Czechoslovak Mathematical Journal

Paul D. Humke
Cluster sets of arbitrary functions defined on plane sets

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 3, 448–457

Persistent URL: http://dml.cz/dmlcz/101418

Terms of use:

© Institute of Mathematics AS CR, 1976

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

CLUSTER SETS OF ARBITRARY FUNCTIONS DEFINED ON PLANE SETS

PAUL D. HUMKE, Macomb

(Received November 28, 1974)

In this paper we investigate the question of "how many" different ambiguities of a function from the Euclidean Plane, P, into the Riemann sphere, Ω, can occur at a point. Here, and throughout the paper unless specified as otherwise, the functions we speak of will be single valued only with no other conditions such as continuity or measurability presumed. Some initial investigations of this question were conducted by McMILLAN [5] and BAGEMIHL and HUMKE [3].

Suppose z is an ambiguous point of a function f with arcs of ambiguity α_1 and α_2. If γ is a third arc at z then either:

1. $C_\gamma(f, z) \cap C_{\alpha_1}(f, z) \neq \phi$ and $C_\gamma(f, z) \cap C_{\alpha_2}(f, z) \neq \phi$

or

2. $C_\gamma(f, z) \cap C_{\alpha_i}(f, z) = \phi$ for at least one of $i = 1$ or $i = 2$.

Here $C_\beta(f, z)$ is the set of limit points of f at z along β. If α_1 and α_2 are specified, then it is easily verified that a third arc γ can be defined such that situation 1 occurs. As such we investigate those points at which there is an arc γ where situation 2 is true. As a special case of situation 2 McMillan has proved [4, Corollary 2, page 447] the following result. (We have couched McMillan's result in our notation.)

Theorem M. *If f is an arbitrary function then the set of ambiguous points z such that both*

$$C_{\alpha_1}(f, z) \cap C_\gamma(f, z) = \phi \quad \textit{and} \quad C_{\alpha_2}(f, z) \cap C_\gamma(f, z) = \phi$$

is of the first Baire Category.

We prove the following theorem under the assumption that α_1, α_2, and γ are as in situation 2 above.

Theorem H. *If f is an arbitrary function then except for a set of the first Baire Category either*:

1. $C_{\alpha_1}(f, z) = C_\gamma(f, z)$

or

2. $C_{\alpha_2}(f, z) = C_\gamma(f, z)$.

McMillans result, then, is a special case of Theorem H. In a restrictive sense Theorem H says that except for a first category set if a point z is an ambiguous point relative to a function f then that ambiguity is the only ambiguity at z. The restriction is of course that in a sense, one arc of ambiguity at z is considered fixed and one searches for new ambiguities relative to the cluster set along that fixed arc at z.

If this restriction of Theorem H is relaxed one obtains examples of functions with many ambiguities at every point. Such examples will be published elsewhere.

The paper is divided into two parts; in the first part we prove a set theoretic theorem concerning arc accessibility to subsets of P and in the second part we utilize this theorem to prove the main result, and then prove a series of corollaries.

1. Denote by P the Euclidean plane with a rectangular Cartesian coordinate system where the x-axis is horizontal and the y-axis is vertical.

The distance between the points z_1 and z_2 in P is denoted $|z_1, z_2|$, and the closed line segment joining z_1 and z_2 is denoted $[z_1, z_2]$. If J is a Jordan arc in P having one endpoint z then $J - \{z\}$ is an *arc at* z or if the terminal point z is not specified $J - \{z\}$ is simply an *arc**. Let $\mathscr{A}$ be a collection of arcs*. A point z is termed *accessible via* $\mathscr{A}$ provided there is an arc at z which is an element of $\mathscr{A}$. Two sets of arcs* $\mathscr{A}$ and $\mathscr{B}$ are *pointwise disjoint* if whenever $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$, then $\alpha \cap \beta = \phi$. Two sets of arcs* $\mathscr{A}$ and $\mathscr{B}$ are *arcwise disjoint* if whenever $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$ and both α and β are arcs at a point z, then $\alpha \cap \beta$ contains no arc at z. Let $\mathscr{A}$, $\mathscr{B}_1$, and $\mathscr{B}_2$ be sets of arcs*, and set $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$. Suppose

1. $\mathscr{A}$ and $\mathscr{B}$ are pointwise disjoint, and
2. $\mathscr{B}_1$ and $\mathscr{B}_2$ are arcwise disjoint.

