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ON INFINITESIMAL ISOMETRIES OF SURFACES IN $E^4$

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To a given surface in $E^4$, there are too many infinitesimal isometries, and we cannot expect to prove reasonable rigidity theorems. In what follows, I restrict the infinitesimal isometries by a simple condition which enables me to prove a direct generalization of the classical rigidity theorem. The calculations are restricted to $E^4$, the general case is to be treated in the same way.

Let $M \subset E^4$ be a surface of class $C^\infty$ with the boundary $\partial M$ such that there is a diffeomorphism $\varphi : D \cup \partial D \to M \cup \partial M$, $D \subset \mathbb{R}^2$ being a bounded domain. Let $T(M)$ and $N(M)$ denote the tangent and normal bundle of $M$ resp. The map

$$II_m : N_m(M) \times T_m(M) \to \mathbb{R}, \quad m \in M,$$

be defined by

$$II_m(n_0, t) = -\langle tm, tn \rangle$$

for any local section $n : M \to N(M)$ around $m$ such that $n_m = n_0$. It will be shown that this is a good definition; for a given $n_0$, $II_m(n_0) \equiv II_m(n_0, \cdot)$ is a quadratic form on $T_m(M)$. Let $v : M \to V^4$ be a $C^\infty$ map into the vector space of $E^4$; $v$ is said to be an infinitesimal isometry of $M$ if

$$\langle tm, tv \rangle = 0 \quad \text{for each} \quad t \in T(M).$$

We are going to prove the following

**Theorem.** Let $n : M \to N(M)$ be a section such that, for each $m \in M$, the form $II_m(n_m)$ is definitive and the vector $n_m$ is not orthogonal to the mean curvature vector $\xi_m$ at $m$. Let $v$ be an infinitesimal isometry of $M$ such that, again for each $m \in M$, the vector $v_m$ is situated in the vector space spanned by $T_m(M)$ and $n_m$. Further, let $v_m \perp T_m(M)$ for each $m \in \partial M$. Then $v = 0$ on $M$. 

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Proof. To each point \( m \in M \), associate an orthonormal frame \( \{m, v_1, v_2, v_3, v_4\} \) such that \( T_m(M) = \{m, v_1, v_2\} \). Then

\[
\begin{align*}
\mathrm{d}M &= \omega^1 v_1 + \omega^2 v_2, \\
\mathrm{d}v_1 &= \omega^2_1 v_2 + \omega^3 v_3 + \omega^4 v_4, \\
\mathrm{d}v_2 &= -\omega^2_2 v_1 + \omega^3_2 v_3 + \omega^4_2 v_4, \\
\mathrm{d}v_3 &= -\omega^2_3 v_1 - \omega^3_3 v_2 + \omega^4_3 v_4, \\
\mathrm{d}v_4 &= -\omega^2_4 v_1 - \omega^3_4 v_2 - \omega^4_4 v_3
\end{align*}
\]

with the well known integrability conditions. From \( \omega^3 = \omega^4 = 0 \),

\[
\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 = 0, \quad \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^4 = 0,
\]

and we get the existence of functions \( a_1, \ldots, b_3 \) such that

\[
\begin{align*}
\omega^3 &= a_1 \omega^1 + a_2 \omega^2, \quad \omega^4 = b_1 \omega^1 + b_2 \omega^2, \\
\omega^3 &= a_2 \omega^1 + a_3 \omega^2, \quad \omega^4 = b_2 \omega^1 + b_3 \omega^2.
\end{align*}
\]

The mean curvature vector of \( M \) is given by

\[
\zeta = (a_1 + a_3) v_3 + (b_1 + b_3) v_4.
\]

Let \( \nu \) be an infinitesimal isometry of \( M \),

\[
\nu = x v_1 + y v_2 + z v_3 + t v_4.
\]

Then

\[
\begin{align*}
\mathrm{d}\nu &= (\mathrm{d}x - y \omega^2_1 - z \omega^3_1 - t \omega^4_1) v_1 + (\mathrm{d}y + x \omega^2_1 - z \omega^3_1 - t \omega^4_1) v_2 + \\
&\quad + (\mathrm{d}z + x \omega^3_1 + y \omega^3_1 - t \omega^4_1) v_3 + (\mathrm{d}t + x \omega^4_1 + y \omega^4_1 + z \omega^4_1) v_4.
\end{align*}
\]

The condition \( \langle \mathrm{d}m, \mathrm{d}\nu \rangle = 0 \) reduces to

\[
\omega^1(\mathrm{d}x - y \omega^2_1 - z \omega^3_1 - t \omega^4_1) + \omega^2(\mathrm{d}y + x \omega^2_1 - z \omega^3_1 - t \omega^4_1) = 0,
\]

and there is a function \( p \) such that

\[
\begin{align*}
\mathrm{d}x - y \omega^2_1 - z \omega^3_1 - t \omega^4_1 &= p \omega^2, \\
\mathrm{d}y + x \omega^2_1 - z \omega^3_1 - t \omega^4_1 &= -p \omega^1.
\end{align*}
\]

Let

\[
n = A v_3 + B v_4.
\]

