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ON INFINITESIMAL ISOMETRIES OF SURFACES IN E^4

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To a given surface in E^n, there are too many infinitesimal isometries, and we cannot expect to prove reasonable rigidity theorems. In what follows, I restrict the infinitesimal isometries by a simple condition which enables me to prove a direct generalization of the classical rigidity theorem. The calculations are restricted to E^4, the general case is to be treated in the same way.

Let $M \subset E^4$ be a surface of class C^∞ with the boundary ∂M such that there is a diffeomorphism $\varphi : D \cup \partial D \to M \cup \partial M$, $D \subset \mathscr{R}^2$ being a bounded domain. Let $T(M)$ and $N(M)$ denote the tangent and normal bundle of M resp. The map

$$(1) \qquad II_m : N_m(M) \times T_m(M) \to \mathscr{R}\,, \quad m \in M\,,$$

be defined by

$$(2) \qquad II_m(n_0, t) = -\langle tm, tn \rangle$$

for any local section $n : M \to N(M)$ around m such that $n_m = n_0$. It will be shown that this is a good definition; for a given n_0, $II_m(n_0) \equiv II_m(n_0, \cdot)$ is a quadratic form on $T_m(M)$. Let $v : M \to V^4$ be a C^∞ map into the vector space of E^4; v is said to be an *infinitesimal isometry* of M if

$$(3) \qquad \langle tm, tv \rangle = 0 \quad \text{for each} \quad t \in T(M)\,.$$

We are going to prove the following

Theorem. *Let $n : M \to N(M)$ be a section such that, for each $m \in M$, the form $II_m(n_m)$ is definitive and the vector n_m is not orthogonal to the mean curvature vector ξ_m at m. Let v be an infinitesimal isometry of M such that, again for each $m \in M$, the vector v_m is situated in the vector space spanned by $T_m(M)$ and n_m. Further, let $v_m \perp T_m(M)$ for each $m \in \partial M$. Then $v = 0$ on M.*

Proof. To each point $m \in M$, associate an orthonormal frame $\{m, v_1, v_2, v_3, v_4\}$ such that $T_m(M) = \{m, v_1, v_2\}$. Then

$$
\begin{aligned}
(4) \qquad \mathrm{d}M &= \omega^1 v_1 + \omega^2 v_2 , \\
\mathrm{d}v_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 , \\
\mathrm{d}v_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 , \\
\mathrm{d}v_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 , \\
\mathrm{d}v_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3
\end{aligned}
$$

with the well known integrability conditions. From $\omega^3 = \omega^4 = 0$,

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0 , \quad \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0 ,$$

and we get the existence of functions $a_1, \ldots, b_3$ such that

$$
\begin{aligned}
(5) \qquad \omega_1^3 &= a_1\omega^1 + a_2\omega^2 , \quad \omega_1^4 = b_1\omega^1 + b_2\omega^2 , \\
\omega_2^3 &= a_2\omega^1 + a_3\omega^2 , \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2 .
\end{aligned}
$$

The mean curvature vector of M is given by

$$(6) \qquad \xi = (a_1 + a_3) v_3 + (b_1 + b_3) v_4 .$$

Let v be an infinitesimal isometry of M,

$$(7) \qquad v = xv_1 + yv_2 + zv_3 + tv_4 .$$

Then

$$
\begin{aligned}
(8) \qquad \mathrm{d}v = {} & (\mathrm{d}x - y\omega_1^2 - z\omega_1^3 - t\omega_1^4) v_1 + (\mathrm{d}y + x\omega_1^2 - z\omega_2^3 - t\omega_2^4) v_2 + \\
& + (\mathrm{d}z + x\omega_1^3 + y\omega_2^3 - t\omega_3^4) v_3 + (\mathrm{d}t + x\omega_1^4 + y\omega_2^4 + z\omega_3^4) v_4 .
\end{aligned}
$$

The condition (3) $\langle \mathrm{d}m, \mathrm{d}v \rangle = 0$ reduces to

$$(9) \qquad \omega^1(\mathrm{d}x - y\omega_1^2 - z\omega_1^3 - t\omega_1^4) + \omega^2(\mathrm{d}y + x\omega_1^2 - z\omega_2^3 - t\omega_2^4) = 0 ,$$

and there is a function p such that

$$(10) \qquad \mathrm{d}x - y\omega_1^2 - z\omega_1^3 - t\omega_1^4 = p\omega^2 , \quad \mathrm{d}y + x\omega_1^2 - z\omega_2^3 - t\omega_2^4 = -p\omega^1 .$$