Then the triple $(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2)$ is called an *accessibility triple* of sets of arcs*. If D is an open disc and $\alpha(z)$ is an arc at a point $z \in D$, then α in D is the arc $\alpha_1(z)$ where

$$\alpha_1(z; t) = \alpha(z; (1 - t^*)\, t + t^*)$$

and

$$t^* = \sup \{t : 0 \leqq t < 1 \text{ and } \alpha(z; t) \notin D\}\,, \quad \text{if} \quad \alpha \nsubseteq D \quad \text{and} \quad t^* = 0 \quad \text{if} \quad \alpha \subseteq D\,.$$

Theorem 1. *If $(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2)$ is an accessibility triple, then the set of points which are simultaneously accessible via each of $\mathscr{A}$, $\mathscr{B}_1$, and $\mathscr{B}_2$ is a set of the first Baire category.*

Proof. Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$. Let S denote the points which are simultaneously accessible via $\mathscr{A}$, $\mathscr{B}_1$, and $\mathscr{B}_2$ and suppose that S is of the second Baire category. For $z \in S$ there are arcs $\alpha(z) \in \mathscr{A}$, $\beta_1(z) \in \mathscr{B}_1$, and $\beta_2(z) \in \mathscr{B}_2$ at z such that

1. $\alpha(z) \cap [\beta_1(z) \cup \beta_2(z)] = \phi$,

and

2. $\beta_1(z) \cap \beta_2(z)$ contains no arc at z.

Let $D(z)$ be a rational disc (i.e. a disc of rational radius having rational coordinates for its center) which contains z and whose boundary meets each of the three arcs of accessibility, $\alpha(z)$, $\beta_1(z)$, and $\beta_2(z)$. If D is a rational disc we define

$$S(D) = \{z \in S : D(z) = D\} .$$

Then $S = \bigcup S(D)$ where the union is taken over the enumerable set of rational discs D. It follows that there is an index D_0 and a disc D_1 such that $S(D_0$ is everywhere of second category in D_1. Let $S_0 = S(D_0) \cap D_1$, $z \in S_0$, and denote respectively by $\alpha^1(z)$, $\beta_1^1(z)$, and $\beta_2^1(z)$ the arcs $\alpha(z)$, $\beta_1(z)$, and $\beta_2(z)$ in D_1. Also, let

$$F = \{t : 0 \leqq t < 1 \text{ and } \beta_1(z; t) \in \beta_2(z)\} .$$

Claim 1. Sup $F \neq 1$.

If sup $F = 1$ then as F is a closed subset of $[0, 1)$ and F contains no interval having 1 as its right end point it follows that there are numbers t_1 and t_2 with

1. $0 \leqq t_1 < t_2 < 1$,
2. $\beta_1^1(z; t_i) \in \beta_2^1(z)$ $(i = 1, 2)$,
3. $\beta_1^1(z; t) \notin \beta_2^1(z)$ for $t_1 < t < t_2$,

There are numbers t_1' and t_2' such that $\beta_1^1(z; t_1) = \beta_2^1(z; t_1')$ and $\beta_1^1(z; t_2) = \beta_2^1(z; t_2')$. Let R_1 denote the region bounded by

$$\beta_1^1(z)/[t_1, t_2] \cup \beta_2^1(z)/[t_1', t_2'] .$$

Then $R_1 \subset D_1$ and as S_0 is dense in D_1 there is a $z_0 \in R_1 \cap S_0$. But $\alpha^1(z_0; 0) \in$ $\in \text{Bd}(D_1)$ $[Bd \equiv \text{boundary}]$ and $\alpha^1(z_0)$ is an arc at z_0. Consequently, $\alpha^1(z_0)$ intersects the boundary of R_1 which is impossible. Claim 1 is verified.

