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ABSOLUTE POINTS IN $\beta N \setminus N$

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The aim of this paper is the study of the space $N^* = \beta N \setminus N$ in the situation when Continuum Hypothesis (CH) not necessarily holds and Martin’s Axiom (MA) is assumed. Now some distinctions of $P$-points are possible. We introduce a notion of absolute points announced as $P(c)$ points by Booth [2] (by CH absolute points coincide with $P$-points). We prove that there exist $2^c$ absolute $P$-points which are minimal in the Rudin-Keisler ordering. Although this result can be obtained in a way analogous to that of Blass [1] (the existence of $2^c$ minimal $P$-points), we get the mentioned result from some theorems of the Baire Category type (Lemmas 2 and 3). These theorems allow to obtain further results concerning the structure of $N^*$. Namely, we prove that each cover of $N^*$ by means of nowhere dense subsets is of the cardinality greater than $c$. In other words, the Novák number (introduced in § 3) of $N^*$ is greater than $c$. It is known to the authors from Professor Novák’s oral communication that, without any extra set-theoretical assumptions, the cardinality of any cover of $N^*$ by disjoint nowhere dense closed subsets is greater than $\aleph_1$.

1. Basic Lemmas. A family $T = \{T_x : x < \beta\}$ of closed-open subsets of $N^*$, where $x$ and $\beta$ are ordinals, is a $\beta$-tower (Hechler [4]) if for all ordinals $x < \gamma < \beta$ we have $T_\gamma \notin T_x$. A tower $T$ is said to be maximal if it is maximal with respect to the length of $T$, i.e., if $\bigcap T$ is a nowhere dense set (Hechler calls such a tower complete). Hechler [4] proved that if MA holds, then each maximal tower has the cardinality $2^{\aleph_0}$.

It is natural to ask whether there exist $P$-points which are maximal towers, i.e., $P$-points with linearly ordered (with respect to the inclusion) base in $N^*$. It is obvious that if CH holds, then each $P$-point in $N^*$ is a tower.

In the sequel, we use the usual convention that a cardinal is an initial ordinal, $c$ is the cardinal of the continuum and free ultrafilters on $N$ are regarded as points of $N^*$.

Lemma 1 (Martin, Solovay). If MA holds and $B$ is a base for a free filter on $N$ such that $\text{card } B < c$, then there exists an infinite subset $T$ of $N$ such that $T \setminus Y$ is finite, for each $Y \in B$. 

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Note. This Lemma follows from the $S_k$ hypothesis which is implied by MA (see Martin, Solovay [5]). Because the Lemma is crucial when applying MA, we give here a direct proof (cf. Both [2]).

Proof. Let $P = \{(F, Y) : F$ is finite and $Y \subseteq B\}$ and for $(F, Y), (F', Y') \in P$ put $(F, Y) \leq (F', Y')$ iff $F \subseteq F' \subseteq F \cup Y$ and $Y' \subseteq Y$. It is obvious that $\leq$ establishes a partial ordering on $P$. Note that, if $(F, Y)$ and $(F', Y')$ are in $P$ and have the same first element and $Y_0 \in B$ is such that $Y_0 \subseteq Y \cap Y'$, then $(F, Y_0) \in P$ and $(F, Y_0)$ is an upper bound for them both. Therefore, if $L \subseteq P$ is an antichain, then elements of $L$ have different first members, hence $L$ is countable. It is to verify that the sets $D_n = \{(F, Y) : (F, Y) \in P; \text{there exists } m \in F \text{ with } m > n\}$ and $D_A = \{(F, Y) : (F, Y) \in P : Y \subseteq A\}$ are dense subsets of $P$ for all $n \in N$ and $A \in B$. Put $A = \{D_n : n \in N\} \cup \{D_A : A \in B\}$. Thus $A$ is a family of dense subsets of $P$ and $\text{card } A = \text{card } B < c$. Let $T = \bigcup\{F : (F, Y) \in G\}$, where $G$ is a generic set for $A$.

