

$$= \sum_{I_i} \int_{G_{x_\alpha x}} f_\alpha(U^{-1} y) \, \mathrm{d}U(m_{x_\alpha x}) = A_i(x, y) , \quad \text{q.e.d.}$$

Summarizing Lemma 2.3 and Lemma 2.8, we obtain

2.9. Theorem. *For every $\varepsilon > 0$ and for every partition of unity $\mathscr{A}$ there exists a symmetric integral partition of unity A such that*

$$H_A(t) \leqq (1 + \varepsilon) H_{\mathscr{A}}(t) \quad \textit{for all} \quad t \geqq 0$$

and $\mathscr{A} \prec \mathscr{U}$ implies $A \prec \mathscr{U}$.

Remark. Using Remark 2.7, one can prove the theorem also for $\varepsilon = 0$.

3.

Now we will study the symmetric integral partitions of unity. In virtue of (3) we can say that A is symmetric if and only if there exists a locally integrable function $f : \langle 0, \infty) \to E_1$ such that

$$(12) \qquad A(x, y) = f(\|x - y\|) \quad \text{for almost all} \quad x, y \in (E \times E, \mu \times \mu) .$$

3.1. Definition. Let A be a symmetric integral partition of unity. Let $1 \geqq s > 0$. Put

$$A^s(x, y) = f\left(\frac{\|x - y\|}{s}\right) .$$

3.2. Lemma. *The following holds:*

i) $$\int_E A^s(x, y)\,\mathrm{d}x(\mu) = s^n$$

(*then A^s/s^n is an integral partition of unity*);

ii) $$H_{A^s/s^n}(t) = H_A\left(\frac{t}{s}\right).$$

Proof is easy (consider the mapping

$$\{x \to s\,.\,x\} : E \to E)\,.$$

3.3. Definition. Let A be an integral symmetric partition of unity. Put

$$\hat{f}(t) = \int_0^1 f\left(\frac{t}{s}\right)\frac{1}{s^n}\,\mathrm{d}s\,, \quad \hat{A}(x, y) = \hat{f}(\|x - y\|)\,.$$

It is easy to show that $\hat{A}$ is a symmetric integral partition of unity.

3.4. Lemma.

$$H_{\hat{A}}(t) \leqq \int_0^1 H_A\left(\frac{t}{s}\right)\mathrm{d}s\,.$$

Proof. Put $\|x - y\| = t$. Then

$$\int_E |\hat{A}(z, x) - \hat{A}(z, y)|\,\mathrm{d}z \leqq \int_E \left(\int_0^1 \left|\frac{A^s}{s^n}(z, x) - \frac{A^s}{s^n}(z, y)\right|\,\mathrm{d}s\right)\mathrm{d}z = \int_0^1 H_A\left(\frac{t}{s}\right)\mathrm{d}s$$

according to 3.2, ii), q.e.d.

3.5. Lemma. *$\hat{f}$ is a decreasing and differentiable function on $\langle 0, \infty)$.*

Proof. Let $t' \geqq t$. Obviously

$$\hat{f}(t') = \int_0^1 f\left(\frac{t'}{s}\right)\frac{1}{s^n}\,\mathrm{d}s = \left(\frac{t}{t'}\right)^{n-1}\int_0^{t/t'} f\left(\frac{t}{s}\right)\frac{1}{s^n}\,\mathrm{d}s < \int_0^1 f\left(\frac{t}{s}\right)\frac{1}{s^n}\,\mathrm{d}s = \hat{f}(t)\,.$$

Analogously one can compute

$$\hat{f}'(t) = -\left(\frac{n-1}{t}\hat{f}(t) + \frac{f(t)}{t^{n+1}}\right), \quad \text{q.e.d.}$$

3.6. Definition. Define a function $\Phi : \Phi(x, y) = 0$ for each x, y satisfying $\|x - y\| \geqq 1$, $\Phi(x, y) = 1$ for each x, y satisfying $\|x - y\| < 1$.

We suppose that the μ-volume of the unit ball equals 1. Then Φ is an integral partition of unity.

3.7. Proposition. *Let A be a symmetric integral partition of unity, $A \prec \mathscr{U}$. Then*

i) $$\hat{A}(x, y) = -\int_0^1 \Phi^{1-s}(x, y)\hat{f}'(1-s)\,ds\,;$$

ii) $$1 = -\int_0^1 (1-s)^n \hat{f}'(1-s)\,ds\,;$$

iii) $$H_{\hat{A}}(t) = -\int_0^1 H_{\Phi^{1-s}}(t)\hat{f}'(1-s)\,ds\,.$$

Proof.

i) Write $\|x - y\| = t$. Then

$$\hat{f}(t) = -\int_t^1 \hat{f}'(s)\,ds = -\int_0^{1-t} \hat{f}'(1-s)\,ds = -\int_0^1 \hat{f}'(1-s)\,\Phi^{1-s}(x, y)\,ds\,.$$

ii) follows easily from i).