If $z \in S_0$ let $D_2(z)$ be a rational subdisc of D_1 such that if $\beta_i^2(z)$ is $\beta_i(z)$ in $D_2(z)$ $(i = 1, 2)$, then $\beta_1^2(z) \cap \beta_2^2(z) = \phi$. Define $\alpha^2(z)$ to be $\alpha(z)$ in $D_2(z)$. Let $\theta(z)$, $\gamma_1(z)$, and $\gamma_2(z)$ be mutually exclusive arcs on the boundary of D_2 such that

1. $\alpha^2(z; 0) \in \theta(z)$,
2. $\beta_i^2(z, 0) \in \gamma_i(z)$ $(i = 1, 2)$, and
3. the radii of D_2 terminating in the endpoints of $\theta(z)$, $\gamma_1(z)$, and $\gamma_2(z)$ all have rational slopes.

Define

$$S_0(D, \theta, \gamma_1, \gamma_2) = \{z \in S_0 : D_2(z) = D, \ \theta(z) = \theta, \ \gamma_1(z) = \gamma_1 \text{ and } \gamma_2(z) = \gamma_2\}$$

Then

$$S_0 = \bigcup S_0(D, \theta, \gamma_1, \gamma_2)$$

where the union is taken over all admissible ordered quadruples. As S_0 is of second Baire category there is an index $(D_2, \theta, \gamma_1, \gamma_2)$ and a disc D_3 such that $D_3 \subset D_2$ and $S_0(D_2, \theta, \gamma_1, \gamma_2)$ is everywhere of second category in D_3. Let $D_3 \cap S_0(D_2, \theta, \gamma_1, \gamma_2) = S_0^*$ and for $z \in S_0^*$ define

1. $\alpha^3(z)$ to be $\alpha(z)$ in D_3, and
2. $\beta_i^3(z)$ to be $\beta_i(z)$ in D_3 $(i = 1, 2)$.

Then define

$$\mathscr{A}^* = \{\alpha^3(z) : z \in S_0^*\}, \quad \mathscr{B}_1^* = \{\beta_1^3(z) : z \in S_0^*\}, \quad \mathscr{B}_2^* = \{\beta_2^3(z) : z \in S_0^*\}.$$

Claim 2. *If $z_0 \in S_0^*$, then, $z_0 \in \alpha^2(z)$ for some $z \in S_0^*$.*

Let $z_0 \in S_0^*$. There is an arc Γ on $\text{Bd}(D_2)$ such that

1. $\Gamma(0) = \beta_1^2(z_0; 0)$,
2. $\Gamma(1) = \beta_2^2(z_0; 0)$;
3. $\Gamma \cap \alpha^2(z_0) = \phi$ (i.e. $\alpha^2(z_0, 0) \notin \Gamma$).

If R is the region bounded by the arcs Γ, $\beta_1^2(z_0)$, and $\beta_2^2(z_0)$, and the point z_0 then $R \cap D_3 \neq \phi$. Let $z \in R \cap D_3 \cap S_0^*$. Then $\alpha^2(z; 0)$ is exterior to R and as $z \in R$ it follows that $\alpha^2(z)$ intersects the boundary of R. But $\mathscr{A}$ and $\mathscr{B}$ are pointwise disjoint and $\alpha^2(z) \cap \Gamma = \phi$; consequently $z_0 \in \alpha^2(z)$ and the claim is verified.

If λ_1 and λ_2 are arcs* and D is a disc, we say that λ_1 *crosses* λ_2 in D if there are numbers t_1, t_2, and t_3 $(0 \leqq t_1 < t_2 < t_3 < 1)$ such that

1. $\lambda_1/[t_1, t_3] \subset D$,
2. $\lambda_1(t_i) \notin \lambda_2$ $(i = 1, 3)$,
3. $\lambda_1(t_2) \in \lambda_2$.