$T$ is an infinite subset of $N$, because for each $n$ there exist $(F, Y) \in G \cap D_n$ and $m \in F \subseteq T$ such that $m > n$. So $T$ is an unbounded subset of $N$.

$T \setminus A$ is finite for each $A \in B$. In fact, let $(F, Y) \in G \cap D_A$, let $(F', Y')$ be an arbitrary element from $G$ and let $(F'', Y'') \in G$ be greater or equal to $(F, Y)$ and $(F', Y')$. Since $F' \subseteq F'' \subseteq F \cup Y$ and $(F', Y')$ is an arbitrary element from $G$ hence $T \subseteq F \cup Y$. This implies that $T \setminus A \subseteq (F \cup Y) \setminus A \subseteq F$, because $Y \subseteq A$. This completes the proof.

**Corollary 1.** Suppose MA holds and $R$ is an infinite family of open subsets of $N^*$ such that $\bigcap R \neq \emptyset$. If $\text{card } R < c$, then $\text{Int } \bigcap R \neq \emptyset$.

**Corollary 2 (Hechler [4]).** If MA holds, then each maximal tower in $N^*$ has the cardinality $c$.

**Lemma 2.** Suppose MA holds and $\mathcal{A}$ is a family of nowhere dense subsets of $N^*$. If $\text{card } \mathcal{A} < c$, then $\bigcup \mathcal{A}$ is a nowhere dense subset of $N^*$.

Proof. Let $\mathcal{A} = \{A_x : x < \gamma\}$, where $\gamma < c$, be a well ordering of $\mathcal{A}$ and suppose $\text{Int } \text{cl } \bigcup \mathcal{A} \neq \emptyset$. Let $V$ be a non-empty closed-open subset of $N^*$ contained in $\text{cl } \bigcup \mathcal{A}$. We define, by transfinite induction, a family $\{V_x : x < \gamma\}$ of non-empty closed-open subsets of $N^*$ such that

(i) $V_\beta \subseteq V_x \subseteq V$, for $x < \beta < \gamma$,

(ii) $V_x \cap A_x = 0$, for $x < \gamma$.

Let $V_i$ be an arbitrary, non-empty and closed-open subset contained in $V$ such that $V_i \cap A_i = 0$. Assume that we have defined $V_x$, for $x < \beta$, which fulfil (i) and (ii). In virtue of compactness of $N^*$ and Corollary 1, $\text{Int } \bigcap \{V_x : x < \beta\} \neq \emptyset$. Let $V_\beta$ be a non-empty, closed-open subset contained in $\text{Int } \bigcap \{V_x : x < \beta\}$ such that $V_\beta \cap A_\beta = 0$. In virtue of Corollary 1 again, we infer $G = \text{Int } \bigcap \{V_\beta : \beta < \gamma\} \neq \emptyset$. This contradicts our assumption, $G$ being a non-empty open set contained in $V$ and disjoint with $\bigcup \mathcal{A}$.

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Lemma 3. Suppose MA holds and $\mathcal{A}$ is a family of nowhere dense subsets of $\mathbb{N}^*$. If $\text{card } \mathcal{A} = c$, then $\mathbb{N}^* \setminus \bigcup \mathcal{A}$ is a dense subset of $\mathbb{N}^*$ of cardinality $2^c$.

Proof. Let $\mathcal{A} = \{A_x : x < c\}$ be a well ordering of $\mathcal{A}$. For each ordinal $x < c$ we define, by transfinite induction, a family $R_x$ of disjoint closed-open subsets of $\mathbb{N}^*$ which fulfil the following conditions:

(i) $\bigcup R_x \cap A_x = \emptyset$.
(ii) the family $R_\beta$ refines the family $R_x$ for $x < \beta < c$, i.e., for every $V \in R_\beta$ there exists $U \in R_x$ such that $V \subseteq U$.
(iii) if $x < \beta < c$ and $U \in R_x$, then $\text{card } \{V \in R_\beta : V \subseteq U\} \geq 2$.
(iv) if $\gamma < c$ is a limit ordinal and $L$ is a $\gamma$-tower consisting of elements of all families $R_x$ for $x < \gamma$, then $\bigcap L$ contains at least two elements of the family $R_\gamma$.