Because of \( \langle \zeta, n \rangle \neq 0 \),

\[
(a_1 + a_3) A + (b_1 + b_3) B \neq 0.
\]
Now,
\begin{equation}
\frac{dn}{\omega_1} = -(A\omega_1^2 + B\omega_1^4)\, v_1 - (A\omega_2^2 + B\omega_2^4)\, v_2 + \\
+ (dA - B\omega_3^4)\, v_3 + (dB + A\omega_3^4)\, v_4 ,
\end{equation}
i.e.,
\begin{equation}
II(n) = \omega^1(A\omega_1^2 + B\omega_1^4) + \omega^2(A\omega_2^2 + B\omega_2^4) = \\
(Aa_1 + Bb_1)(\omega^1)^2 + 2(Aa_2 + Bb_2)\, \omega^1 \omega^2 + (Aa_3 + Bb_3)(\omega^2)^2 .
\end{equation}
The form (14) being definitive, we have
\begin{equation}
(Aa_1 + Bb_1)(Aa_3 + Bb_3) - (Aa_2 + Bb_2)^2 > 0 .
\end{equation}
Because of \( v \in \{ v_1, v_2, n \} \), there is a function \( q \) such that \( z = Ag, t = Bq \), and the equations (10) reduce to
\begin{equation}
\begin{align*}
dx - y\omega_1^2 &= (Aa_1 + Bb_1)\, q\omega^1 + \{(Aa_2 + Bb_2)\, q + p\} \, \omega^2 , \\
dy + x\omega_1^2 &= \{(Aa_2 + Bb_2)\, q - p\} \, \omega^1 + (Aa_3 + Bb_3)\, q\omega^2 .
\end{align*}
\end{equation}
Over \( M \), choose the isothermic coordinates \((u, v)\) such that
\begin{equation}
l = r^2(du^2 + dv^2), \quad r(u, v) > 0 ; \quad \omega^1 = r\, du , \quad \omega^2 = r\, dv .
\end{equation}
Then
\begin{equation}
\omega_1^2 = r^{-1}(-r_v\, du + r_u\, dv)
\end{equation}
because of \( d\omega^1 = -\omega^2 \wedge \omega_1^2, \, d\omega^2 = \omega^1 \wedge \omega_1^2 \), and we have
\begin{equation}
\begin{align*}
\frac{\partial x}{\partial u} + r^{-1}r_v y &= (Aa_1 + Bb_1)\, qr , \quad \frac{\partial x}{\partial v} - r^{-1}r_u y = (Aa_2 + Bb_2)\, qr + pr , \\
\frac{\partial y}{\partial u} - r^{-1}r_u x &= (Aa_2 + Bb_2)\, qr - pr , \quad \frac{\partial y}{\partial v} + r^{-1}r_v x = (Aa_3 + Bb_3)\, qr
\end{align*}
\end{equation}
from (16). The elimination of \( p \) and \( q \) yields
\begin{equation}
\begin{align*}
(Aa_3 + Bb_3)\, \frac{\partial x}{\partial u} - (Aa_1 + Bb_1)\, \frac{\partial y}{\partial v} &= \\
&= (Aa_1 + Bb_1)\, r^{-1}r_u x - (Aa_3 + Bb_3)\, r^{-1}r_v y ,
\end{align*}
\end{equation}
\begin{equation}
2(Aa_2 + Bb_2)\left(\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\right) - (A(a_1 + a_3) + B(b_1 + b_3))\left(\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\right) = \\
&= -2(Aa_2 + Bb_2)(r_u x + r_v y)\, r^{-1} - \\
&- (A(a_1 + a_3) + B(b_1 + b_3))\, (r_v x + r_u y)\, r^{-1} .
\end{equation}
Recall [1] that the system

\[ a_{11} \frac{\partial x}{\partial u} + a_{12} \frac{\partial x}{\partial v} + b_{11} \frac{\partial y}{\partial u} + b_{12} \frac{\partial y}{\partial v} + c_1 x + e_1 y = f_1, \]

\[ a_{21} \frac{\partial x}{\partial u} + a_{22} \frac{\partial x}{\partial v} + b_{21} \frac{\partial y}{\partial u} + b_{22} \frac{\partial y}{\partial v} + c_2 x + e_2 y = f_2 \]

is called elliptic if

\[ \Delta := 4(a_{12}b_{22} - a_{22}b_{12})(a_{11}b_{21} - a_{21}b_{11}) - (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11})^2 > 0. \]

In our case,

\[ \Delta = 4\{(Aa_1 + Bb_1)(Aa_3 + Bb_3) - (Aa_2 + Bb_2)^2\}. \]

\[ \Delta = \{A(a_1 + a_3) + B(b_1 + b_3)\}^2. \]

and \( \Delta > 0 \) because of (12) and (15). On the boundary \( \partial M \), we have \( x = y = 0 \), and the maximum principle for the solutions of (20) implies \( x = y = 0 \) on \( M \). The equations (16) imply

\[ (Aa_1 + Bb_1)q = (Aa_2 + Bb_2)q = (Aa_3 + Bb_3)q = 0; \]

because of (15), \( q = 0 \), i.e., \( z = t = 0 \). Thus \( v = 0 \) on \( M \). QED.

\textit{Bibliography}


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