Claim 3. *If z_1 and z_2 are in S_0^*, then $\beta_1^2(z_1)$ does not cross either $\beta_1^2(z_2)$ or $\beta_2^2(z_2)$ in D_3.*

Suppose to the contrary that there are points z_1 and z_2 in S_0^* such that $\beta_1^2(z_1)$ crosses either $\beta_1^2(z_2)$ or $\beta_2^2(z_2)$ in D_3. For definiteness we suppose $\beta_1^2(z_1)$ crosses $\beta_2^2(z_2)$ in D_3. That is, there are numbers t_1, t_2. and t_3 $(0 \leqq t_1 < t_2 < t_3 < 1)$ such that

1. $\beta_2^2(z_1)/[t_1, t_3] \subset D_3$,
2. $\beta_1^2(z_1; t_i) \notin \beta_2^2(z_2)$ $(i = 1, 3)$, and
3. $\beta_1^2(z_1; t_2) \in \beta_2^2(z_2)$.

Let

$$t_2' = \inf\{t : t_1 < t < t_2 \text{ and } \beta_1^2(z_1; t) \in \beta_2^2(z_2)\}, \text{ and}$$
$$t_1' = \inf\{t < t_1 : \beta_1^2(z_1)/[t, t_1] \cap \beta_2^2(z_2) = \phi\}.$$

It follows that $\beta_1^2(z_1)/(t_1', t_2') \cap \beta_2^2(z_2) = \phi$. Let s be such that $\beta_2^2(z_2; s) = \beta_1^2(z_2; t_2')$. We consider two cases depending on whether $\beta_1^2(z_1; t_1') \in \beta_2^2(z_2)$ or not.

Case 1. $\beta_1^2(z_1; t_1') \in \beta_2^2(z_2)$.

As $\beta_1^2(z_1; t_1') \notin \beta_2^2(z_2)$ it must be that $t_1' = 0$, and there is an arc Γ on $\mathrm{Bd}(D_2)$ such that

1. $\Gamma(0) = \beta_1^2(z_1; 0)$,
2. $\Gamma(1) = \beta_2^2(z_2; 0)$,
3. Γ does not contain $\alpha^2(z; 0)$ for $z \in S_0^*$.

Let R denote the region bounded by Γ, $\beta_1^2(z_1)/[0, t_2']$, and $\beta_2^2(z_2)/[0, s]$. As $\beta_1^2(z_1, t_2') \in$ $\in D_3$ it follows that $R \cap D_3 \neq \phi$. But if $z_0 \in R \cap D_3$ then $\alpha^2(z_0)$ must meet the boundary of R and this entails a contradiction.

Case 2. $\beta_1^2(z; t_1') \in \beta_2^2(z_2)$.

A contradiction similar to that in Case 1 is reached in Case 2, and the proof of Claim 3 is complete.

Claim 4. *If z_1 and z_2 are in S_0^* then $\beta_1^3(z_1) \cap \beta_1^3(z_2) = \phi$ and $\beta_1^3(z_1) \cap \beta_2^3(z_2) = \phi$.*

Suppose Claim 4 is false and there exist z_1 and z_2 in S_0^* such that either $\beta_1^3(z_1) \cap$ $\cap \beta_1^3(z_2) \neq \phi$ or $\beta_1^3(z_1) \cap \beta_2^3(z_2) \neq \phi$. For notational simplicity we assume the following

1. z_1 is the origin,
2. D_2 is the disc of radius 1 centered at the origin,
3. D_3 is the disc of radius $\frac{1}{2}$ centered at the origin,
4. $\beta_1^2(z_1) = [(0, 0), (1, 0)]$,
5. $\beta_2^2(z_1) = [(-1, 0), (0, 0)]$,
6. $\alpha^2(z_1) = [(0, 1), (0, 0)]$.

The arcs* $\beta_1^2(z_1)$, $\beta_2^2(z_1)$, and $\alpha^2(z_1)$ divide D_3 into three subregions:

1. R_1 is the lower half plane intersected with D_3.
2. R_2 is the upper right quarter plane intersected with D_3.
3. R_3 is the upper left quarter plane intersected with D_3.

We consider three subcases depending on the subregion within which z_2 resides.

Case 1. $z_2 \in R_1$.