Let $R_0$ be a family consisting of two disjoint, non-empty, closed-open sets which are also disjoint with $A_0$.

Assume that we have defined the families $R_x$ for $x < \beta$.

If $\beta = x + 1$, then for every $U \in R_x$ take two disjoint, non-empty closed-open sets contained in $U$ and disjoint with $A_{x+1}$. Let $R_{x+1}$ be the family of all these sets, for each $U \in R_x$.

If $\beta$ is a limit ordinal and $L$ is a $\beta$-tower consisting of elements of all families $R_x$ for $x < \beta$, then, by Corollary 1, $\text{Int } \bigcap L \neq \emptyset$. Take for every such $\beta$-tower two disjoint non-empty closed-open sets contained in $\text{Int } \bigcap L$ and disjoint with $A_\beta$. Let $R_\beta$ be the family of all these sets.

Conditions (i)−(iv) are in both cases obviously fulfilled.

Now, conditions (ii), (iii) and (iv) imply that the cardinality of the family of all $c$-towers of elements of all families $R_x$ for $x < c$, is $2^c$. Moreover, if we take two such different $c$-towers, then their intersections are non-empty and disjoint. Condition (i) implies that the intersection of such $c$-tower is disjoint with $\bigcup \mathcal{A}$. The elements of $R_0$ can be chosen as subsets of an arbitrary open set which implies the density of $\mathbb{N}^* \setminus \bigcup \mathcal{A}$.

Remark. A more detailed version (although without further applications in this paper) of Lemma 3 can be stated: $\mathbb{N}^* \setminus \bigcup \mathcal{A}$ contains the space $2^c$ with the box-topology as a dense subspace.

2. Minimal and absolute points in $\mathbb{N}^*$. Let $\aleph$ be a cardinal and let $X$ be a space. A point $p \in X$ is said to be an $\aleph$-point if $\aleph$ is the supremum of all cardinals such that the intersection of each family of the cardinality less than $\aleph$ of neighbourhoods of $p$ is a neighborhood of $p$.

Let $X$ be a space and let $w(X, x)$ denote the weight of $X$ at the point $x$. A point $x$ which is $w(X, x)$-point is called an absolute point of $X$.

In the sequel, $F_\aleph$ denotes the set of all $\aleph$-points of $\mathbb{N}^*$ and $F$ denotes the set of all absolute points of $\mathbb{N}^*$, i.e., if MA is assumed, the set of all $c$-points.
It is obvious that absolute points of $N^*$ can be characterized in terms of towers as follows:

a point $p \in N^*$ is an absolute point iff there exists a tower $T$ such that $\{p\} = \bigcap T$.

Note that the set of all non-$P$-points of $N^*$ coincides with $F_{\aleph_0}$.

**Theorem 1.** Suppose $MA$ holds. If $\aleph_\kappa < \kappa$, then the set $F_\kappa$ can be covered by $\kappa$ closed and nowhere dense subsets of $N^*$.

**Proof.** Let $B$ be a base in $N^*$ consisting of closed-open sets and card $B = \kappa$. If $p$ is an $\aleph_\kappa$-point, then there exists a family $R$ of neighborhoods of $p$ with card $R = \aleph_\kappa$ and $p \in \bigcap R \setminus \text{Int} \bigcap R$. We can assume that $R \subset B$.

For each family $R \subset B$ with card $R = \aleph_\kappa$, let $A_R = \bigcap R \setminus \text{Int} \bigcap R$. The cardinality of the set of all subfamilies of the cardinality $\aleph_\kappa$ from $B$ is equal to $2^{\aleph_\kappa \cdot \aleph_\kappa} = 2^{\aleph_\kappa}$. In virtue of $MA$, we have $2^{\aleph_\kappa} = 2^{\aleph_\kappa}$ (see Martin, Solovay [5]). Thus the family of all such sets $A_R$ gives the required cover of $F_\kappa$.