Let

$$(11) \qquad n = Av_3 + Bv_4 .$$

Because of $\langle \xi, n \rangle \neq 0$,

$$(12) \qquad (a_1 + a_3) A + (b_1 + b_3) B \neq 0 .$$

Now,

$$\text{(13)} \qquad \mathrm{d}n = -(A\omega_1^3 + B\omega_1^4)\,v_1 - (A\omega_2^3 + B\omega_2^4)\,v_2 + \\ + (\mathrm{d}A - B\omega_3^4)\,v_3 + (\mathrm{d}B + A\omega_3^4)\,v_4\,,$$

i.e.,

$$\text{(14)} \qquad II(n) = \omega^1(A\omega_1^3 + B\omega_1^4) + \omega^2(A\omega_2^3 + B\omega_2^4) = \\ = (Aa_1 + Bb_1)(\omega^1)^2 + 2(Aa_2 + Bb_2)\,\omega^1\omega^2 + (Aa_3 + Bb_3)(\omega^2)^2\,.$$

The form (14) being definitive, we have

$$\text{(15)} \qquad (Aa_1 + Bb_1)(Aa_3 + Bb_3) - (Aa_2 + Bb_2)^2 > 0\,.$$

Because of $v \in \{v_1, v_2, n\}$, there is a function q such that $z = Aq$, $t = Bq$, and the equations (10) reduce to

$$\text{(16)} \qquad dx - y\omega_1^2 = (Aa_1 + Bb_1)\,q\omega^1 + \{(Aa_2 + Bb_2)\,q + p\}\,\omega^2\,, \\ \mathrm{d}y + x\omega_1^2 = \{(Aa_2 + Bb_2)\,q - p\}\,\omega^1 + (Aa_3 + Bb_3)\,q\omega^2\,.$$

Over M, choose the isothermic coordinates (u, v) such that

$$\text{(17)} \qquad I = r^2(\mathrm{d}u^2 + \mathrm{d}v^2)\,, \quad r(u, v) > 0\,; \qquad \omega^1 = r\,\mathrm{d}u\,, \quad \omega^2 = r\,\mathrm{d}v\,.$$

Then

$$\text{(18)} \qquad \omega_1^2 = r^{-1}(-r_v\,\mathrm{d}u + r_u\,\mathrm{d}v)$$

because of $\mathrm{d}\omega^1 = -\omega^2 \wedge \omega_1^2$, $\mathrm{d}\omega^2 = \omega^1 \wedge \omega_1^2$, and we have

$$\text{(19)} \qquad \frac{\partial x}{\partial u} + r^{-1}r_v y = (Aa_1 + Bb_1)\,qr\,, \quad \frac{\partial x}{\partial v} - r^{-1}r_u y = (Aa_2 + Bb_2)\,qr + pr\,, \\ \frac{\partial y}{\partial u} - r^{-1}r_v x = (Aa_2 + Bb_2)\,qr - pr\,, \quad \frac{\partial y}{\partial v} + r^{-1}r_u x = (Aa_3 + Bb_3)\,qr$$

from (16). The elimination of p and q yields

$$\text{(20)} \qquad (Aa_3 + Bb_3)\frac{\partial x}{\partial u} - (Aa_1 + Bb_1)\frac{\partial y}{\partial v} = \\ = (Aa_1 + Bb_1)\,r^{-1}r_u x - (Aa_3 + Bb_3)\,r^{-1}r_v y\,,$$

$$2(Aa_2 + Bb_2)\left(\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\right) - \{A(a_1 + a_3) + B(b_1 + b_3)\}\left(\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\right) = \\ = -2(Aa_2 + Bb_2)(r_u x + r_v y)\,r^{-1} - \\ - \{A(a_1 + a_3) + B(b_1 + b_3)\}(r_v x + r_u y)\,r^{-1}\,.$$

Recall [1] that the system

$$(21) \qquad a_{11}\frac{\partial x}{\partial u} + a_{12}\frac{\partial x}{\partial v} + b_{11}\frac{\partial y}{\partial u} + b_{12}\frac{\partial y}{\partial v} + c_1 x + e_1 y = f_1\,,$$

$$a_{21}\frac{\partial x}{\partial u} + a_{22}\frac{\partial x}{\partial v} + b_{21}\frac{\partial y}{\partial u} + b_{22}\frac{\partial y}{\partial v} + c_2 x + e_2 y = f_2$$

is called elliptic if

$$(22) \qquad \Delta := 4(a_{12}b_{22} - a_{22}b_{12})(a_{11}b_{21} - a_{21}b_{11}) - $$
$$- (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11})^2 > 0\,.$$

In our case,

$$(23) \qquad \Delta = 4\{(Aa_1 + Bb_1)(Aa_3 + Bb_3) - (Aa_2 + Bb_2)^2\}\,.$$
$$.\{A(a_1 + a_3) + B(b_1 + b_3)\}^2\,,$$

and $\Delta > 0$ because of (12) and (15). On the boundary ∂M, we have $x = y = 0$, and the maximum principle for the solutions of (20) implies $x = y = 0$ on M. The equations (16) imply

$$(24) \qquad (Aa_1 + Bb_1)\,q = (Aa_2 + Bb_2)\,q = (Aa_3 + Bb_3)\,q = 0\,;$$

because of (15), $q = 0$, i.e., $z = t = 0$. Thus $v = 0$ on M. QED.

Bibliography

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