Either $\beta_1^3(z_1)$ intersects $\beta_1^3(z_2)$ or $\beta_2^3(z_2)$; suppose the former and define

$$t^* = \sup \{t : \beta_1^3(z_1; t) \in \beta_1^3(z_2)\},$$

$$z^* = \beta_1^3(z_1; t^*), \quad \text{and } s^* \text{ such that } \quad \beta_1^3(z_2; s^*) = z^*.$$

There is an arc Γ on the boundary of D_2 such that Γ extends between $\beta_1^2(z_2; 0)$ and $\beta_2^2(z_2; 0)$, and Γ does not contain $\alpha^2(z_2; 0)$. Let R_4 be the region bounded by the arcs $\beta_1^2(z_2)$, $\beta_2^2(z_2)$, and Γ, and the point z_2. As $z_2 \in D_3$ it follows that $R_1 \cap R_4 \neq \phi$. Let $z_3 \in R_1 \cap R_4 \cap S_0^*$. As $\alpha^2(z_3)$ extends from the boundary of D_2 to R_4 it must be that $\alpha^2(z_3)$ intersects the boundary of R_4. As the arcs defining a portion of R_4 necessarily miss $\alpha^2(z_3)$ it follows that there is a t_2 such that $\alpha^2(z_3; t_2) = z_2$. In an analogous manner it is evident that $z_1 \in \alpha^2(z_3)$ and consequently there is a t_1 such that $\alpha^2(z_3; t_1) = z_1$. Let R_5 be the region bounded by the arcs $\alpha^2(z_3)/[t_2, t_1]$, $\beta_1^3(z_2)/[s^*, 1)$, and $\beta_1^3(z_1)/[t^*, 1)$. Again, as $z_2 \in D_3$ it follows that there is a $z_4 \in R_5 \cap S_0^*$. As $\beta_1^2(z_4)$ is on the boundary of D_2 and $z_4 \in R_5 \subset D_2$ it follows that $\beta_1^2(z_4) \cap \text{Bd}(R_5) \neq \phi$. But $\{z_1, z_2\} \cup \alpha^2(z_3) \subset \mathscr{A}$ and thus either

$$\beta_1^2(z_4) \cap \beta_1^3(z_2)/[s^*, 1) \neq \phi, \quad \text{or} \quad \beta_1^2(z_4) \cap \beta_1^3(z_1)/[t^*, 1] \neq \phi.$$

In either case $\beta_1^2(z_4)$ does not cross the intersected arc and consequently must contain z^*. In a completely analogous manner it can be shown that $\beta_2^2(z_4)$ also contains z^*. This, however, contradicts the fact that $\beta_1^2(z_4)$ and $\beta_2^2(z_4)$ are mutually exclusive. If $\beta_2^3(z_2) \cap \beta_1^3(z_1) \neq \phi$ a similar contradiction is obtained.

Case 2. $z_2 \in R_2$.

Again, either $\beta_1^3(z_2) \cap \beta_1^3(z_1) \neq \phi$ or $\beta_2^3(z_2) \cap \beta_1^3(z_1) \neq \phi$. As in Case 1 these subcases are similar and we consider only the latter. Let

$$t^* = \sup \{t : \beta_1^3(z_1; t) \in \beta_2^3(z_2)\},$$

$$z^* = \beta_1^3(z_1; t^*), \quad \text{and} \quad s^* = \text{be such that } z^* = \beta_2^3(z_2; s^*).$$

As both $\alpha^2(z_1; 0)$ and $\alpha^2(z_2; 0)$ lie on θ, there is an arc Γ on θ which extends from $\alpha^2(z_1; 0)$ to $\alpha^2(z_2; 0)$. Let R_4 be the region bounded by the arcs $\beta_2^3(z_2)/[s^*, 1)$, $\beta_1^3(z_1)/[t^*, 1)$, Γ, $\alpha^2(z_2)$, and the points z_1 and z_2. (If $\alpha^2(z_1) \cap \alpha^2(z_2) \neq \phi$ then subarcs of $\alpha^2(z_1)$ and $\alpha^2(z_2)$ must be used to define R_4. For definiteness we have supposed $\alpha^2(z_1)$ and $\alpha^2(z_2)$ to be disjoint.) As $R_4 \cap D_3 \neq \phi$ there is a point $z_4 \in R_4 \cap S_0^*$. Further, as $\beta_1^2(z_4; 0) \in \gamma_1$ and consequently is exterior to R_4 it follows that $\beta_1^2(z_4)$ intersects the boundary of R_4. But, $\beta_1^2(z_4)$ misses the arcs $\alpha^2(z_1)$ and $\alpha^2(z_2)$ and as both z_1 and z_2 lie on arcs contained in $\mathscr{A}$ neither z_1 nor z_2 is on $\beta_1^2(z_4)$. It follows that either

$$\beta_1^2(z_4) \cap \beta_1^3(z_1) \neq \phi \quad \text{or} \quad \beta_1^2(z_4) \cap \beta_2^3(z_2) \neq \phi.$$