Recall that an ultrafilter $p \in N^*$ is a $P$-point iff each map $f : N \to N$ is either constant or finite-to-one on an element of $p$. An ultrafilter $p \in N^*$ is minimal (with respect to the Rudin-Keisler ordering) iff each map $f : N \to N$ is either constant or one-to-one on an element of $p$. It is obvious that the minimal points of $N^*$ are $P$-points. The definition implies the following characterization of minimal $P$-points:

**Lemma 4.** A $P$-point $p \in N^*$ is minimal iff for each finite-to-one map $f : N \to N$ there exists a neighborhood $U$ of $p$ in $\beta N$ such that $\beta f \upharpoonright U$ is a homeomorphism onto $(\beta f)(U)$, where $\beta f$ is the extension of $f$ onto $\beta N$.

Let $f : N \to N$ be a finite-to-one map. Denote by $O_f$ the family $\{c|_{\beta \chi} M \setminus N : M \subset N$ and $f \upharpoonright M$ is one-to-one}. It is easy to prove the following

**Lemma 5.** If $f : N \to N$ is finite-to-one, then $\bigcup O_f$ is dense and open in $N^*$.

**Theorem 2.** Suppose $MA$ holds. The set of all non-minimal points of $N^*$ can be covered by $\kappa$ closed and nowhere dense subsets of $N^*$.

**Proof.** Let $F$ be the set of all finite-to-one maps from $N$ into $N$. The family $\{N^* \setminus \bigcup O_f : f \in F\}$ is a family of closed and nowhere dense subsets of $N^*$ which covers the set of all non-minimal $P$-points. The cardinality of this family is $\kappa$. In virtue of Theorem 1, the set of all non-$P$-points is covered by $\kappa$ closed and nowhere dense subsets of $N^*$. Both these families give the required cover of all non-minimal points of $N^*$.

**Theorem 3.** There exists a dense subset $D$ of $N^*$ of cardinality $2^\kappa$ consisting of points which are both absolute and minimal, and such that $N^* \setminus D$ can be covered by $\kappa$ closed and nowhere dense subsets of $N^*$.

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Proof. In virtue of Theorem 1, the set $\bigcup \{F_n : \aleph < c\}$ of all non-absolute points of $N^*$ can be covered by a family $R_1$ of closed and nowhere dense subsets of $N^*$ with $\text{card} \; R_1 = c$. From Theorem 2 it follows that there exists a family $R_2$ of closed and nowhere dense subsets of $N$ of cardinality $c$ which covers the set of all non-minimal points of $N$. Thus the family $R = R_1 \cup R_2$ is a family of closed and nowhere dense subsets of $N^*$ of cardinality $c$ and $D = N^* \setminus \bigcup R$ is contained in the set consisting of points which are both absolute and minimal. In virtue of Lemma 3, the set $D$ is a dense subset of $N^*$ of cardinality $2^c$.

Remark. It can be proved, using Remark to Lemma 3, that the space $2^c$ with the box-topology can be embedded as a dense set in the set of points which are both absolute and minimal.

Question 1. Assume MA. Do there exist $\aleph$-points in $N^*$ for $\aleph_0 < \aleph < c$?

Question 2. Are all absolute points of the same type in $N^*$, i.e., does there exist for any absolute points $p$ and $q$ in $N^*$ a homeomorphism $f$ of $N^*$ onto itself such that $f(p) = q$?

3. Novák Number of subspaces of $N^*$. The Novák number $nX$ of a dense in itself space $X$ is the least infinite cardinal being the cardinal of a covering of $X$ by nowhere dense sets.

In this section we establish the Novák number of some subspaces of $N^*$ and we discuss the cardinality of some special families of nowhere dense subsets of $N^*$. All theorems in this section depend on Martin’s Axiom.

First we state some consequences of previous theorems.