iii) $$\int_E |\hat{A}(z, x) - \hat{A}(z, y)|\,dz(\mu) =$$

$$= \int_E \left|\int_0^1 (\Phi^{1-s}(z, x) - \Phi^{1-s}(z, y))\hat{f}'(1-s)\,ds\right| dz =$$

$$= -\int_E \left(\int_0^1 \hat{f}'(1-s)\,|\Phi^{1-s}(z, x) - \Phi^{1-s}(z, y)|\,ds\right) dz =$$

$$= -\int_0^1 \hat{f}'(1-s)\,H_{\Phi^{1-s}}(t)\,ds\,, \quad \text{q.e.d.}$$

3.8. Combining ii) with iii) we obtain $H_{\Phi^s/s^n}(t) \leqq H_{\hat{A}}(t)$ for some $s \leqq 1$. But it is obvious that $H_\Phi(t) \leqq H_{\Phi^s/s^n}$. Then $H_\Phi(t) \leqq H_{\hat{A}}(t)$ holds for all $t \geqq 0$. Using Lemma 3.4, we obtain

(13) $$H_\Phi(t) \leqq \int_0^1 H_A\left(\frac{t}{s}\right) ds\,.$$

An application of Theorem 2.9 yields

3.9. Corollary. *Let $\mathscr{A}$ be a partition of unity, let $\mathscr{A} \prec \mathscr{U}$. Then*

$$H_\Phi(t) \leqq \int_0^1 H_{\mathscr{A}}\left(\frac{t}{s}\right) ds$$

holds for all $t \geqq 0$.

In order to emphasize the dimension of $E = E_n$, write

$$H_{\Phi_n}(t) \leqq \int_0^1 H_{\mathscr{A}_n}\left(\frac{t}{s}\right) ds . \tag{14}$$

3.10. Lemma. $\lim_{n\to\infty} H_{\Phi_n}(t) = 2$ for all $t > 0$.

Proof. Denote

$$K_0^n = \{x \in E_n, \ \|x\| \leqq 1\} ,$$
$$K_t^n = \{x \in E_n, \ \|x - (t, 0, \ldots, 0)\| \leqq 1\} ,$$
$$K_t^{n'} = \{x \in E_n, \ \|x - (\tfrac{1}{2}t, 0, \ldots, 0)\| \leqq \sqrt{(1 - \tfrac{1}{4}t^2)}\} .$$

It is easy to check $K_0^n \cap K_t^n \subset K_t^{n'}$. We obtain

$$H_{\Phi_n}(t) = \mu(K_0^n \Delta K_t^n) = 2 - \mu(K_0^n \cap K_t^n) > 2 - \mu(K_t^{n'}) = \tag{15}$$
$$= 2 - (1 - \tfrac{1}{4}t^2)^{n/2} , \quad \text{q.e.d.}$$

4.

In this section we prove 1.4 using the following theorem of Dvoretzky:

Let $(X, \| \ \|)$ be an infinite dimensional normed space, let n be an integer, let $\varepsilon > 0$. There exists an n-dimensional subspace X_n of X and a Euclidean norm $\| \ \|^E$ on X_n such that

$$\| \ \|^E \leqq \| \ \| \leqq (1 + \varepsilon) \| \ \|^E \quad \text{holds on} \quad X_n \tag{16}$$

(see [5], p. 123).

Now we can complete the proof of 1.4. Let $\mathscr{A}$ be a partition of unity on X, let $\mathscr{A} \prec \mathscr{U}$. Denote by $\mathscr{A}_n$ and $\mathscr{U}_n$ respectively the restriction of $\mathscr{A}$ and $\mathscr{U}$ to X_n. Obviously $\mathscr{A}_n \prec \mathscr{U}_n$. Each element of $\mathscr{U}_n$ has the $\| \ \|$ diameter less or equal to 2. Using (16), we obtain that $\mathscr{U}_n$ refines the covering of all $\frac{1}{2}\| \ \|^E$ open balls with radius 1. We use (14) and (15) for X_n provided with $\frac{1}{2}\| \ \|^E$ and obtain

$$2 - \left(1 - \frac{t^2}{4}\right)^{n/2} < \int_0^1 H^E_{\mathscr{A}_n}\left(\frac{t}{s}\right) ds ,$$

where H^E denotes the modulus of continuity in $(X_n, \frac{1}{2}\| \ \|^E)$. (16) implies $H^E(t) \leqq \leqq H(2(1 + \varepsilon) t)$. Thus

$$2 - \left(1 - \frac{t^2}{4}\right)^{n/2} < \int_0^1 H_{\mathscr{A}_n}\left(\frac{2(1 + \varepsilon) t}{s}\right) ds .$$

In virtue of (2),

$$2 - \left(1 - \frac{t^2}{4}\right)^{n/2} < \int_0^1 H_{\mathscr{A}}\left(\frac{2(1 + \varepsilon)\, t}{s}\right) \mathrm{d}s$$

for all n.

This implies $\int_0^1 H_{\mathscr{A}}([2(1 + \varepsilon)\, t]/s)\, \mathrm{d}s = 2$, then $H_{\mathscr{A}}(t) = 2$ for all $t > 0$, q.e.d.

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