In either case $\beta_1^2(z_4)$ does not cross the intersected arc and hence must contain z^*. In a similar manner we obtain that z^* also lies on $\beta_2^2(z_4)$ contradicting the fact that $\beta_1^2(z_4) \cap \beta_2^2(z_4) = \phi$.

Case 3. $z_2 \in R_3$.

This case is disposed of in a manner quite analogous to Case 2, and the proof of Claim 4 has been completed. Claim 4 then provides that the sets $\mathscr{B}_2^*$ and $\mathscr{B}_1^*$ are pointwise disjoint, and together with that which was previously noted we conclude that the sets $\mathscr{A}^*$, $\mathscr{B}_1^*$, and $\mathscr{B}_2^*$ are pointwise disjoint. Further, every point of the second category set S_0^* is accessible via each of $\mathscr{A}^*$, $\mathscr{B}_1^*$, and $\mathscr{B}_2^*$. However, in [4, Corollary 2, page 447] McMillan proves that the set of points accessible via three mutually exclusive sets is a set of first Baire category and the theorem obtains.

2. In the remainder of this paper we will extend Theorem 1 to a result concerning arbitrary functions from P into the Riemann sphere Ω. (Although our proof requires only that the range be a second countable compact Hausdorff space our primary interest is the special case where the range of f is a subset of Ω.) Let $f(z) = w$ be an arbitrary single valued function of $z \in P$ with values on the Riemann sphere. If Λ is an arc at $z \in P$ then the *cluster set of f at z along Λ*, denoted $C_\Lambda(f, z)$, is defined to be the set of all points $w \in \Omega$ having the property that there exists a sequence, $\{z_n\}$ on Λ converging to z such that $\{f(z_n)\}$ converges to w. It is easily seen that $C_\Lambda(f, z)$ is a compact subset of Ω.

A point $z \in P$ is said to be an *ambiguous point* of the function f if there exist arcs Λ and Γ at z such that $C_\Lambda(f, z) \cap C_\Gamma(f, z) = \phi$. The arcs Λ and Γ are called *arcs of ambiguity* at z. It is evident that the arcs of ambiguity at a point $z \in P$ may be taken to be disjoint. A point $z \in P$ is termed *a doubly ambiguous point* of the function f if there exist three arcs Λ, Γ_1, and Γ_2 at z such that

1. $C_\Lambda(f, z) \cap [C_{\Gamma_1}(f, z) \cup C_{\Gamma_2}(f, z)] = \phi$,

and

2. $C_{\Gamma_1}(f, z) \neq C_{\Gamma_2}(f, z)$.

Theorem 2. *If f is an arbitrary function from P into the Riemann sphere, then the set of doubly ambiguous points of f is a set of first Baire category.*

Proof. We use a technique similar to that developed by Bagemihl in [1]. In this paper Bagemihl defined an open spherical cap of Ω whose bounding circle is a rational disatance from the center, and whose center is either the point at infinity or a point with both rational real and rational imaginary parts to be a rational cap. We adopt his notation.

If $\mathscr{G}$ is the set of finite unions of rational caps, then $\mathscr{G}$ is enumerable and we can write $\mathscr{G}$ as a sequence

$$\mathscr{G} = G_1, G_2, \ldots, G_n \ldots .$$

Denote by Q the set of doubly ambiguous points of f, and let $z \in Q$. As $z \in Q$ there are arcs $\Lambda(z)$, $\Gamma_1(z)$, and $\Gamma_2(z)$ at z such that

1. $C_{\Lambda(z)}(f, z) \cap [C_{\Gamma_1(z)}(f, z) \cup C_{\Gamma_2(z)}(f, z)] = \phi$,
2. $C_{\Gamma_1(z)}(f, z) \neq C_{\Gamma_2(z)}(f, z)$.