Theorem 4. $nN^* > c$. If $D$ is a dense subset of $N^*$, then $nD \geq c$.

Proof. The former inequality follows from Lemmas 2 and 3, the latter from Lemma 2.

Theorem 5. $n(F \cap M) > c$ and $n(N^* \setminus (F \cap M)) = c$, where $F$ denotes the set of all absolute points and $M$ the set of all minimal points of $N^*$.

Proof. Let $D \subset F \cap M$ be the same as in Theorem 3. If $n(F \cap M) \leq c$, then $N^* = (N^* \setminus D) \cup F \cap M$ can be covered by a family of closed and nowhere dense subsets of $N^*$ of cardinality $\leq c$ (nowhere dense subsets in a subspace are nowhere dense subsets in the whole space). This contradicts Lemma 3.

The set $N^* \setminus F \cap M$ is a dense subset of $N^*$, so from Theorem 4 it follows that $n(N^* \setminus F \cap M) \geq c$. In virtue of Theorem 3, $N^* \setminus F \cap M \subset N^* \setminus D$ can be covered by a family of closed and nowhere dense subsets of $N^* \setminus F \cap M$ of cardinality $c$ (nowhere dense subsets in the whole space are nowhere dense in a dense subspace).
A cover $\mathcal{A}$ of $X$ by closed and disjoint nowhere dense subsets of a space $X$ is called upper semicontinuous (lower semicontinuous) if for every open set $U \subset X$ the set $\bigcup \{A \in \mathcal{A} : A \subset U\}$ (the set $\bigcup \{A \in \mathcal{A} : A \cap U \neq \emptyset\}$) is open.

A cover $\mathcal{A}$ of a space $X$ is said to be regular if for each non-empty open set $G \subset X$ there exist disjoint and non-empty open sets $U$, $V$ contained in $G$ such that the sets $\{A \in \mathcal{A} : A \cap U \neq \emptyset\}$ and $\{A \in \mathcal{A} : A \cap V \neq \emptyset\}$ are disjoint.

**Theorem 6.** If $\mathcal{A}$ is an upper semicontinuous cover of a normal space $X$, then $\mathcal{A}$ is regular. Assume MA. If $\mathcal{A}$ is lower semicontinuous cover of $N^*$, then $\mathcal{A}$ is regular.

**Proof.** Let $\mathcal{A}$ be an upper semicontinuous cover of a normal space $X$, and let $U$ be a non-empty open subset of $X$. Since the elements of $\mathcal{A}$ are nowhere dense and $\mathcal{A}$ is a cover, hence there exist $A_1, A_2 \in \mathcal{A}$ such that $A_1 \cap U \neq \emptyset \neq A_2 \cap U$ and $A_1 \neq A_2$. Since $A_1$ and $A_2$ are disjoint and closed subsets of a normal space $X$, hence there exist disjoint open sets $V_1$ and $V_2$ such that $A_i \subset V_i$ for $i = 1, 2$. Since $\mathcal{A}$ is upper semicontinuous hence $B_i = \bigcup \{A \in \mathcal{A} : A \subset V_i\}$ for $i = 1, 2$ are non-empty and open. Moreover, since $A_i \subset B_i$ hence $B_i \cap U \neq \emptyset$ for $i = 1, 2$. It is obvious that then $U_i = B_i \cap U$, $i = 1, 2$, are the open subsets of $U$ desired for $\mathcal{A}$ to be regular.