Consequently, there are two indices $n(z)$ and $m(z)$ such that

1. $C_{\Lambda(z)}(f, z) \subset G_{n(z)}$,
2. $C_{\Gamma_1(z)}(f, z) \subset G_{m(z)}$,
3. $G_{n(z)} \cap G_{m(z)} = \phi$,
4. $C_{\Gamma_2(z)}(f, z) \subset \text{Int}\,[\Omega - G_{n(z)}]$ [Int $\equiv$ interior],
5. $C_{\Gamma_2(z)}(f, z) \not\subset \text{cl}(G_{m(z)})$ [cl $\equiv$ closure].

There are subarcs Λ^*, Γ_1^*, and Γ_2^* or Λ, Γ_1, and Γ_2 such that

1. $f(\Lambda^*) \subset G_{n(z)}$,
2. $f(\Gamma_1^*) \subset G_{m(z)}$,
3. $f(\Gamma_2^*) \subset \text{Int}\,[\Omega - G_{n(z)}]$.

Note that as $C_{\Gamma_2{}^*(z)}(f, z) \not\subset \text{cl}(G_{m(z)})$ it follows that $f(\Gamma_2^*) \not\subset G_{m(z)}$. Define

$$\begin{aligned} \mathscr{A}(n, m) &= \{\Lambda^*(z) : n(z) = n \text{ and } m(z) = m\}\,, \\ \mathscr{B}_1(n, m) &= \{\Gamma_1^*(z) : n(z) = n \text{ and } m(z) = m\}\,, \\ \mathscr{B}_2(n, m) &= \{\Gamma_2^*(z) : n(z) = n \text{ and } m(z) = m\}\,, \\ Q(n, m) &= \{z \in Q : n(z) = n \text{ and } m(z) = m\}\,. \end{aligned}$$

Then every point of $Q(n, m)$ is accessible via $\mathscr{A}(n, m)$, $\mathscr{B}_1(n, m)$, and $\mathscr{B}_2(n, m)$. But, $(\mathscr{A}(n, m), \mathscr{B}_1(n, m), \mathscr{B}_2(n, m))$ is an accessibility triple and hence $Q(n, m)$ is of first Baire category. It is evident that $Q = \bigcup Q(n, m)$ where n and m are natural numbers, and it follows that Q is the enumerable union of first category sets and is itself of first Baire category.

A number of examples of functions having ambiguities at a "large" set of points have been constructed (e.g. see [1, Theorem 2, page 206] where the function to which we refer is the characteristic function of the constructed set, S.) Many of these functions, however, map into either the real numbers, the unit interval, or a finite set considered as subsets of Ω. For these functions, and for those whose range is linearly orderable in a fashion compatible with the subspace topology inherited from Ω, certain natural multiple ambiguities are impossible at a second category subset of P. The following definitions and corollaries are to this point.

Let $S \subseteq \Omega$ be linearly ordered by $<$. Then we say $<$ is a *compatible order* if $(a, b) = \{s \in S : a < s < b\}$ is an open subset of S whenever $a < b$ in S. If $f : P \to \to S \subseteq \Omega$ and S is linearly ordered by $<$ then a point z is called a *separated ambiguous point* of f relative to $<$ if there are arcs α and β at z such that $C_\alpha(f, z) < < C_\beta(f, z)$ (i.e. If $x \in C_\alpha(f, z)$ and $y \in C_\beta(f, z)$ then $x < y$.) A *continuously ambiguous* point of a function f is an ambiguous point where f is continuous along the arcs* of ambiguity. As the thrust of this paper is to discuss multiple ambiguities we make the following definition. If f is function and $\mathfrak{m}$ is a cardinal number, a point $z \in P$ is a $\mathfrak{m}$-*ambiguous* point of f if there are $\mathfrak{m}$ pairs of arcs at z $\{(\alpha_x, \beta_x) : x \in X$ and $|X| = \mathfrak{m}\}$ satisfying

a) $C_{\alpha_x}(f, z) \cap C_{\beta_x}(f, z) = \phi$ for $x \in X$,

b) $C_{\alpha_x}(f, z) \neq C_{\alpha_y}(f, z)$ and $C_{\beta_x}(f, z) \neq C_{\beta_y}(f, z)$ for $x \neq y$,

c) $C_{\alpha_x}(f, z) \neq C_{\beta_y}(f, z)$ for $x \neq y$.

In addition, we will combine these terms so that a $\mathfrak{m}$-*separated ambiguous point*, z, of a function f is a $\mathfrak{m}$-ambiguous point of f and z is a separted ambiguous point with respect to each of the $\mathfrak{m}$ pairs of arcs at z.

Corollary 1. *If $S \subseteq \Omega$ is compact and has a compatible ordering $<$ and f is a function from P onto S then the set 2-separated ambiguous points of f is of the first Baire category.*

Proof. Let z be a 2-separated ambiguous point of f. Then there are pairs of arcs (α, β) and $(\alpha*, \beta*)$ at z such that

(i) $C_\alpha(f, z) < C_\beta(f, z)$,

(ii) $C_{\alpha*}(f, z) < C_{\beta*}(f, z)$.

It follows that z is a doubly ambiguous point of f using either the arcs $(\alpha, \beta, \beta*)$, or $(\alpha*, \beta, \beta*)$ and hence the 2-separated ambiguous points of f is a set of the first Baire category.

As every $\mathfrak{m}$-separated ambiguous point, $\mathfrak{m} \geqq 2$, is also a 2-separated ambiguous point we obtain the following.

Corollary 2. *If $S \subseteq \Omega$ is compact and has a compatible ordering $<$ and f is a function from P into S then the set of $\mathfrak{m}$-separated ambiguous points is of the first Baire category for $\mathfrak{m} \geqq 2$.*

Bagemihl has shown [1, p. 206, Theorem 2] (under the convention that we use the characteristic function of the constructed set S) that there is a function h such that every point of P is a 1-separated ambiguous point relative to h. Indeed, h has the additional property that every point of Ω is continuously ambiguous relative to h.

However, if $f : P \to S$ and S is a compact subset of Ω with a compatible ordering $<$, and f is continuous along an arc, α, at z, then $C_\alpha(f, z)$ is a closed interval in S or a point in S. Consequently, a continuously ambiguous point is a separated ambiguous point and we have the following corollary.

Corollary 3. *If $S \subseteq \Omega$ is compact and has a compatible ordering $<$ and f is a function from P into S then the set of $\mathfrak{m}$-continuously ambiguous points is of the first Baire category for $\mathfrak{m} \geqq 2$.*

To be published elsewhere is an example of a function $f : \dot{P} \to \Omega$ such that the set of $\aleph_0$-ambiguous points of f is a dense G_δ set of full measure in every disc.

References

[1] *F. Bagemihl,* Ambiguous points of arbitrary planar sets and functions. Zeitschr. f. math. Logik and Grundlagen d. Math. Bd. *12*, S. 205—217 (1966).

[2] *F. Bagemihl,* Curvilinear cluster sets of arbitrary functions. Proc. Nat. Acad. Sci. *41*, 379—382 (1955).

[3] *F. Bagemihl* and *P. Humke,* Rectifiably ambiguous points of planar sets. J. Australian Math. Soc. Vol. *XX*-(Series)-part 1, pp. 85—109 (1975).

[4] *H. Hahn,* Reelle Funktionen. Leipzig, 1932.

[5] *P. Humke,* Samely ambiguous points of arbitrary planar sets and functions. Zeitsch. f. math. Logik und Grundlagen d. Math. Bd. *19*, S. 427—433 (1973).

[6] *P. Humke,* An example of a function with multiple ambiguities. Zeitschr. f. math. Logik und Grundlagen d. Math. Bd. *21*, S. 413—416 (1975).

[7] *J. E. McMillan,* Arbitrary functions defined on plane sets. Michigan Math. J. *14*, 445—447 (1967).

Author's address: Department of Mathematics, Western Illinois University, Macomb, Illinois 61455, U.S.A.