Assume MA. Let $\mathcal{A}$ be a lower semicontinuous cover of $N^*$ and let $U$ be a non-empty open subset of $N^*$. Let us suppose, on the contrary, that for any open sets $V_1, V_2 \subset U$ there is

$$\left(\bigcup \{A \in \mathcal{A} : A \cap V_1 \neq \emptyset\}\right) \cap \left(\bigcup \{A \in \mathcal{A} : A \cap V_2 \neq \emptyset\}\right) = \emptyset.$$  

The last assumption implies that for each open set $V \subset U$ the set $D_V = \bigcup \{A \in \mathcal{A} : A \cap V \neq \emptyset\}$ is a dense and open subset of $U$. Hence for each open $V \subset U$ we have that $U \setminus D_V$ is a nowhere dense subset of $N^*$. Now, let $B$ be a base in $N^*$ consisting of closed-open sets and card $B = c$. The family $R = \{U \setminus D_W : W \in B, W \subset U\}$ is a family of nowhere dense subsets of $N^*$ of cardinality $c$. $R$ is a cover of $U$. To see this, let $A \in \mathcal{A}$ be such that $A \cap U \neq \emptyset$. Since $A$ is a nowhere dense subset of $N^*$ hence there exists $W \in B$ such that $W \subset U$ and $W \cap A = \emptyset$. This means $A \cap U \subset \subset U \setminus D_W$ and hence $U \subset \bigcup R$. The last inclusion contradicts Lemma 3.

**Theorem 7.** If $\mathcal{A}$ is a regular cover of $N^*$, then card $\mathcal{A} = 2^c$.

**Proof.** For each $\alpha < c$ we define a family $R_\alpha$ of closed-open subsets of $N^*$ which are disjoint and which fulfil the following conditions:

(i) if $U, V \in R_\alpha$ and $U \neq V$, then the sets $\{A \in \mathcal{A} : A \cap U \neq \emptyset\}$ and $\{A \in \mathcal{A} : A \cap V \neq \emptyset\}$ are disjoint,

(ii) if $\alpha < \beta < c$, then $R_\beta$ refines $R_\alpha$,

(iii) if $\alpha < c$ and $L = \{\gamma : \gamma < \alpha\}$ is an $\alpha$-tower consisting of elements of families $R_\gamma$ for $\gamma < \alpha$, then card $\{V \in R_\alpha : V \subset \bigcap L\} \geq 2$.

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Let $R_0$ consist of two arbitrary disjoint, closed-open sets $U$, $V$ which fulfil (i) (the existence is implied by regularity of $\mathcal{A}$). Assume that we have defined the families $R_\gamma$ for each $\gamma < \alpha$ which fulfil conditions (i)–(iii). Take an arbitrary $\alpha$-tower $L$ consisting of elements of families $R_\gamma$ for $\gamma < \alpha$. In virtue of MA we have $\text{Int} \cap L = \emptyset$. Since $\mathcal{A}$ is regular hence there exist closed-open and disjoint sets $U$, $V$ contained in $\text{Int} \cap L$ such that the sets $\{ A \in \mathcal{A} : A \cap U \neq \emptyset \}$ and $\{ A \in \mathcal{A} : A \cap V \neq \emptyset \}$ are disjoint. Put $R_\alpha$ to be the family of all such $U$, $V$ for all $\alpha$-towers consisting of elements of all families $R_\gamma$ for $\gamma < \alpha$. It is obvious that $\{ R_\gamma : \gamma \leq \alpha \}$ fulfills conditions (i)–(iii).

Now, conditions (ii) and (iii) imply that the set of all $c$-towers consisting of elements of all families $R_\alpha$ for $\alpha < c$ has cardinality $2^c$. For such a $c$-tower $L$, denote by $A_L$ the element of $\mathcal{A}$ such that $A_L \cap \cap L \neq \emptyset$ (such an element $A_L$ exists because $\cap L \neq \emptyset$ and $\mathcal{A}$ is a cover of $N^\ast$). Moreover, if $L$ and $L'$ are distinct such $c$-towers, then there exists an ordinal $\beta < c$ and sets $U_\beta \in L \cap R_\beta$, $V_\beta \in L' \cap R_\beta$ such that $U_\beta \cap V_\beta = \emptyset$. In virtue of condition (i), we have $A_L = A_{L'}$. Hence card $\mathcal{A} = 2^c$.

Added in proof. Question 1 was answered positively by the second author, On the existence of $P(N)$-points for $N_0 < N < c$, Colloquium Mathematicum (in print).

References


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